Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1735-1747
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22299.2346

Entire functions and some of their growth properties on the basis of generalized order (α, β)

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(Communicated by Choonkil Park)

Abstract

For any two entire functions f, g defined on finite complex plane $\mathbb{C}$, the ratios $\frac{M_{f\circ g}(r)}{M_f(r)}$ and $\frac{M_{f\circ g}(r)}{M_g(r)}$ as $r \to \infty$ are called the growth of composite entire function $f \circ g$ with respect to f and g respectively in terms of their maximum moduli. Several authors have worked about growth properties of functions in different directions. In this paper, we have discussed about the comparative growth properties of $f \circ g$, f and g, and derived some results relating to the generalized order (α, β) after revised the original definition introduced by Sheremeta, where α, β are slowly increasing continuous functions defined on $(-\infty, +\infty)$. Under different conditions, we have found the limiting values of the ratios formed from the left and right factors on the basis of their generalized order (α, β) and generalized lower order (α, β), and also established some inequalities in this regard.

Keywords: Entire function, growth, composition, generalized order (α, β), generalized lower order (α, β).
2010 MSC: Primary 30D35; Secondary 30B30.

1. Introduction, Definitions and Notations

A function f which is analytic in the entire finite complex plane $\mathbb{C}$, is called an entire function, which may be represented by an everywhere convergent power series $\sum_{n=0}^{\infty} a_n z^n$, e.g. $\sin z$, $\cos z$, $\exp z$

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Received: January 2021 *Accepted:* March 2021

etc. Rolf Nevanlinna initiated the value distribution theory of entire functions in 1926, which is a prominent branch of Complex Analysis and is the prime concern of our paper. We use the standard notations and definitions of the theory of entire functions which are available in [10, 11, 17, 19], thus we do not explain those in details.

The ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli, where the maximum modulus function $M_f(r)$ of f on $|z| = r$ is defined as $M_f = \max\limits_{|z|=r}|f(z)|$. It can be also defined with maximum term function $\mu_f(r) = \max\limits_{n\geq 0}(|a_n|r^n)$ in place of maximum modulus. The order of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function.

Juneja et al. [9] introduced the definitions of (p, q)-th order and (p, q)-th lower order of an entire function, where p and q always denote positive integers with $p \geq q$. These definitions extended the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function considered in [12] for each integer $l \geq 2$. During the past decades, several authors made close investigations on the properties of entire functions related to (p, q)-order in some different direction (e.g. see, [4, 5, 6, 18]).

Recently, Chyzhykov et al. [7] showed that both generalized order and (p, q)-order have the disadvantage that they do not cover arbitrary growth (see [7, Example 1.4]). Considering this, let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. Moreover we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0 unless otherwise specifically stated. The quantity

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

introduced by Sheremeta {see, [16]} in 1967, which is called generalized order (α, β) of f. Some studies are made on the properties of entire functions related to generalized order (α, β) in some different direction during the past decades (e.g. see, [13]). For the purpose of further applications, Biswas et al.[2, 3] rewrite the definition of the generalized order (α, β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [16]).

Definition 1.1. *[2, 3] The generalized order (α, β) and generalized lower order (α, β) of an entire function f denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R)\{cf.\ [15]\},$$

it is easy to see that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)}.$$

Here, in this paper, we investigate certain interesting results associated with the comparative growth properties of composite entire functions using generalized order (α, β) and generalized lower order (α, β). In fact some works in this direction have already been explored in [1, 2, 3].

2. Known Results

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[8] Let f and g are any two entire functions with $g(0) = 0$. Also let b satisfy $0 < b < 1$ and $c(b) = \frac{(1-b)^2}{4b}$. Then for all sufficiently large values of r,*

$$M_f(c(b)M_g(br)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

In addition if $b = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2.2. *[14] Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\delta}{\delta - 1}\mu_f\left(\frac{\delta R}{R-r}\mu_g(R)\right).$$

Lemma 2.3. *[14] If f and g are any two entire functions. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f and g be any two entire functions such that $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim\limits_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, then

$$\lim_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

Proof . Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]$ we can find $\varepsilon(> 0)$ is such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon. \tag{3.1}$$

From of Lemma 2.1, we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \leqslant (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \tag{3.2}$$

Now the following three cases may arise .

Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then from (3.2) for all sufficiently large positive numbers of r, we have

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \leqslant B(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\exp(\alpha_2(M_g(\beta_2^{-1}(\log r))))$$

$$i.e.,\ \alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \leqslant B(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.3}$$

Case II. Let $\lim\limits_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$. For all sufficiently large positive numbers of r, we get from (3.2),

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) < (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.4}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then we get from (3.2), for all sufficiently large positive numbers of r,

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))) \leq (1+o(1))\alpha_2(M_g(\beta_2^{-1}(\log r)))$$

$$i.e.,\ \exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \leqslant r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.5}$$

Also from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, for all sufficiently large positive numbers of r,

$$\exp(\alpha_1(M_f(\beta_1^{-1}(\log r)))) \geqslant r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}. \tag{3.6}$$

Now combining (3.3) of Case I and (3.6) we get for all sufficiently large positive numbers of r,

$$\frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} \leq \frac{B(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}. \tag{3.7}$$

Hence from (3.1) and (3.7), we have

$$\lim_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

Similar conclusion can also be derived from (3.4) of Case II and (3.6).

Hence the first part of the theorem is completed.

Further combining (3.5) of Case III and (3.6) we obtain for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} \leq \frac{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}. \tag{3.8}$$

Therefore in view of (3.1) we get from above that

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

Hence the second part of the theorem follows from above.

Thus the theorem follows. □

Theorem 3.2. *Let f and g be any two entire functions such that $\lambda_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty$.*

(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim\limits_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$*, then*

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = 0.$$

The proof of Theorem 3.2 is omitted as it can be carried out in the line of Theorem 3.1.

Theorem 3.3. *Let f and g be any two entire functions such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ *where B is any positive constant or* $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$*, then*

$$\lim_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = +\infty.$$

Proof . Let us choose $0 < \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]$. Now for all sufficiently large positive numbers of r we get from Lemma 2.1 that

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \geqslant (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1\left(M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right). \tag{3.9}$$

Now the following three cases may arise.
Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then from (3.9) we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \geqslant B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.10}$$

Case II. Let $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$. Now from (3.9) it follows that for all sufficiently large positive numbers of r,

$$\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) > (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.11}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then from (3.9), for all sufficiently large positive numbers of r,

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))) \geq (1+o(1))\alpha_2\left(M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right)$$

$$i.e., \ \exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \geq r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.12}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}[f]$ we get for all sufficiently large positive numbers of r that

$$\exp(\alpha_1(M_f(\beta_1^{-1}(\log r)))) \leq r^{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)}. \tag{3.13}$$

Now combining (3.10) of Case I and (3.13) we get for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} \geq \frac{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)}}.$$

Since $\rho_{(\alpha_1,\beta_1)}[f] < \lambda_{(\alpha_2,\beta_2)}[g]$, it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = +\infty.$$

Similar conclusion can also be derived from (3.11) of Case II and (3.13).

Therefore the first part of the theorem is proved.

Again combining (3.12) of Case III and (3.13) we obtain for all sufficiently large positive numbers of r that

$$\begin{aligned}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} &\geq \frac{r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)}}\\ i.e., \lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} &= +\infty,\end{aligned}$$

Therefore the second part of the theorem follows from above.

Hence the theorem follows. □

Theorem 3.4. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim\limits_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r))))} = +\infty.$$

The proof of Theorem 3.4 is omitted as it can be carried out in the line of Theorem 3.3.

Theorem 3.5. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty$ and $0 < \lambda_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If $\beta_1(r) = \alpha_2(r)$, then

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\\ &\leqslant \min\left\{\rho_{(\alpha_2,\beta_2)}[g], \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}\right\}\end{aligned}$$

and

$$\begin{aligned}\max\left\{\lambda_{(\alpha_2,\beta_2)}[g], \frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} &\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\\ &\leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.\end{aligned}$$

(ii) If $\beta_1(\alpha_2^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_1,\beta_1)}[f]} \leq \liminf_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \leqslant 1 \leqslant \limsup_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

(iii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]} \leq \liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leqslant \min\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \leq \max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \leq \limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Proof . From the definitions of generalized order (α_1, β_1) and generalized lower order (α_1, β_1) of f, we have for all sufficiently large positive numbers of r that

$$\alpha_1(M_f(r)) \leq (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(r), \tag{3.14}$$
$$\alpha_1(M_f(r)) \geq (\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)\beta_1(r). \tag{3.15}$$

and also for a sequence of positive numbers of r tending to infinity, we get

$$\alpha_1(M_f(r)) \geq (\rho_{(\alpha_1,\beta_1)}[f] - \varepsilon)\beta_1(r), \tag{3.16}$$

$$\alpha_1(M_f(r)) \leq (\lambda_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(r). \tag{3.17}$$

Now in view of Lemma 2.1, for all sufficiently large positive numbers of r,

$$\alpha_1(M_{f\circ g}(r)) \leqslant (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(r)), \tag{3.18}$$

$$\alpha_1(M_{f\circ g}(r)) \geq (1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)\beta_1\left(M_g\left(\frac{r}{2}\right)\right). \tag{3.19}$$

and also for a sequence of positive numbers of r tending to infinity,

$$\alpha_1(M_{f\circ g}(r)) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(r)), \tag{3.20}$$

$$\alpha_1(M_{f\circ g}(r)) \geq (1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] - \varepsilon)\beta_1\left(M_g\left(\frac{r}{2}\right)\right). \tag{3.21}$$

Now the following two cases may arise:
Case I. Let $\beta_1(r) = \alpha_2(r)$.

Then for all sufficiently large positive numbers of r, we get from (3.18),

$$\alpha_1(M_{f\circ g}(r)) \leqslant (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r), \tag{3.22}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r)) \leqslant (\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r). \tag{3.23}$$

Also we obtain from (3.20) for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r)) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r). \tag{3.24}$$

Further it follows from (3.19), for all sufficiently large positive numbers of r,

$$\alpha_1(M_{f\circ g}(r)) \geq (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r), \tag{3.25}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r)) \geq (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r). \tag{3.26}$$

Moreover, we obtain from (3.21)for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r)) \geq (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r). \tag{3.27}$$

Therefore from (3.15) and (3.22), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \frac{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_2(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.28}$$

Now from (3.16) and (3.22), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \frac{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_2(r)}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.29}$$

In the same way also from (3.15) and (3.23),we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.30}$$

Similarly from (3.15) and (3.24), we get that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.31}$$

Thus from (3.29), (3.30) and (3.31), it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \leqslant \min\left\{\rho_{(\alpha_2,\beta_2)}[g], \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}\right\}. \tag{3.32}$$

Further from (3.14) and (3.25), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_2(r)}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{3.33}$$

Applying similar procedure, we obtain from (3.17) and (3.25),

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\lambda_{(\alpha_2,\beta_2)}[g], \tag{3.34}$$

From (3.14) and (3.26),

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}, \tag{3.35}$$

And from (3.14) and (3.27),

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.36}$$

Hence from (3.34), (3.35) and (3.36), it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\max\left\{\lambda_{(\alpha_2,\beta_2)}[g],\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\}. \tag{3.37}$$

Therefore the first part of the theorem follows from (3.28), (3.32), (3.33) and (3.37).

Case II. Let $\beta_1(\alpha_2^{-1}(r))\in L^0$.

Then for all sufficiently large positive numbers of r, we have from (3.18),

$$\begin{aligned}\alpha_1(M_{f\circ g}(r)) &\leqslant (\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}((\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)))\\ i.e.,\ \alpha_1(M_{f\circ g}(r)) &\leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))),\end{aligned} \tag{3.38}$$

And from (3.19),

$$\alpha_1(M_{f\circ g}(r))\geq(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))). \tag{3.39}$$

Now from (3.15) and (3.38), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))}\leqslant\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))}\leqslant\frac{\rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.40}$$

Again from (3.16) and (3.38), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))}\leqslant\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))}\leqslant 1. \tag{3.41}$$

Further from (3.14) and (3.39), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{3.42}$$

Also from (3.17) and (3.39), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\alpha_2^{-1}(\beta_2(r))))} \geq 1. \tag{3.43}$$

Hence the second part of the theorem follows from (3.40), (3.41), (3.42) and (3.43).

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$.

Then we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))) \leqslant (1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r), \tag{3.44}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))) \leqslant (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r). \tag{3.45}$$

Further, it follows from (3.19) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))) \geq (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r), \tag{3.46}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))) \geq (1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r). \tag{3.47}$$

Now from (3.15) and (3.44), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_2(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leqslant \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.48}$$

Also from (3.16) and (3.44) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon)\log^{[n]} r}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))} \leqslant \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{3.49}$$

Similarly from (3.15) and (3.45), we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\leqslant\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.50}$$

Thus from (3.49) and (3.50) it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\leqslant\min\left\{\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]},\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}\right\}. \tag{3.51}$$

Further from (3.14) and (3.46), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_2(r)}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{3.52}$$

Also from (3.17) and (3.46) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_2(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{3.53}$$

Similarly from (3.14) and (3.47), we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{3.54}$$

Thus from (3.53) and (3.54), it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))}\geq\max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]},\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\}. \tag{3.55}$$

Thus the third part of the theorem follows from (3.48), (3.51), (3.52) and (3.55). □

Theorem 3.6. *Let f and g be any two entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<+\infty$ and $0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\rho_{(\alpha_2,\beta_2)}[g]<+\infty$.*
(i) If $\beta_1(r)=\alpha_2(r)$, then

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(r))}\\ &\leqslant \min\left\{\rho_{(\alpha_1,\beta_1)}[f],\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]}\right\}\end{aligned}$$

and

$$\begin{aligned}\max\left\{\lambda_{(\alpha_1,\beta_1)}[f],\frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\} &\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(r))}\\ &\leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.\end{aligned}$$

(ii) If $\beta_1(\alpha_2^{-1}(r)) \in L^0$*, then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_2,\beta_2)}[g]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))}\\ &\leqslant \min\left\{\frac{\rho_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_2,\beta_2)}[g]}, \frac{\lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}\right\}\end{aligned}$$

and

$$\begin{aligned}\max\left\{\frac{\rho_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_2,\beta_2)}[g]}, \frac{\lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}\right\} &\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))}\\ &\leq \frac{\rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.\end{aligned}$$

(iii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$*, then*

$$\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_2(M_g(r))} \leqslant 1 \leq \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_2(M_g(r))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

The proof of Theorem 3.6 is omitted as it can be carried out in the line of Theorem 3.5.

Remark 3.7. *The same results of above theorems in terms of maximum terms of entire functions can also be deduced with the help of Lemma 2.2 and Lemma 2.3.*

Conclusion

According to the Fundamental Theorem of Classical Algebra "If $f(z)$ is a polynomial of degree n with real or complex coefficients, then the equation $f(z) = 0$ has at least one root" is the most well known value distribution theorem, and hence any such given polynomial can take any given value, real or complex. In the value distribution theory, one studies how an entire function assumes some values and, conversely, what is the influence in some specific manner of taking certain values on a function. Also it deals with various aspects of the behavior of entire functions, one of which is the study of their comparative growth. In this paper we deals with the extension of the works on the growth properties of composite entire functions on the basis of their generalized order (α, β) where α and β are continuous non-negative functions on $(-\infty, +\infty)$. The technique used to define generalized order (α, β) and generalized lower order (α, β) is newly developed, so there are no works in this regard. The concept of generalized order (α, β) is more generalized concept and its related work is very much significant in the study of growth properties of entire functions. This works will be helpful for the future researchers.

Acknowledgement

The authors are thankful to the referee for his / her valuable suggestions towards the improvement of the paper.

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