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TWO COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS

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Abstract. Recently, Zhang and Song [Q. Zhang, Y. Song, Fixed point theory for generalized φ-weak contractions, Appl. Math. Lett. 22(2009) 75-78] proved a common fixed point theorem for two maps satisfying generalized φ-weak contractions. In this paper, we prove a common fixed point theorem for a family of compatible maps. In fact, a new generalization of Zhang and Song's theorem is given.

1. Introduction and preliminaries

Let X be a metric space. A map $T : X \to X$ is a contraction if there exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$.
A map $T : X \to X$ is a φ-weak contraction if there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ such that φ is positive on $(0, +\infty)$, $\varphi(0) = 0$ and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)). \tag{1.1}$$

The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. Actually in [1], the authors defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [20] showed that most results of [1] are still true for any Banach spaces. Also, Rhoades [20] proved an interesting fixed point theorem which is one of generalizations of the Banach contraction principle because it contains contractions as special cases ($\varphi(t) = (1 - k)t$).

Theorem 1.1. [20] *Let (X, d) be a complete metric space and A be a $\varphi-$ weak contraction on X. If φ is continuous and nondecreasing function, then A has a unique fixed point.*

In fact, the weak contractions are also closely related to maps of Boyd and Wong's type [4] and Reich's type [19]. Namely, if φ is a lower semi-continuous function from the right, then $\psi(t) = t - \varphi(t)$ is an upper semi-continuous function from the right and moreover, (1.1) turns into $d(Tx, Ty) \leq \psi(d(x, y))$. Therefore, the φ-weak contraction with a function φ is of Boyd and Wong [4]. if we define $K(t) = \frac{\varphi(t)}{t}$ for

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$t > 0$ and $K(0) = 0$, then (1.1) is replaced by $d(Tx, Ty) \leq K(d(x, y))d(x, y)$. Thus the φ-weak contraction becomes a Reich type one.

During the last few decades, a number of hybrid contractive mapping results have been obtained by many mathematical researchers. For example, Song [25, 26], Al-Thagafi and Shahzad [2], Shahzad [21] and Hussain and Junck [11] obtained the common fixed pint theorems of f-contraction $(T(d(Tx, Ty) \leq kd(fx, fy)))$, generalized f-contraction

$$(T(d(Tx, Ty) \leq k \max\{d(fx, fy), d(Tx, fx), d(Ty, fy), \frac{1}{2}[d(fx, Ty) + d(Tx, fy)]\}))$$

and generalized (f, g)-contraction

$$(T(d(Tx, Ty) \leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), \frac{1}{2}[d(fx, Ty) + d(Tx, gy)]\})),$$

respectively.
Song [24] extended the above results to f-weak contraction $(d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)))$.

Recently, Zhang and Song [30] proved the following theorem.

Theorem 1.2. [30] *Let (X, d) be a complete metric space and $T, S : X \to X$ two mappings such that for all $x, y \in X$,*

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t > 0, \varphi(0) = 0$ and

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}.$$

Then, there exists a unique point $u \in X$ such that $Tu = Su = u$.

The object of this paper is to prove a common fixed point theorem for a family of compatible maps in a metric space.

2. Main result

In this section, we shall prove a common fixed point theorem for any even number of compatible maps in a complete metric space. In fact, it is a generalization of Zhang and Song's common fixed point theorem (Theorem 1.2).

Let (X, d) be a metric space and T a self-mapping on X. In [7], Ćirić introduced and investigated a class of self-mappings on X satisfying the following condition:

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \quad (c)$$

where $0 < k < 1$. In [8] Ćirić proved the following common fixed point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space and let $\{T_\alpha\}_{\alpha \in J}$ be a family of self-mappings on X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$ and all $x, y \in X$*

$$d(T_\alpha x, T_\beta y) \leq \lambda \max\{d(x, y), d(x, T_\alpha x), d(y, T_\beta y), \frac{1}{2}[d(x, T_\beta y) + d(y, T_\alpha x)]\},$$

where $\lambda = \lambda(\alpha) \in (0, 1)$, then all T_α have a unique common fixed point in X.

The class of mappings satisfying the contractive definition of type of (c), as well as its generalization, has proved useful in fixed and common fixed point theory (see [3, 18, 23]).

Definition 2.2. [13] Self-maps A and S of a metric space (X, d) are said to be compatible if $d(ASp_n, SAp_n) \to 0$ whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.

Definition 2.3. [15] Self-maps A and S of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e. if $Ap = Sp$ for some $p \in X$, then $ASp = SAp$.

This concept is most general among all the commutativity concepts in this field, as every pair of weakly commuting self-maps is compatible and each pair of compatible self-maps is weakly compatible, but the reverse is not true always. Many authors have proved common fixed point theorems for a variety of commuting self-mappings on usual metric, as well as on different kinds of generalized metric spaces([3, 5, 6, 8],[9]-[17], [22, 23],[27]-[29]).

Theorem 2.4. [22] *Let A,B,S,T,L and M be self-maps of a complete metric space (X, d), satisfying the conditions:*
(1) $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$;
(2) $AB = BA, ST = TS, LB = BL, MT = TM$;
(3) For all $x, y \in X$ and for some $k \in (0, 1)$,

$$\begin{aligned} d(Lx, My) \leq \quad & k \max\{d(Lx, ABx), d(My, STy), d(ABx, STy), \\ & \tfrac{1}{2}[d(Lx, STy) + d(My, ABx)]\}; \end{aligned}$$

(4) The pair (L, AB) is compatible and the pair (M, ST) is weakly compatible;
(5) Either AB or L is continuous.
Then, A, B, S, T, L and M have a unique common fixed point.

Define $\Phi = \{\varphi : [0, +\infty) \to [0, +\infty)\}$ where each $\varphi \in \Phi$ satisfies the following conditions:
(a) φ is lower semi-continuous on $[0, +\infty)$,
(b) φ is non-decreasing,
(c) $\varphi(0) = 0$, and
(d) $\varphi(t) > 0$ for each $t > 0$.

Now, we prove our main result.

Theorem 2.5. *Let $P_1, P_2, \cdots, P_{2n}, Q_0$ and Q_1 be self-maps on a complete metric space (X,d), satisfying conditions:*
(1) $Q_0(X) \subseteq P_1P_3, \cdots P_{2n-1}(X), Q_1(X) \subseteq P_2P_4, \cdots P_{2n}(X)$;

(2)

$$\begin{aligned}
P_2(P_4\cdots P_{2n}) =& \ (P_4\cdots P_{2n})P_2,\\
P_2P_4(P_6\cdots P_{2n}) =& \ (P_6\cdots P_{2n})P_2P_4,\\
&\vdots\\
P_2\cdots P_{2n-2}(P_{2n}) =& \ (P_{2n})P_2\cdots P_{2n-2},\\
Q_0(P_4\cdots P_{2n}) =& \ (P_4\cdots P_{2n})Q_0,\\
Q_0(P_6\cdots P_{2n}) =& \ (P_6\cdots P_{2n})Q_0,\\
&\vdots\\
Q_0P_{2n} =& \ P_{2n}Q_0,\\
P_1(P_3\cdots P_{2n-1}) =& \ (P_3\cdots P_{2n-1})P_1,\\
P_1P_3(P_5\cdots P_{2n-1}) =& \ (P_5\cdots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\cdots P_{2n-3}(P_{2n-1}) =& \ (P_{2n-1})P_1\cdots P_{2n-3},\\
Q_1(P_3\cdots P_{2n-1}) =& \ (P_3\cdots P_{2n-1})Q_1,\\
Q_1(P_5\cdots P_{2n-1}) =& \ (P_5\cdots P_{2n-1})Q_1,\\
&\vdots\\
Q_1P_{2n-1} =& \ P_{2n-1}Q_1;
\end{aligned}$$

(3) $P_2\cdots P_{2n}$ *or* Q_0 *is continuous;*
(4) The pair $(Q_0, P_2\cdots P_{2n})$ *is compatible and the pair* $(Q_1, P_1\cdots P_{2n-1})$ *is weakly compatible;*
(5) There exists $\varphi\in\Phi$ *such that*

$$d(Q_0u, Q_1v)\le M(u,v)-\varphi(M(u,v)), \forall u,v\in X,$$

where

$$\begin{aligned}
M(u,v) \ &= \max\{d(P_2P_4\cdots P_{2n}u, Q_0u), d(P_1P_3\cdots P_{2n-1}v, Q_1v),\\
&\quad d(P_2P_4\cdots P_{2n}u, P_1P_3\cdots P_{2n-1}v),\\
&\quad \tfrac{1}{2}[d(P_1P_3\cdots P_{2n-1}v, Q_0u)+d(P_2P_4\cdots P_{2n}u, Q_1v)]\}
\end{aligned}$$

for all $u,v\in X$*. Then* $P_1, P_2, \cdots, P_{2n}, Q_0$ *and* Q_1 *have a unique common fixed point in* X*.*

Proof. Let $x_0\in X$, from condition (1) there exist $x_1, x_2\in X$ such that $Q_0x_0 = P_1P_3\cdots P_{2n-1}x_1 = y_0$ and $Q_1x_1 = P_2P_4\cdots P_{2n}x_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X:

$$Q_0x_{2k} = P_1P_3\cdots P_{2n-1}x_{2k+1} = y_{2k}$$

and

$$Q_1x_{2k+1} = P_2P_4\cdots P_{2n}x_{2k+2} = y_{2k+1},$$

for $k\in\mathbb{N}$.

Putting $u = x_p = x_{2k}, v = x_{q+1} = x_{2m+1}, G_1 = P_2P_4\cdots P_{2n}$ and $G_2 = P_1P_3\cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned}
d(Q_0x_{2k}, Q_1x_{2m+1}) \le& \ M(x_{2k}, x_{2m+1}) - \varphi(M(x_{2k}, x_{2m+1}))\\
\le& \ M(x_{2k}, x_{2m+1})\\
=& \ \max\{d(G_1x_{2k}, Q_0x_{2k}), d(G_2x_{2m+1}, Q_1x_{2m+1}),\\
&\quad d(G_1x_{2k}, G_2x_{2m+1}),\\
&\quad \tfrac{1}{2}[d(G_2x_{2m+1}, Q_0x_{2k}) + d(G_1x_{2k}, Q_1x_{2m+1})]\}
\end{aligned}$$

i.e.,

$$d(y_{2k}, y_{2m+1}) \leq \max\{d(y_{2k-1}, y_{2k}), d(y_{2m}, y_{2m+1}), d(y_{2k-1}, y_{2m}), \frac{1}{2}[d(y_{2m}, y_{2k}) + d(y_{2k-1}, y_{2m+1})]\}.$$

Thus

$$d(y_p, y_{q+1}) \leq \max\{d(y_{p-1}, y_p), d(y_q, y_{q+1}), d(y_{p-1}, y_q), \frac{1}{2}[d(y_q, y_p) + d(y_{p-1}, y_{q+1})]\}.$$

If $q = p$, then

$$\begin{aligned} \frac{1}{2}[d(y_p, y_p) + d(y_{p-1}, y_{p+1})] &\leq \frac{1}{2}[d(y_{p-1}, y_p) + d(y_p, y_{p+1})] \\ &\leq \max\{d(y_{p-1}, y_p), d(y_p, y_{p+1})\}. \end{aligned}$$

Thus $(y_p, y_{p+1}) \leq d(y_{p-1}, y_p)$ as the inequality $d(y_p, y_{p+1}) > d(y_{p-1}, y_p)$ implies $M(x_p, x_{p+1}) = d(y_p, y_{p+1})$ and furthermore,

$$d(y_p, y_{p+1}) \leq d(y_p, y_{p+1}) - \varphi(d(y_p, y_{p+1})).$$

So $\varphi(d(y_p, y_{p+1})) = 0$. This is a contradiction. Hence

$$d(y_{2k}, y_{2k+1}) \leq M(x_{2k}, x_{2k+1}) \leq d(y_{2k}, y_{2k-1}).$$

Similarly,

$$d(y_{2k+1}, y_{2k+2}) \leq M(x_{2k+1}, x_{2k+2}) \leq d(y_{2k}, y_{2k+1}).$$

Therefore, for all $n \in \mathbb{N}$, even or odd,

$$d(y_n, y_{n+1}) \leq M(x_n, x_{n+1}) \leq d(y_{n-1}, y_n).$$

Thus $\{d(y_n, y_{n+1})\}$ is a decreasing and bounded below sequence. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}) = r.$$

Then (by semi-continuity of φ)

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(M(x_n, x_{n+1})).$$

We claim that $r = 0$. We know

$$d(y_n, y_{n+1}) \leq M(x_n, x_{n+1}) - \varphi(M(x_n, x_{n+1})).$$

So

$$r \leq r - \liminf_{n\to\infty} \varphi(M(x_n, x_{n+1})) \leq r - \varphi(r),$$

i.e., $\varphi(r) \leq 0$. Thus $\varphi(r) = 0$ by the property of the function φ and furthermore,

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Next, we show that $\{y_n\}$ is a cauchy sequence. Let

$$C_n = \sup\{d(y_j, y_k) : k, j \geq n\}.$$

Then $\{C_n\}$ is decreasing. If $\lim_{n\to\infty} C_n = 0$, then we are done. Assume that $\lim_{n\to\infty} C_n = C > 0$. Choose $\varepsilon < \frac{C}{8}$ small enough and select N such that for all $n \geq N$,

$$d(y_n, y_{n+1}) < \varepsilon \text{ and } C_n < C + \varepsilon.$$

By the definition of C_{N+1}, there exist $m, n \geq N+1$ such that $d(y_m, y_n) > C_n - \varepsilon \geq C - \varepsilon$. Replace y_m by y_{m+1} if necessary. We may assume that m is even, n is odd and $d(y_m, y_n) > C - 2\varepsilon$. Then $d(y_{m-1}, y_{n-1}) > C - 4\varepsilon$ and

$$\begin{aligned} d(y_m, y_n) \leq\ & M(x_m, x_n) - \varphi(M(x_m, x_n)) \\ \leq\ & \max\{d(y_{m-1}, y_m), d(y_{n-1}, y_n), d(y_{m-1}, y_{n-1}), \\ & \tfrac{1}{2}[d(y_{n-1}, y_m) + d(y_{m-1}, y_n)]\} - \varphi(\tfrac{C}{2}). \end{aligned}$$

i.e.,

$$C - 2\varepsilon < d(y_m, y_n) \leq \max\{\varepsilon, \varepsilon, d(y_{m-1}, y_{n-1}), C_N\} - \varphi(\frac{C}{2}).$$

So

$$C - 2\varepsilon < C_N - \varphi(\frac{C}{2}) \leq C + \varepsilon - \varphi(\frac{C}{2}).$$

This is impossible if ε be small enough. Thus, we must have $c = 0$. Therefore, the sequence $\{y_n\}$ is a cauchy sequence. Since X is complete, there exists some $z \in X$ such that $y_n \to z$. Also, for it's subsequence we have

$$Q_0 x_{2k} \to z, P_2 P_4 \cdots P_{2n} x_{2k} \to z$$

and

$$Q_1 x_{2k+1} \to z, P_1 P_3 \cdots P_{2n-1} x_{2k+1} \to z.$$

Case 1. $P_2 P_4 \cdots P_{2n}$ is continuous.

Define $G_1 = P_2 P_4 \cdots P_{2n}$. Since G_1 is continuous, $G_1^2 x_{2k} \to G_1 z$ and $G_1 Q_0 x_{2k} \to G_1 z$. Also, as (Q_0, G_1) is compatible, this implies that $Q_0 G_1 x_{2k} \to G_1 z$.

(a) Putting $u = P_2 P_4 \cdots P_{2n} x_{2k} = G_1 x_{2k}, v = x_{2k+1}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0 G_1 x_{2k}, Q_1 x_{2k+1}) \leq\ & M(G_1 x_{2k}, x_{2k+1}) - \varphi(M(G_1 x_{2k}, x_{2k+1})) \\ =\ & \max\{d({G_1}^2 x_{2k}, Q_0 G_1 x_{2k}), d(G_2 x_{2k+1}, Q_1 x_{2k+1}), \\ & d(G_1^2 x_{2k}, G_2 x_{2k+1}), \\ & \tfrac{1}{2}[d(G_2 x_{2k+1}, Q_0 G_1 x_{2k}) + d(G_1^2 x_{2k}, Q_1 x_{2k+1})]\} \\ & -\varphi(M(G_1 x_{2k}, x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$ (taking lower limit), we get

$$\begin{aligned} d(G_1 z, z) \leq\ & \max\{d(Gz, Gz), d(z, z), d(z, G_1 z), \tfrac{1}{2}[d(G_1 z, z) + d(G_1 z, z)]\} \\ & - \liminf_{n\to\infty} \varphi(M(G_1 x_{2k}, x_{2k+1})) \\ \leq\ & d(G_1 z, z) - \varphi(d(G_1 z, z)). \end{aligned}$$

So $G_1 z = z$. Thus $P_2 P_4 \cdots P_{2n} z = z$.

(b) Putting $u = z, v = x_{2k+1}, G_1 = P_2 P_4 \cdots P_{2n}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0 z, Q_1 x_{2k+1}) \leq\ & M(z, x_{2k+1}) - \varphi(M(z, x_{2k+1})) \\ =\ & \max\{d(G_1 z, Q_0 z), d(G_2 x_{2k+1}, Q_1 x_{2k+1}), d(G_1 z, G_2 x_{2k+1}), \\ & \tfrac{1}{2}[d(G_2 x_{2k+1}, Q_0 z) + d(G_1 z, Q_1 x_{2k+1})]\} - \varphi(M(z, x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$ (taking lower limit), we get

$$\begin{aligned} d(Q_0 z, z) \leq\ & \max\{d(z, Q_0 z), d(z, z), d(z, z), \tfrac{1}{2} d(z, Q_0 z)\} \\ & -\varphi(M(z, Q_0 z)). \end{aligned}$$

So $d(Q_0 z, z) \leq d(z, Q_0 z) - \varphi(M(z, Q_0 z))$. Hence $Q_0 z = z$. Therefore $Q_0 z = P_2 P_4 \cdots P_{2n} z = z$.

(c) Putting $u = P_4 \cdots P_{2n}z, v = x_{2k+1}, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5) and using the condition $P_2(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})P_2$ and $Q_0(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})Q_o$ in condition (2), we get

$$\begin{aligned} d(Q_0P_4 \cdots P_{2n}z, Q_1x_{2k+1}) \leq\ & M(P_4 \cdots P_{2n}z, x_{2k+1}) - \varphi(M(P_4 \cdots P_{2n}z, x_{2k+1})) \\ =\ & \max\{d(G_1P_4 \cdots P_{2n}z, G_2x_{2k+1}), d(G_2x_{2k+1}, Q_1x_{2k+1}), \\ & d(G_1P_4 \cdots P_{2n}z, Q_0P_4 \cdots P_{2n}), \\ & \tfrac{1}{2}[d(G_2x_{2k+1}, Q_0P_4 \cdots P_{2n}z) + d(G_1P_4 \cdots P_{2n}z, \\ & Q_1x_{2k+1})]\} - \varphi(M(P_4 \cdots P_{2n}z, x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$, (taking lower limit) we get

$$\begin{aligned} d(P_4 \cdots P_{2n}z, z) \leq\ & \max\{d(P_4 \cdots P_{2n}z, P_4 \cdots P_{2n}z), d(z, z), d(P_4 \cdots P_{2n}z, z), \\ & \tfrac{1}{2}[d(z, P_4 \cdots P_{2n}z) + d(P_4 \cdots P_{2n}z, z)]\} \\ & -\varphi(M(P_4 \cdots P_{2n}z, z)). \end{aligned}$$

Hence, it follows that $P_4 \cdots P_{2n}z = z$. Then $P_2(P_4 \cdots P_{2n})z = P_2z = z$. Continuing this procedure, we obtain $Q_0z = P_2z = P_4z = \cdots = P_{2n}z = z$.

(d) As $Q_0(X) \subseteq P_1P_3 \cdots P_{2n-1}(X)$, there exists $v \in X$ such that $P_1P_3 \cdots P_{2n-1}v = Q_0z = z$. Putting $u = x_{2k}, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0x_{2k}, Q_1v) \leq\ & M(x_{2k}, v) - \varphi(M(x_{2k}, v)) \\ =\ & \max\{d(G_1x_{2k}, Q_0x_{2k}), d(G_2v, Q_1v), d(G_1x_{2k}, G_2v), \\ & \tfrac{1}{2}[d(G_2v, Q_0x_{2k}) + d(G_1x_{2k}, Q_1v)]\} - \varphi(M(x_{2k}, v)). \end{aligned}$$

Letting $k \to \infty$, (taking lower limit) we get

$$\begin{aligned} d(z, Q_1v) \leq\ & \max\{d(z, z), d(z, Q_1v), d(z, z), \tfrac{1}{2}[d(z, z) + d(z, Q_1v)]\} \\ & -\varphi(d(z, Q_1v)). \end{aligned}$$

So $Q_1v = z$. Hence $P_1P_3 \cdots P_{2n-1}v = Q_1v = z$. As $(Q_1, P_1P_3 \cdots P_{2n-1})$ is weakly compatible, we have

$$P_1P_3 \cdots P_{2n-1}Q_1v = Q_1P_1P_3 \cdots P_{2n-1}v.$$

Thus $P_1P_3 \cdots P_{2n-1}z = Q_1z$.

(e) Putting $u = x_{2k}, v = z, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0x_{2k}, Q_1z) \leq\ & M(x_{2k}, z) - \varphi(M(x_{2k}, z)) \\ =\ & \max\{d(G_1x_{2k}, Q_0x_{2k}), d(G_2z, Q_1z), d(G_1x_{2k}, G_2z), \\ & \tfrac{1}{2}[d(G_2z, Q_0x_{2k}) + d(G_1x_{2k}, Q_1z)]\} - \varphi(M(x_{2k}, z)). \end{aligned}$$

Letting $k \to \infty$, (taking lower limit) we get

$$\begin{aligned} d(z, Q_1z) \leq\ & \max\{d(z, z), d(Q_1z, Q_1z), d(z, Q_1z), \tfrac{1}{2}[d(Q_1z, z) + d(z, Q_1z)]\} \\ & -\varphi(d(Q_1z, z)). \end{aligned}$$

Therefore $Q_1z = z$. Hence $P_1P_3 \cdots P_{2n-1}z = Q_1z = z$.

(f) Putting $u = x_{2k}, v = P_3 \cdots P_{2n-1}z, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5) and using the conditions $P_1(P_3 \cdots P_{2n-1}) = (P_3 \cdots P_{2n-1})P_1$ and

$Q_1(P_3\cdots P_{2n-1}) = (P_3\cdots P_{2n-1})Q_1$ in condition (2), we get

$$\begin{aligned} d(Q_0x_{2k}, Q_1P_3\cdots P_{2n-1}z) \leq\ & M(x_{2k}, P_3\cdots P_{2n-1}z) - \varphi(M(x_{2k}, P_3\cdots P_{2n-1}z)) \\ =\ & \max\{d(G_1x_{2k}, Q_0x_{2k}), d(G_1x_{2k}, G_2P_3\cdots P_{2n-1}z), \\ & d(G_2P_3\cdots P_{2n-1}z, Q_1P_3\cdots P_{2n-1}z), \\ & \tfrac{1}{2}[d(G_2P_3\cdots P_{2n-1}z, Q_0x_{2k}) + d(G_1x_{2k}, \\ & Q_1P_3\cdots P_{2n-1}z)] \\ & -\varphi(M(x_{2k}, P_3\cdots P_{2n-1}z)). \end{aligned}$$

Letting $k\to\infty$, (taking lower limit) we get

$$\begin{aligned} d(z, P_3\cdots P_{2n-1}z) \leq\ & \max\{d(P_3\cdots P_{2n-1}z, P_3\cdots P_{2n-1}z), d(z, P_3\cdots P_{2n-1}z), \\ & d(z,z), \tfrac{1}{2}[d(P_3\cdots P_{2n-1}z, z) + d(z, P_3\cdots P_{2n-1}z)]\} \\ & -\varphi(d(z, P_3\cdots P_{2n-1}z)). \end{aligned}$$

So $P_3\cdots P_{2n-1}z = z$. Therefore $P_1(P_3\cdots P_{2n-1}z) = P_1z = z$. Continuing this procedure, we have

$$Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z.$$

Thus, we have proved

$$Q_0z = Q_1z = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z.$$

Case 2. Q_0 is continuous.

Since Q_0 is continuous, $Q_0^2x_{2k} \to Q_0z$. As $(Q_0, P_2P_4\cdots P_{2n})$ is compatible, we have

$$P_2P_4\cdots P_{2n}Q_0x_{2k} \to Q_0z.$$

(g) Putting $u = Q_0x_{2k}, v = x_{2k+1}, G_1 = P_2P_4\cdots P_{2n}$ and $G_2 = P_1P_3\cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0^2x_{2k}, Q_1x_{2k+1}) \leq\ & M(Q_0x_{2k}, x_{2k+1}) - \varphi(M(Q_0x_{2k}, x_{2k+1})) \\ =\ & \max\{d(G_1Q_0x_{2k}, Q_0^2x_{2k}), d(G_2x_{2k+1}, Q_1x_{2k+1}), \\ & d(G_1Q_0x_{2k}G_2x_{2k+1}), \\ & \tfrac{1}{2}[d(G_2x_{2k+1}, Q_0^2x_{2k}) + d(G_1oQ_0x_{2k}, Q_1x_{2k+1})]\} \\ & -\varphi(M(Q_0x_{2k}, x_{2k+1})). \end{aligned}$$

Letting $k\to\infty$, (taking lower limit) we get

$$\begin{aligned} d(Q_0z, z) \leq\ & \max\{d(Q_0z, Q_0z), d(z,z), d(Q_0z, z), \tfrac{1}{2}[d(z, Q_0z) + d(Q_0z, z)]\} \\ & -\varphi(d(Q_0z, z)). \end{aligned}$$

Therefore $Q_0z = z$. Now using step (d), (e), (f) and continuing step (f) gives us $Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z$

(h) As $Q_1(X) \subseteq P_2P_4\cdots P_{2n}(X)$, there exists $w \in X$ such that $P_2P_4\cdots P_{2n}w = Q_1z = z$. Putting $u = w, v = x_{2k+1}, G_1 = P_2P_4\cdots P_{2n}$ and $G_2 = P_1P_3\cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0w, Q_1x_{2k+1}) \leq\ & M(w, x_{2k+1}) - \varphi(M(w, x_{2k+1})) \\ =\ & \max\{d(G_1w, Q_0w), d(G_2x_{2k+1}, Q_1x_{2k+1}), d(G_1w, G_2x_{2k+1}), \\ & \tfrac{1}{2}[d(G_2x_{2k+1}, Q_0w) + d(G_1w, Q_1x_{2k+1})]\} \\ & -\varphi(M(w, x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$, (taking lower limit) we get

$$\begin{aligned} d(Q_0w, z) \leq \; & \max\{d(z, Q_0w), d(z, z), d(z, z), \tfrac{1}{2}[d(z, Q_0w) + d(z, z)]\} \\ & -\varphi(M(z, Q_0w)). \end{aligned}$$

So $Q_0w = z$. Hence $Q_0w = P_2P_4 \cdots P_{2n}w = z$. As $(Q_0, P_2P_4 \cdots P_{2n})$ is weakly compatible, we have

$$Q_0P_2P_4 \cdots P_{2n}w = P_2P_4 \cdots P_{2n}Q_0w.$$

Hence $Q_0z = P_2P_4 \cdots P_{2n}z = z$. Similarly to in step (c) it can be shown that $Q_0z = P_2z = \cdots = P_{2n}z = z$. Thus, we have proved that

$$Q_0z = Q_1z = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z.$$

To prove the uniqueness property of z, let z' be another common fixed point of the aforementioned maps; then

$$Q_0z' = Q_1z' = P_1z' = P_2z' = \cdots = P_{2n-1}z' = P_{2n}z' = z'.$$

Putting $u = z, v = z', G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

$$\begin{aligned} d(Q_0z, Q_1z') \leq \; & M(z, z') - \varphi(M(z, z')) \\ = \; & \max\{d(G_1z, Q_0z), d(G_2z', Q_1z'), d(G_1z, G_2z'), \\ & \tfrac{1}{2}[d(G_2z', Q_0z) + d(G_1z, Q_1z')]\} - \varphi(M(z, z')). \end{aligned}$$

Then $d(z, z') \leq d(z, z') - \varphi(d(z, z'))$. So $z = z'$ and this shows that z is a unique common fixed point of the maps. □

Remark 2.6. Theorem 1.2 is a special case of Theorem 2.5 with $Q_0 = S, Q_1 = T$ and $P_i = I$(identity map) for all $1 \leq i \leq 2n$. Also, Theorem 2.5 is a generalization of Theorem 2.4 with $\varphi(t) = (1 - k)t$.

Theorem 2.7. *Let (X, d) be a complete metric space and let $\{T_\alpha\}_{\alpha \in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self-mappings on X. Suppose, there exists a fixed $\beta \in J$ such that*
(1) $T_\alpha(X) \subseteq P_2P_4, \cdots P_{2n}(X)$ for each $\alpha \in J$ and $T_\beta(X) \subseteq P_1P_3, \cdots P_{2n-1}(X)$;
(2)

$$
\begin{aligned}
P_2(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})P_2,\\
P_2P_4(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})P_2P_4,\\
&\vdots\\
P_2\cdots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\cdots P_{2n-2},\\
T_\beta(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})T_\beta,\\
T_\beta(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})T_\beta,\\
&\vdots\\
T_\beta P_{2n} &= P_{2n}T_\beta,\\
P_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})P_1,\\
P_1P_3(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\cdots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\cdots P_{2n-3},\\
T_\alpha(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})T_\alpha,\\
T_\alpha(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})T_\alpha,\\
&\vdots\\
T_\alpha P_{2n-1} &= P_{2n-1}T_\alpha, (\forall \alpha\in J);
\end{aligned}
$$

(3) $P_2\cdots P_{2n}$ or T_β is continuous;
(4) The pair $(T_\beta, P_2\cdots P_{2n})$ is compatible and the pairs $(T_\alpha, P_1\cdots P_{2n-1})$ are weakly compatible;
(5) There exists $\varphi\in\Phi$ such that
$d(T_\beta u, T_\alpha v)\le M(u,v)-\varphi(M(u,v))$, for all $u,v\in X$ and for all $\alpha\in J$, where

$$
\begin{aligned}
M(u,v) = \;&\max\{d(P_2P_4\cdots P_{2n}u, T_\beta u), d(P_1P_3\cdots P_{2n-1}v, T_\alpha v),\\
&d(P_2P_4\cdots P_{2n}u, P_1P_3\cdots P_{2n-1}v),\\
&\tfrac{1}{2}[d(P_1P_3\cdots P_{2n-1}v, T_\beta u)+d(P_2\cdots P_{2n}u, T_\alpha v)]\}
\end{aligned}
$$

Then, all P_i and T_α have a unique common fixed point in X.

Proof. Let T_{α_0} be a fixed element of $\{T_\alpha\}_{\alpha\in J}$. By Theorem 2.5 with $Q_0 = T_\beta$ and $Q_1 = T_{\alpha_0}$ it follows that there exists some $z\in X$ such that $T_\beta z = T_{\alpha_0}z = P_1P_3\cdots P_{2n-1}z = P_2P_4\cdots P_{2n}z = z$. Let $\alpha\in J$ be arbitrary. Then from condition (5),

$$
\begin{aligned}
d(T_\beta z, T_\alpha z)\le \;&\max\{d(P_2P_4\cdots P_{2n}z, T_\beta z), d(P_1P_3\cdots P_{2n-1}z, T_\alpha z),\\
&d(P_2P_4\cdots P_{2n}z, P_1P_3\cdots P_{2n-1}z),\\
&\tfrac{1}{2}[d(P_1P_3\cdots P_{2n-1}z, T_\beta z)+d(P_2\cdots P_{2n}z, T_\alpha z)]\}-\varphi(M(z,z)).
\end{aligned}
$$

So $d(z,T_\alpha z)\le d(z,T_\alpha z)-\varphi(d(z,T_\alpha z))$. Thus $T_\alpha z = z$ for each $\alpha\in J$. Since condition (5) implies the uniqueness of the common fixed point, Theorem 2.7 is proved. □

Remark 2.8. Theorem 2.1 is a special case of Theorem 2.7 with $P_i = I$ (identity map), for all $1\le i\le 2n$ and $\varphi(t) = (1-\lambda)t$.

Now, we prove a common fixed point for any number of mappings.

Corollary 2.9. *Let $P_0, P_1, P_2, \cdots, P_n$ be self-maps on a complete metric space (X,d) satisfying conditions:*
(1) $P_0(X)\subseteq P_1P_2,\cdots P_n(X)$;

(2)

$$\begin{aligned} P_1(P_2\cdots P_n) = & \; (P_2\cdots P_n)P_1, \\ P_1P_2(P_3\cdots P_n) = & \; (P_3\cdots P_n)P_1P_2, \\ & \vdots \\ P_1\cdots P_{n-1}(P_n) = & \; (P_n)P_1\cdots P_{n-1}; \end{aligned}$$

(3) There exists $\varphi \in \Phi$ such that
$d(P_0u, v) \leq M(u, v) - \varphi(M(u, v))$, *for all $u, v \in X$ where*

$$\begin{aligned} M(u,v) = & \; \max\{d(u, P_0u), d(P_1P_2\cdots P_nv, v), \\ & \; d(u, P_1P_2\cdots P_nv), \tfrac{1}{2}[d(P_1P_2\cdots P_nv, P_0u) + d(u,v)]\}. \end{aligned}$$

Then, $P_0, P_1, P_2, \cdots, P_n$ have a unique common fixed point in X.

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