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Walsh functions and their applications for solving nonlinear fractional-order Volterra integro-differential equation

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Abstract

In this article, we extended an efficient computational method based on Walsh operational matrix to find an approximate solution of nonlinear fractional order Volterra integro-differential equation. First, we present the fractional Walsh operational matrix of integration and differentiation. Then by applying this method, the nonlinear fractional Volterra integro-differential equation is reduced into a system of algebraic equation. The benefits of this method are the low-cost of setting up the equations without applying any projection method such as collocation, Galerkin, etc. The results show that the method is very accuracy and efficiency.

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1. Introduction

Many problems in sciences, economics, and engineering are modeled by a fractional differential equation and fractional integral equation. Nonlinear fractional-order Volterra integro differential equations arise in physics, biology, reactor dynamics and visco-elasticity [10, 11, 8, 22]. Many researchers have studied operational matrix of various orthogonal functions and polynomials, for example, Block-pulse functions [9, 1], Bernoulli wavelet [14], Hat function [3], Triangular function

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[13, 21], Boubaker functions [15], Bernstein polynomials [2] and Legendre function [24]. Haar wavelet operational matrix method has been applied for fractional Bagley-Torvik equation [25]. The authors have recently applied a fractional operational matrix for solving two-dimensional (2D) nonlinear integro-differential equations by BPFs [18]. E. Hesameddini et al. used shifted Legendre polynomials operational matrix to solve two dimensional fractional integral equations [12]. In [5] the single term Walsh series (STWS) techniques also developed to solve the system of Volterra integral equations. In [7, 6], researchers have extended the STWS method for the nonlinear Volterra integral equations and system of linear Volterra integro-differential equations.

Consider the following nonlinear fractional-order Volterra integro-differential equations

$$D^{\alpha}u(t) = \sum_{j=1}^{n} a_j(t)D^{\beta_j}u(t) + a_0(t)u(t) + g(t) + \int_0^t k(t,x)F(u(x))dx, \quad t \in [0,1] \tag{1.1}$$

with the supplementary conditions

$$u^{(i)}(0) = \delta_i \quad i = 0, 1, \cdots, \lceil \alpha \rceil - 1, \tag{1.2}$$

where $a_j(t)$ for $j = 0, 1, \cdots, n$ and functions $g(x)$, $k(t,x)F(u(x))$ are known and belong to $\Omega^2 \in [0,1]$. D^{α} is the Caputo fractional derivative operator of order α. The unknown function $u(x)$ needs to be determined. In this work, we consider that,

$$F(u(x)) = (u(x))^q,$$

where q is a positive integer number. This paper introduces a new operational method to solve the nonlinear fractional-order Volterra integro-differential equation (1.1). The method is based on reducing the equation to a system of algebraic equations by expanding the solution as Walsh functions.

2. Preliminaries and Basic Definitions

In this section, we initially recall some basic definitions and properties of the fractional integral and derivative.

Definition 2.1. *[19] A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0,\infty)$. Clearly, $C_\mu \in C_\beta$ if $\beta < \mu$.*

Definition 2.2. *[19] A function $f(x)$, $x > 0$ is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.*

Definition 2.3. *[19] The Riemann-Liouville fractional integral operator I^{θ_1} of order $\theta_1 \geq 0$, of a function $f \in C_\mu, \mu \geq 1$, is defined as*

$$(I^{\theta_1})f(x) = \begin{cases} \frac{1}{\Gamma(\theta_1)} \int_0^x \frac{f(s)}{(x-s)^{1-\theta_1}} ds, & \theta_1 > 0, \\ f(x), & \theta_1 = 0, \end{cases}$$

for $\theta_2 \geq -1$, the property of the operator I^{θ_1} that is needed in this article as

$$I^{\theta_1} x^{\theta_2} = \frac{\Gamma(\theta_2+1)}{\Gamma(\theta_2+\theta_1+1)} x^{\theta_1+\theta_2}.$$

Definition 2.4. *[19] The Caputo fractional derivative D^{θ_1} of order θ_1 is defined as*

$$(D^{\theta_1}f)(x)=\frac{1}{\Gamma(n-\theta_1)}\int_0^x \frac{f^{(n)}(s)}{(x-s)^{\theta_1+1-n}}ds, \quad \theta_1>0,$$

for $n-1<\theta_1\le n$, $n\in N$ and $f\in C_{-1}^{n}$, where $D=\frac{d}{dx}$ and $\Gamma(.)$ is the Gamma function.

A relation between Riemann-Liouville and Caputo fractional differentiation operator can be defined as follows:

Lemma 2.5. *[19] If $m-1<\alpha\le m$, $m\in\mathbb{N}$, then $D^{\alpha}I^{\alpha}u(x)=u(x)$, and:*

$$I^{\alpha}D^{\alpha}\mathrm{u}(x)=\mathrm{u}(x)-\sum_{k=0}^{m-1}\mathrm{u}^{(k)}(0^{+})\frac{x^k}{k!}, \quad x>0.$$

3. Definition and properties of Walsh function

Let f be an integrable function defined in $[0,1)$. The expansion of $f(x)$ with respect to the Walsh series is as follows:

$$f(x)=\sum_{i=0}^{\infty} f_i\Phi_i(x), \tag{3.1}$$

where $\Phi_i(x)$ is the ith Walsh function (WF), and f_i is the corresponding coefficient [23]. In the Walsh series approach, we consider only a finite number of terms. Then,

$$f(x)\simeq F^T\Phi(x), \tag{3.2}$$

where $F=[f_0,\dots,f_{m-1}]^T$ and $\Phi_m(x)=[\phi_0(x),\phi_1(x),\dots,\phi_{m-1}(x)]^T$.
The coefficients f_i are chosen to minimize the mean integrated squared error

$$\varepsilon=\int_0^1\Big[f(x)-F^T\Phi_m(x)\Big]^2dx, \tag{3.3}$$

and are given by

$$f_i=\int_0^1 f(x)\Phi_i(x)dx. \tag{3.4}$$

It has been proved that

$$\int_0^t f(x)dx=F^T\Upsilon\Phi_m(t), \tag{3.5}$$

where Υ is the operational matrix of the integration of the Walsh series. In Single Term Walsh Series, $\Upsilon_{1\times1}=\dfrac{1}{2}$ [5, 4]. The operational matrix of integration of $\Phi_m(t)$ is defined as

$$\int_0^t \Phi_m(x)dx\cong P_{m\times m}\Phi_m(t), \tag{3.6}$$

where $P_{m\times m}$ is the operational matrix of WFs [8]. This matrix can be expressed as follows:

$$P_{m\times m}=\begin{bmatrix} \frac{1}{2} & & -\frac{2}{m}I_{(\frac{m}{8})} & -\frac{1}{m}I_{(\frac{m}{4})} & -\frac{1}{2m}I_{(\frac{m}{2})} \\ & \ddots & & & \\ \frac{2}{m}I_{(\frac{m}{8})} & O_{(\frac{m}{8})} & & & \\ \frac{1}{m}I_{(\frac{m}{4})} & & & O_{(\frac{m}{4})} & \\ \frac{1}{2m}I_{(\frac{m}{2})} & & & & O_{(\frac{m}{2})} \end{bmatrix}. \tag{3.7}$$

Let A be a m-vector and B be a $m \times m$ matrix, then, it can be concluded that

$$\Phi_m(t)\Phi_m^T(t)A = \tilde{A}\Phi_m(t), \tag{3.8}$$

and

$$\Phi_m(t)B\Phi_m^T(t) = \hat{B}\Phi_m(t), \tag{3.9}$$

in which $\tilde{A} = diag(A)$ and B is a m vector with elements equal to the diagonal entries of B [8]. Let $\Psi_m = [b_0, b_1, \cdots, b_{m-1}]^T$. Clearly we can define

$$\Phi_m(t) = W_{m\times m}\Psi_m(t), \tag{3.10}$$

where $W_{m\times m}$ is the Walsh matrix, and Ψ_i are Block-pulse functions (BPFs) with unity height and $1/m$ width. BPFs are a set of piecewise constant orthogonal functions, defined in the time interval $[0, T_1]$:

$$b_i = \begin{cases} 1, & (i-1)\frac{T_1}{m} \le t < \frac{iT_1}{m}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, \cdots, m-1$ with m as a positive integer. The $W_{m\times m}$ matrix has the following properties that will be considered:

$$W_{m\times m}^2 = mI_m$$

or

$$W_{m\times m}^{-1} = \frac{1}{m}W_{m\times m}. \tag{3.11}$$

Substituting (3.10) into (3.6), yields

$$\int_0^t W_{m\times m}\Psi_m(x)dx = P_{m\times m}W_{m\times m}\Psi_m(t) \tag{3.12}$$

Therefore,

$$\int_0^t \Psi_m(x)dx = W_{m\times m}^{-1}P_{m\times m}W_{m\times m}\Psi_m(t). \tag{3.13}$$

Let

$$W_{m\times m}^{-1}P_{m\times m}W_{m\times m} = \Upsilon_{m\times m}. \tag{3.14}$$

Using (3.11), we have

$$\Upsilon_{m\times m} = \frac{1}{m}WPW. \tag{3.15}$$

Evaluating the similarity transformation yields:

$$\Upsilon_{m\times m} = \frac{1}{m}\begin{pmatrix} \frac{1}{2} & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{2} & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} \end{pmatrix}, \tag{3.16}$$

where $\Upsilon_{m\times m}$ is an operational matrix of integration for BPFs. Inspecting the $\Upsilon_{m\times m}$ matrix, the following decomposition can be made:

$$\Upsilon_{m\times m} = \frac{1}{m}(\frac{1}{2}I_m + Q_{m\times m} + Q_{m\times m}^2 + \cdots + Q_{m\times m}^{m-1})$$

$$
\begin{aligned}
&= \frac{1}{m}\left(\frac{1}{2}I_m + \sum_{i=1}^{\infty} Q^i_{m\times m}\right) && (3.17)\\
&= \frac{1}{m}\left(-\frac{1}{2}I_m + (I_m - Q_{m\times m})^{-1}\right)\\
&= \frac{1}{2m}(I_m + Q_{m\times m}(I_m - Q^{-1}_{m\times m})),
\end{aligned}
$$

where

$$
Q_{m\times m} = \frac{1}{m}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.18}
$$

Also, the following property can be concluded for $Q_{m\times m}$

$$
Q^i_{m\times m} = \begin{pmatrix} O & I_{m-i}\\ O & O \end{pmatrix} \quad for \quad i < m, \tag{3.19}
$$

and

$$
Q^i_{m\times m} = O_m \quad for \quad i \geq m. \tag{3.20}
$$

3.1. Operational Matrix of Differentiation

In this section, we want to derive an explicit formula for the Walsh function of the *mth* degree operational matrix of differentiation. Let us denote the operational matrix of differentiation as $\Upsilon_{m\times m}$ (see [8]).

$$
\begin{aligned}
\Upsilon^{-1}_{m\times m} &= 2m(I_m + Q_{m\times m}(I_m - Q_{m\times m})\\
&= 2m(I_m - 2Q_{m\times m} + 2Q^2_{m\times m} + \cdots (-1)^{m-1}Q^{m-1}_{m\times m})\\
&= 4m(\frac{1}{2}I_m + \sum_{i=1}^{m-1}(-1)^i Q^i_{m\times m}). && (3.21)
\end{aligned}
$$

Similarly, transformation back to the Walsh domain yields the operational matrix of differentiation, denoted by $D_{m\times m}$

$$
D_{m\times m} = P^{-1}_{m\times m} = \frac{1}{m}W_{m\times m}\Upsilon^{-1}_{m\times m}W_{m\times m}. \tag{3.22}
$$

In general, the formula is

$$
D_{m\times m} = 2m\left[\begin{array}{c:c} O_{(\frac{m}{2})} & I_{(\frac{m}{2})}\\ \hdashline -I(\frac{m}{2}) & \begin{array}{c:c} \begin{array}{c:c} \begin{array}{c:c} m & \\ \hdashline & \ddots \end{array} & -4I_{\frac{m}{8}}\\ \hdashline 4I_{(\frac{m}{8})} & O_{(\frac{m}{4})} \end{array} & -2I_{(\frac{m}{4})}\\ \hdashline 2I_{(\frac{m}{4})} & O\left(\frac{m}{4}\right) \end{array} \end{array}\right]. \tag{3.23}
$$

From (3.21) the eigenvalue, h^{-1}, of the $\Upsilon^{-1}_{m\times m}$ matrix can be expressed as the eigenvalue, q, of $Q_{m\times m}$

$$b = 4m(\frac{1}{2} + \sum_{i=1}^{m-1} (-1)^i q^i) \tag{3.24}$$

$$b = 2m\frac{1-q}{1+q} \tag{3.25}$$

3.2. Operational Matrices of Fractional Differentiation

Now we try to find the operational matrix of fractional differentiation. The general form of (3.25) could be written as follows:

$$b^\alpha = \left(2m\frac{1-q}{1+q}\right)^\alpha. \tag{3.26}$$

Equation (3.26) can be developed into polynomial of q and terminated at q^{m-1}. As a result, Eq. (3.26) becomes

$$b^\alpha = (2m)^\alpha \Lambda_{l,m}(q), \tag{3.27}$$

where $\Lambda_{l,m}$ is the polynomial of order $m-1$ for α differentiation. Thus the operational matrix for α differentiation from (3.21) is given by

$$B^\alpha_{m\times m} = (2m)^\alpha \Lambda_{l,m}(Q_{m\times m}). \tag{3.28}$$

In the Walsh domain, the corresponding α differentiation operational matrix is

$$D^\alpha_{m\times m} = (2m)^\alpha W^{-1}_{m\times m} \Lambda_{l,m}(Q_{m\times m}) W_{m\times m}. \tag{3.29}$$

3.3. Operational Matrices for Fractional Integration

We can rewrite (3.17) by expressing $\Upsilon_{m\times m}$ as a polynomial $Q_{m\times m}$

$$\Upsilon_{m\times m} = h_m(Q_{m\times m}), \tag{3.30}$$

where

$$h_m(x) = \frac{1}{m}(\frac{1}{2} + x + x^2 + \cdots + x^{m-1}). \tag{3.31}$$

If q is an eigenvalue of $Q_{m\times m}$, it is known (3.7) that corresponding eigenvalue for $\Upsilon_{m\times m}$ is

$$h = h_m(q) = \frac{1}{2m}\frac{1+q}{1-q}. \tag{3.32}$$

Therefore, it can be stated that the eigenvalues $\Upsilon_{m\times m}$ are $1/2m$ with multiplicity m. To find the operational matrix of fractional integration, we can use the same reasoning applied in the fractional differentiation case. Generalizing Eq. (3.32), yields

$$h = \left[\frac{1-q}{2m(1+q)}\right]^\alpha = (\frac{1}{2m}\rho_{l,m}(q))^\alpha \tag{3.33}$$

where $\rho_{l,m}$ is the polynomial of order $m-1$ for α integration.

The operational matrix for α-integration in terms of the BPF is given by

$$\Upsilon^\alpha_{(m\times m)} = \frac{1}{(2m)^\alpha}\rho_{l,m}(Q_{m\times m}) \tag{3.34}$$

and the corresponding α-integration operational matrix in the Walsh domain is easily found as

$$\begin{aligned} P^{\alpha}_{m\times m} &= \frac{1}{(2m)^{\alpha}} W^{-1}_{m\times m}\rho_{l,m}(Q_{m\times m})W_{m\times m} \\ &= \frac{1}{m(2m)^{\alpha}} W_{m\times m}\rho_{l,m}(Q_{m\times m})W_{m\times m}, \end{aligned} \tag{3.35}$$

Therefore, we have the following nonlinear system.

$$(I^{\alpha}\Phi_m)(t) = P^{\alpha}_{m\times m}\Phi_m(t). \tag{3.36}$$

4. Applying the method

In this section, nonlinear Volterra integro-differential equations are solved using WFs. As demonstrated before, we can write

$$\begin{aligned} g(t) &= G^T\Phi_m(t), \\ D^{\alpha}u(t) &= C^T\Phi_m(t), \\ a_j(t) &= A_j^T\Phi_m(t), \\ k(t,x) &= \Phi_m^T(t)K\Phi_m(x), \end{aligned} \tag{4.1}$$

where $A_j = [a_0^j, a_1^j, \cdots, a_{m-1}^j]^T$ and $G = [g_0, g_1, \cdots, g_{m-1}]^T$ are known m-vectors. However $C = [c_0, c_1, \cdots, c_{m-1}]^T$ is an unknown $m-$vector. Consider Eq. (1.1)

$$D^{\alpha}u(t) = \sum_{j=1}^{n} a_j(t)D^{\beta_j}u(t) + a_0(t)u(t) + g(t) + \int_0^t k(t,x)F(u(x))dx, \quad t\in[0,1] \tag{4.2}$$

subject to the initial conditions

$$u^{(k)}(0) = 0, \quad k = 0, 1, \cdots, \lceil\alpha\rceil - 1. \tag{4.3}$$

Using Eq. (4.1) together with the property of fractional calculus, we have

$$D^{\beta_j}u(t) = I^{\alpha-\beta_j}[D^{\alpha}u(t)] = I^{\alpha-\beta_j}[C^T\Phi_m(t)] = C^T P^{\alpha-\beta_j}_{m\times m}\Phi_m(t). \tag{4.4}$$

Substituting Eqs. (4.1) and (4.4) into (4.2), we have

$$\begin{aligned} C^T\Phi_m(t) &= \sum_{j=1}^{n}\Phi_m^T(t)A_jC^TP^{\alpha-\beta_j}_{m\times m}\Phi_m(t) + \Phi_m^T(t)A_0CP^{\alpha}_{m\times m}\Phi_m(t) \\ &\quad + G^T\Phi_m(t) + \int_0^t k(t,x)F(u(x))dx, \end{aligned} \tag{4.5}$$

By using (3.9), we can write

$$C^T\Phi_m(t) = \sum_{j=1}^{n}\hat{\Theta}_j^T\Phi_m(t) + \hat{\Lambda}^T\Phi_m(t) + G^T\Phi_m(t) + \int_0^t k(t,x)F(u(x))dx, \tag{4.6}$$

where $\Theta_j = A_jC^TP^{\alpha-\beta_j}_{m\times m}$ and $\Lambda = A_0CP^{\alpha}_{m\times m}$. By using Eqs.(4.1) and (3.35) and Lemma 2.5, we have:

$$u(t) \cong C^TP^{\alpha}_{m\times m}\Phi_m(t) + \sum_{k=0}^{m-1}u^{(k)}(0^+)\frac{t^k}{k!}. \tag{4.7}$$

Hence, by substituting the supplementary initial conditions (1.2) in the above summation of the above equations, we have:

$$u(t) \cong (C^T P^{\alpha}_{m\times m} + C_1^T)\Phi_m(t). \tag{4.8}$$

It can be written as:

$$u(t) \simeq e^T \Phi_m(t)$$

where $e = CP^{\alpha}_{m\times m} + C_1$ and C_1 is the corresponding vector of the function $\sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{t^k}{k!}$ in the Walsh function basis. Now, we approximate $F(u(x)) = u(x)^q$ in the following way:

$$(u(x))^2 = e^T\Phi_m(t)\Phi_m^T(x)e = e^T\tilde{e}\Phi_m(t) = e_2^T\Phi_m(x), \tag{4.9}$$

where $\tilde{e}$ is the product operational matrix for the vector e. Also,

$$\begin{aligned}
(u(x))^3 &= e^T\Phi_m(x)\Phi_m^T(x)e_2 = e^T\tilde{e}_2\Phi_m(x) = e_3^T\Phi_m(x),\\
&\vdots\\
(u(x))^q &= e^T\tilde{e}_{q-1}\Phi_m(x) = e_q^T\Phi_m(x),
\end{aligned} \tag{4.10}$$

where $\tilde{e}_{q-1}$ is the product operational matrix for the vector e_{q-1}, by assuming $e^T\tilde{e}_{q-2} = e_{q-1}^T$. Using Eqs. (4.1), and (4.10), we have

$$\begin{aligned}
\int_0^t k(t,x)F(u(x))dx &= \int_0^t \Phi_m^T(t)K\Phi_m(x)\Phi_m^T(x)e_q dx = \Phi_m^T(t)K\int_0^t \Phi_m(x)\Phi_m^T(x)e_q dx\\
&= \Phi_m^T(t)K\tilde{e_q}\int_0^t \Phi_m(x)dx = \Phi_m^T(t)K\tilde{e_q}P_{m\times m}\Phi_m(t).
\end{aligned}$$

Substituting Eq. (4.11) into Eq. (4.6), yields:

$$C^T\Phi_m(t) = \sum_{j=1}^{n}\hat{\Theta}_j^T\Phi_m(t) + \hat{\Lambda}^T\Phi_m(t) + G^T\Phi_m(t) + (\widehat{K\tilde{e}_q P_{m\times m}})\Phi_m(t),$$

or

$$C^T = \sum_{j=1}^{n}\hat{\Theta}_j^T + \hat{\Lambda}^T + G^T + (\widehat{K\tilde{e}_q P_{m\times m}}),$$

which is a system of algebraic equations. By solving this system we can obtain the approximate solution of Eq. (1.1) by using

$$u(t) \cong (C^T P^{\alpha}_{m\times m} + C_1^T)\Phi_m(t). \tag{4.11}$$

5. Numerical Examples

In this section, to demonstrate the validity and applicability of the numerical scheme, we apply the present method for the following illustrative examples.

Example 5.1. *Consider the following fractional integro differential equation [20]*

$$D^{\alpha}u(t) = g(t) - tu(t) + \frac{1}{\Gamma(6.5)}\int_0^t (t-x)^{5.5}(u(t))^3 dx, \quad 0 \le t \le 1, \tag{5.1}$$

$$g(t) = \Gamma(\frac{8}{3}) + x^{\frac{8}{3}} - \frac{0.000252451x^{11.5}}{\Gamma(6.5)}$$

with the initial condition

$$u(0) = \acute{u}(0) = 0. \tag{5.2}$$

The exact solution of this example for $\alpha = \frac{5}{3}$ *is* $u(t) = t^{\frac{5}{3}}$. *Table 1 displays the absolute error obtained between the approximate solution and the exact solution* ($|u(t) - u_m(t)|$) *for* $m = 5$ *and different values of* α. *Also, the numerical results for* $u(t)$ *with* $m = 5$ *and* $\alpha = 0.75, 1, 1.5$ *and* 1.6 *are shown Fig 1 and Fig 2.*

Example 5.2. *As the second example, we consider the following linear fourth-order fractional integro-differential equation*

$$D^{\alpha}u(t) = t(1 + e^t) + 3e^t + u(t) - \int_0^t u(x)dx, \tag{5.3}$$

with the following boundary conditions

$$u(0) = 1, \quad u^{'}(0) = 1, \quad u^{''}(0) = 2, \quad u^{'''}(0) = 3. \tag{5.4}$$

The exact solution, when $\alpha = 4$, *is* $u(t) = 1 + te^t$. *Numerical results are presented in the Table 2 which illustrate the absolute errors for* $m = 12$. *Also, the numerical results for* $u(t)$ *with* $m = 12$ *and* $\alpha = 3.25, 3.5, 3.75$ *and* 4 *are shown Fig 3 and Fig 4.*

Example 5.3. *As the third example, we consider the following nonlinear fractional-order integro-differential equation [16]*

$$D^{\alpha}u(t) = g(t)u(t) + h(t) + \sqrt{t}\int_0^t u^2(x)dx, \tag{5.5}$$

where

$$g(t) = 2\sqrt{t} + 2t^{\frac{3}{2}} - (\sqrt{t} + t^{\frac{3}{2}})Ln(1 + t), \quad h(t) = \frac{2Arcsinh(\sqrt{t})}{\sqrt{\pi}\sqrt{1 + t}} - 2t^{\frac{3}{2}} \tag{5.6}$$

with the initial condition

$$u(0) = 0. \tag{5.7}$$

The exact solution of this example for $\alpha = 0.5$ *is* $u(t) = Ln(1 + t)$. *Table 3 displays the absolute error obtained between the approximate solution and the exact solution* ($|u(t) - u_m(t)|$) *for* $m = 12$ *and different values of* α. *Also, the numerical results for* $u(t)$ *with* $m = 12$ *and* $\alpha = 0.25, 0.5, 0.75$ *and* 1 *are shown Fig 5 and Fig 6.*

Table 1: The absolute errors with $m = 5$ and different value of α for Example 1

	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.5$
t	The method of [20]	Present method	The method of [20]	Present method
0.0	2.56×10^{-3}	4.02×10^{-4}	8.25×10^{-5}	5.04×10^{-5}
0.1	1.91×10^{-4}	5.20×10^{-4}	1.26×10^{-5}	5.48×10^{-6}
0.2	1.19×10^{-3}	3.54×10^{-3}	1.44×10^{-5}	2.78×10^{-6}
0.3	1.78×10^{-3}	1.27×10^{-3}	8.99×10^{-6}	4.54×10^{-6}
0.4	1.59×10^{-3}	2.71×10^{-3}	8.93×10^{-6}	4.35×10^{-6}
0.5	1.02×10^{-3}	1.51×10^{-4}	1.94×10^{-5}	3.54×10^{-5}
0.6	4.51×10^{-4}	6.57×10^{-4}	1.47×10^{-5}	6.50×10^{-5}
0.7	2.45×10^{-4}	7.36×10^{-3}	1.25×10^{-6}	4.24×10^{-5}
0.8	6.97×10^{-4}	9.09×10^{-3}	1.52×10^{-5}	2.35×10^{-5}
0.9	2.06×10^{-3}	3.81×10^{-3}	6.82×10^{-6}	9.51×10^{-6}
1	4.55×10^{-3}	3.31×10^{-3}	5.00×10^{-5}	3.67×10^{-6}

Table 2: The absolute errors with $m = 12$ and different values of α for Example 2

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
t	$u_m(t)$	$u_m(t)$	$u_m(t)$	$u_m(t)$
0.0	4.41×10^{-6}	5.41×10^{-6}	2.01×10^{-5}	7.21×10^{-8}
0.1	1.91×10^{-6}	5.20×10^{-4}	1.26×10^{-6}	5.24×10^{-8}
0.2	2.74×10^{-7}	5.57×10^{-5}	6.37×10^{-6}	9.87×10^{-8}
0.3	6.57×10^{-7}	4.93×10^{-5}	3.58×10^{-6}	9.01×10^{-7}
0.4	9.35×10^{-7}	6.00×10^{-6}	7.29×10^{-7}	4.35×10^{-7}
0.5	7.02×10^{-7}	8.74×10^{-6}	6.17×10^{-7}	4.78×10^{-7}
0.6	6.91×10^{-6}	1.27×10^{-6}	4.27×10^{-8}	6.90×10^{-8}
0.7	6.68×10^{-6}	3.74×10^{-6}	5.14×10^{-8}	4.24×10^{-8}
0.8	2.69×10^{-6}	6.09×10^{-7}	6.27×10^{-8}	2.35×10^{-9}
0.9	3.74×10^{-7}	8.84×10^{-7}	3.89×10^{-7}	9.51×10^{-9}
1	4.40×10^{-7}	4.97×10^{-6}	6.06×10^{-7}	2.54×10^{-8}

Table 3: The absolute errors with $m = 12$ and different values of α for Example 3

	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.5$
t	The method of [20]	Present method	The method of [20]	Present method
0.0	3.31×10^{-7}	2.01×10^{-5}	3.97×10^{-11}	5.80×10^{-7}
0.1	1.47×10^{-6}	1.22×10^{-6}	2.51×10^{-5}	7.74×10^{-9}
0.2	1.14×10^{-6}	3.81×10^{-6}	2.59×10^{-9}	7.78×10^{-9}
0.3	1.67×10^{-6}	8.61×10^{-7}	2.92×10^{-9}	3.31×10^{-10}
0.4	1.91×10^{-6}	3.30×10^{-7}	3.73×10^{-9}	4.87×10^{-10}
0.5	2.78×10^{-6}	5.61×10^{-7}	5.18×10^{-9}	1.75×10^{-8}
0.6	4.48×10^{-6}	7.01×10^{-7}	7.95×10^{-9}	8.64×10^{-9}
0.7	7.27×10^{-6}	8.91×10^{-7}	1.32×10^{-8}	1.01×10^{-9}
0.8	1.29×10^{-5}	7.01×10^{-7}	2.39×10^{-8}	7.74×10^{-8}
0.9	2.59×10^{-5}	9.11×10^{-6}	4.71×10^{-8}	2.42×10^{-9}
1	5.50×10^{-5}	3.31×10^{-6}	1.01×10^{-7}	2.31×10^{-8}

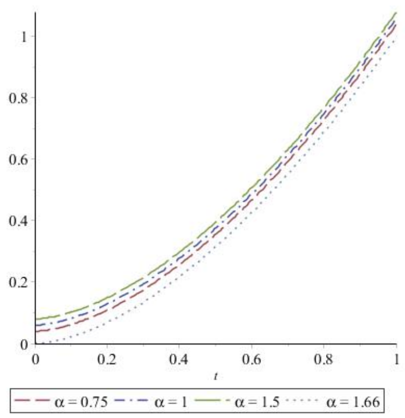

Figure 1: Approximate solution of Example 1, with $m = 5$ and some $0.75 \leq \alpha \leq 1.6$

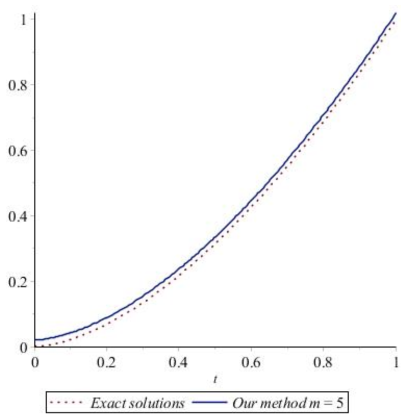

Figure 2: A comparison between the approximate and exact solution of Example 1.

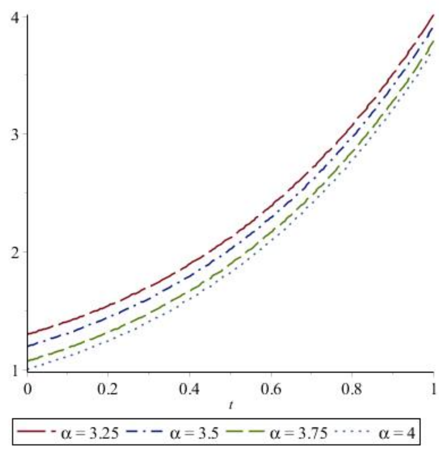

Figure 3: Approximate solution of Example 2, with $m = 12$ and some $3.25 \leq \alpha \leq 4$

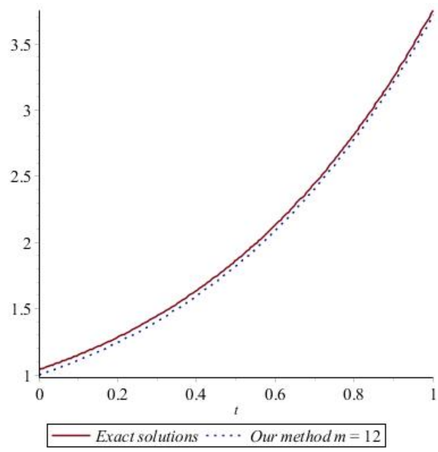

Figure 4: A comparison between the approximate and exact solution of Example 2.

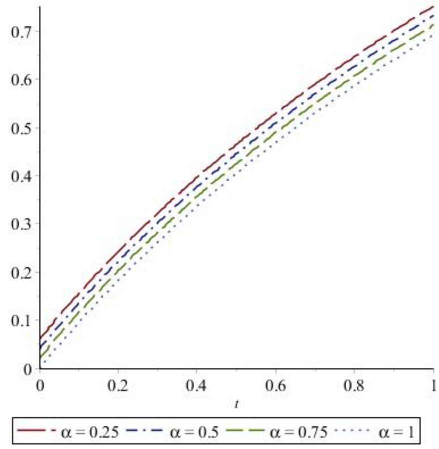

Figure 5: Approximate solution of Example 3, with $m = 12$ and some $0.25 \leq \alpha \leq 1$

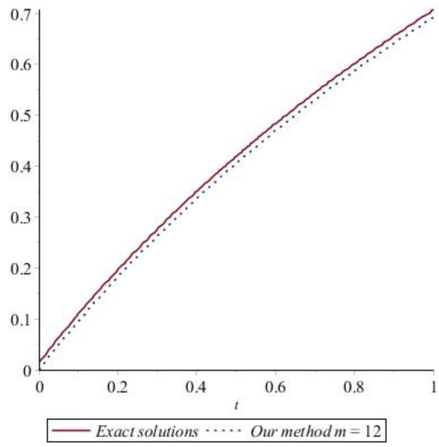

Figure 6: A comparison between the approximate and exact solution of Example 3.

6. Conclusion

This paper aimed to extend a Walsh functions for obtaining approximate solution of nonlinear fractional-order Volterra integro-differential equations. First, the Walsh function fractional operational matrix of differentiation and integration was presented. Using this matrix, the nonlinear fractional-order Volterra integro-differential equation was reduced to a system of algebraic equations. The benefits of this method are the low cost of setting up equations without applying a projection method such as collocation, Galerkin etc. The numerical results indicated the high accuracy and efficiency of the proposed method.

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