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Essential-small fully stable modules

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Abstract

We will add new types of modules over a ring $\mathfrak{R}$ called $g-$ small completely stable module and $g-$ small duo module, assuming $\mathfrak{R}$ is a ring with identity and $\mathcal{M}$ is a unitary left $\mathfrak{R}$-module. We also present the $g-$ small dual stable module, which is a duality. We show that our new ideas have characterization and a few properties.

Keywords: $g-$ small fully modules, $g-$ small duo modules, $g-$ small dual stable.

1. Introduction

In this article all rings are associative with identity and all modules are unital left modules. Let $\mathcal{M}$ be an $\mathfrak{R}$-module and $\mathcal{N} \leq \mathcal{M}$, $\mathcal{M}$ is called small, if for every submodule $\mathcal{K}$ of $\mathcal{M}$ with $\mathcal{N}+\mathcal{K} = \mathcal{M}$ implies $\mathcal{K} = \mathcal{M}$ [3]. A submodule $\mathcal{N}$ of an $\mathfrak{R}$ module $\mathcal{M}$ is called an essential submodule and denoted by $\mathcal{N} \leq_e \mathcal{M}$ in case $\mathcal{K} \cap \mathcal{N} \neq 0$ for every submodule $\mathcal{K} \neq 0$[3]. A module $\mathcal{M}$ is called uniform, if every submodule of $\mathcal{M}$ is essential. A submodule $\mathcal{K}$ of $\mathcal{M}$ is called $g-$ small submodule of $\mathcal{M}$ denoted by $\mathcal{K} \ll_g \mathcal{M}$ (in [11] it is denoted by $\mathcal{K} \ll_e \mathcal{M}$), if for every essential submodule $\mathcal{T}$ of $\mathcal{M}$ with the property $\mathcal{M} = \mathcal{K} + \mathcal{T}$ implies that $\mathcal{T} = \mathcal{M}$ [9]. It is clear that every small submodule of $\mathcal{M}$ is g -small ,but the converse in generals not true , for example $\mathcal{Z}_6$ *as* $\mathcal{Z}$-module $\{0^-, 2^-, 4^-\}$ and $\{0^-, 3^-\}$ are g-small but not small. Anon-zero module $\mathcal{M}$ is called g-hollow, if every proper submodule of $\mathcal{M}$ is g- small (in [6] it is denoted by $e-hollow$).
The intersection of maximal essential submodules of an $\mathfrak{R}$-module $\mathcal{M}$ is called a generalized Radical of $\mathcal{M}$ which represented $Radg(\mathcal{M})$, if $\mathcal{M}$ have no maximal essential submodules, then $Rad_g\mathcal{M} = \mathcal{M}$[9, 5].
A submodule $\mathcal{N}$ of $\mathfrak{R}-$module $\mathcal{M}$ is called stable , if $f(\mathcal{N}) \leq \mathcal{N}$ for each $\mathfrak{R}-$homomorphism f : $\mathcal{N} \to \mathcal{M}$, and $\mathcal{M}$ is called fully stable, if every submodule of $\mathcal{M}$ is stable, A submodule $\mathcal{N}$ of an

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$\mathfrak{R}-$module $\mathcal{M}$ is called a fully invariant , if for every $\hbar \in End(\mathcal{M})$, $f(\mathcal{N}) \leq \mathcal{N}$ .And $\mathcal{M}$ is called duo module , if every submodule of $\mathcal{M}$ is fully invariant [7].It is clear that the concepts stable and fully stable submodules are strong then the concepts of fully invariant submodules and duo module respectively.

In this article summarized these two concepts with respec to $g-$ small submodules called $g-$ small fully stable and $g-$ small duo modules. We also introduce $g-$ small dual stable modules, and we proved that each $g-$ small fully dual stable modules is a $g-$ small duo modules .

2. $g-$ Small Fully Stable Modules

In this section we introduce $g-$ small fully module as a generalization of fully stable modules appeared in[1]. We illustrate the concept with examples and the same properties.
Recall that a submodule $\mathcal{N}$ of an $\mathfrak{R}$-module $\mathcal{M}$ is called generalized small (for short $g-$ small) denoted by $\mathcal{N} \ll_g \mathcal{M}$, if whenever $\mathcal{N} + \mathcal{K} = \mathcal{M}$ for $\mathcal{K} \leq_e \mathcal{M}$ impels $\mathcal{K} = \mathcal{M}$ [11].

The following are some properties of $g-$ small submodule appeared in [11].

Lemma 2.1. *Let $\mathcal{M}$ be an $\mathfrak{R}$-module and $\mathcal{K}, l, \mathcal{N}, \mathcal{T} \leq \mathcal{M}$, then it holds .*

1. *Let $f : \mathcal{M} \to \mathcal{N}$ be $\mathfrak{R}$- module homomorphism, if $\mathcal{K} \ll_g \mathcal{M}$ then $f(\mathcal{K}) \ll_g \mathcal{M}$.*
2. *If $\mathcal{K} \leq \mathcal{N}$ and $\mathcal{N} \ll_g \mathcal{M}$ then $\mathcal{K} \ll_g \mathcal{M}$*
3. *Let $\mathcal{K} \leq \mathcal{N} \leq \mathcal{M}$, if $\mathcal{N} \ll_g \mathcal{M}$ then $\mathcal{N}\ \mathcal{K} \ll_g \mathcal{M}\ /\mathcal{K}$.*
4. *If $\mathcal{K} \leq \mathcal{N}$ and $\mathcal{K} \ll_g \mathcal{M}$,$\mathcal{N}/\mathcal{K} \ll_g \mathcal{M}/\mathcal{K}$, then $\mathcal{N} \ll_g \mathcal{M}$.*
5. *Let $\mathcal{K} \ll_g \mathcal{M}$ and $l \leq \mathcal{M}$, then $(\mathcal{K} + l)/\mathcal{K} \ll_g \mathcal{M}/\mathcal{K}$.*
6. *If $\mathcal{K} \ll_g l$ and $\mathcal{N} \ll_g \mathcal{T}$, then $\mathcal{K} + \mathcal{N} \ll_g l + \mathcal{T}$.*

Recall that a submodule $\mathcal{K}$ of $\mathcal{M}$ is called generalized maximal, if $\mathcal{K}$ is maximal essential submodule of $\mathcal{M}$. The intersection of all maximal essential submodule represented by $Rad_g(\mathcal{M})$ and it is called generalized radical of $\mathcal{M}$ .If $\mathcal{M}$ has no maximal essential submodule , then $Rad_g(\mathcal{M}) = \mathcal{M}$, [5].

Lemma 2.2. *Let M be an $\mathfrak{R}-$module, then* $Rad_g(\mathcal{M}) = \sum_{l \ll_g \mathcal{M}} l$

Lemma 2.3. *For $\mathfrak{R}$-module $\mathcal{M}$ The following assertions holds*

1. *If $\mathcal{M}$ is an $\mathfrak{R}-$module then $\mathfrak{R}m \ll_g \mathcal{M}$ for every $m \in Rad_g(\mathcal{M})$.*
2. *If $f : \mathcal{M} \to \mathcal{N}$ is an $\mathfrak{R}-$module homomorphism, then $f(Rad_g(\mathcal{M})) \leq Rad_g(\mathcal{N})$.*
3. *If $\mathcal{N} \leq \mathcal{M}$, then $Rad_g(\mathcal{N}) \leq Rad_g(\mathcal{M})$.*
4. *If $\mathcal{K}$, $l \leq \mathcal{M}$, then $Rad_g\ \mathcal{K} + Rad_g l \leq Rad_g(\mathcal{K} + l)$.*

Lemma 2.4. *[4] Let $\mathcal{M}$ be a supplemented $\mathfrak{R}-$module, and let $\mathcal{N}$ be any submodule of $\mathcal{M}$, then $Rad_g(\mathcal{N}) = Rad_g(\mathcal{M}) \cap \mathcal{N}$.*

As a generalization of fully stable module we introduce the following.

Definition 2.5. *$\mathcal{M}$ is called $g-$ small fully stable module , if for every $g-$ small submodule $\mathcal{N}$ of $\mathcal{M}$ and for each $\mathfrak{R}-$homomorphism $f : \mathcal{N} \to \mathcal{M}$, $f(\mathcal{N}) \leq \mathcal{N}$.*

The following is a characterization for $g-$ small fully stable module.

Proposition 2.6. *Let $\mathcal{M}$ be $\mathfrak{R}-$module ,then the following are equivalent:*

(i) *$\mathcal{M}$ is $g-$ small fully stable module.*

(ii) *If $\mathcal{K}, \mathcal{N} \leq \mathcal{M}$ so that $\mathcal{K} \ll_g \mathcal{M}$ and $\mathcal{N}$ epimorphic image of $\mathcal{K}$, then $\mathcal{N} \leq \mathcal{K}$.*

Proof .
$(i) \Longrightarrow (ii)$ *Let $\mathcal{N} \leq \mathcal{M}$, and Let $\alpha : \mathcal{K} \to \mathcal{N}$ be $\mathfrak{R}$-epimorphism. Let $x \in \mathcal{N}$, then $\exists\, y \in \mathcal{K}$ such that $\alpha(y) = x$, consider $i : \mathcal{N} \to \mathcal{M}$ to be the inclusion homomorphism ,then $(i \circ \alpha)(\mathcal{K}) \subseteq \mathcal{K}$, because $\mathcal{K} \ll_g \mathcal{M}$ then by(1), $\alpha(\mathcal{K}) \leq \mathcal{K}$ but $\alpha(\mathcal{K}) = \mathcal{N}$ therefore $\mathcal{N} \leq \mathcal{K}$.*
$(ii) \Longrightarrow (i)$ *Let $\mathcal{C} \subseteq \mathcal{M}$, $\mathcal{C} \ll_g \mathcal{M}$. Let $\alpha : \mathcal{C} \to \mathcal{M}$ be an $\mathfrak{R}$- homomorphism, then $\alpha : \mathcal{C} \to \alpha(\mathcal{C})$ is an epimorphism , let $\alpha(\mathcal{C}) = \mathcal{N}$, then by $\mathcal{N} \ll_g \mathcal{M}$, Hence so assume $\alpha(\mathcal{C}) \subseteq \mathcal{C}$.* □

Proposition 2.7. *Let M be $\mathfrak{R}-$module, then $\mathcal{M}$ is $g-$ small fully stable if and only if for every $g-$ small cyclic submodule $\mathcal{N}$and every homomorphism $f : \mathcal{N} \to \mathcal{M}$, $f(\mathcal{N}) \leq \mathcal{N}$.*
Proof .
$\Longrightarrow$ *Clear*
$\Longleftarrow$ *Let $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{N} \ll_g \mathcal{M}$, let $f : \mathcal{N} \to \mathcal{M}$ be a homomorphism then $\forall x \in \mathcal{N}, \mathfrak{R}_x \leq N$ and $\mathfrak{R}_x \ll_g \mathcal{N} \ll_g \mathcal{M}$, hence $\mathfrak{R}_x \ll_g \mathcal{M}$. then by assumption $f(\mathfrak{R}_x) \leq \mathfrak{R}_x, \forall x \in \mathcal{M}$, therefore $f(\mathcal{N}) \leq \mathcal{N}$.* □

3. Examples and Remarks

1. It is clear that every fully stable module is $g-$ small fully stable , but usually this is not case. Consider as $\mathcal{Z}-$ module(0) is the only $g-$ small submodule of $\mathcal{M}$. which is stable, but $\mathcal{Z}$ is not fully stable since $\exists\, f : 2\mathcal{Z} \to 3\mathcal{Z}$ definee by $f(2n) = 3n$, $f(2\mathcal{Z}) \nsubseteq 2\mathcal{Z}$.
2. If $\mathrm{Rad}_g\mathcal{M} = \mathcal{M}$, then every $g-$ small fully stable is fully stable .
3. Regarding Q as $\mathcal{Z}-$ module , it is known that $\mathbb{Q}$ as $\mathcal{Z}-$ module is not fully stable and since $\mathrm{Rad}_g(\mathbb{Q}) = \mathbb{Q}$ according by (2), it is not $g-$ small fully stable.
4. It is known that $\mathcal{Z}_p^\infty$ as $\mathcal{Z}-$ module is fully stable and hence by(2), $\mathcal{Z}_p^\infty$ as $\mathcal{Z}-$ module is $g-$ small fully stable , Notice that $\mathrm{Rad}_g(\mathcal{Z}_p^\infty) = \mathcal{Z}_p^\infty$.

Recall that anon-zero $\mathfrak{R}-$module $\mathcal{M}$ is called $g-$hollow , if every proper submodule of $\mathcal{M}$ is $g-$ small in [6] denoted by $e - hollow$.

Remark 3.1. *If $\mathcal{M}$ is $g-$ small module, then $\mathcal{M}$ is fully stable if and only if $\mathcal{M}$ is $g-$ small fully stable.*
Proof . *Since $\mathcal{M}$ is $g-$ small, then $Rad_g(\mathcal{M}) = \sum_{l \ll_g \mathcal{M}} l = \mathcal{M}$ and by through (2.2) , $\mathcal{M}$ is fully stable if and only if $\mathcal{M}$ is $g-$ small fully stable .* □

4. $g-$ Small Duo and $g-$ Small Dual Stable Modules

In this section we introduce $g-$ small duo module as a generalization of duo module in [7].In addition we will introduce a duality named $g-$ small duo stable.

Definition 4.1. *An $\mathfrak{R}-$module $\mathcal{M}$ is called $g-$ small duo, if every $g-$ small submodule of $\mathcal{M}$ is fully invariant.*

Example 4.2. *dual module are $g-$ small duo. On the contrary, $\mathcal{Z}$ is usually regardad as $\mathcal{Z}-$ module, $Rad_g(\mathcal{Z}) = 0$ and 0 is fully invariant , but $\mathcal{Z}$ is not duo module because $\exists\, f : \mathcal{Z} \to 3\mathcal{Z}$, $f(n) = 3\mathcal{Z} \notin 2\mathcal{Z}$, when $n = 2n$.*

If $\mathcal{M}$ is $g-$ hollow, then each $g-$ small duo is duo module .

Proposition 4.3. *An $\mathfrak{R}-$module $\mathcal{M}$ is $g-$ small duo if and only if each cyclic $g-$ small submodule of $\mathcal{M}$ is fully invariant.*
Proof . *Clear.* □

Proposition 4.4. *Let $\mathcal{M}$ be $\mathfrak{R}-$module, $\mathcal{M}$ is $g-$ small duo if and only if for each $f \in End(\mathcal{M})$, and for each $x \in Rad_g(\mathcal{M}), \exists r \in \mathfrak{R}$ such that $f(x) = rx$.*
Proof . *Let $f : \mathcal{M} \to \mathcal{M}$ be $\mathfrak{R}$-homomorphism and let $x \in Rad_g(\mathcal{M})$, then $\mathfrak{R}_x \ll_g \mathcal{M}$ by Lemma (2.1 (4)) ,so $f(\mathfrak{R}(x)) \leq \mathfrak{R}_x$, and $f(1.x) \in \mathfrak{R}_x$, thus $\exists r \in \mathfrak{R}$ such that $f(x) = rx$,Conversely, Let $\mathcal{N} \ll_g M$, then $\mathcal{N} \leq Rad_g(\mathcal{M})$, hence $\forall x \in \mathcal{N}, x \in Rad_g(\mathcal{M})$ and by assumption $\exists r \in \mathfrak{R}$ such that $f(x) = r\, x$, hence $f(\mathcal{N}) \leq \mathcal{N}$.* □

Proposition 4.5. *Every direct summand of $g-$ small duo module is also $g-$ small duo.*
Proof . *Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{M}_1$,$\mathcal{M}_2 \leq \mathcal{M}$ and let $f : \mathcal{M}_1 \to \mathcal{M}_1$, and $\mathcal{N} \leq_g \mathcal{M}_1$ then consider $\mathcal{M}_1\ \mathcal{M}_2 \overset{p_1}{\to} \mathcal{M}_1 \overset{f}{\to} N \overset{j_1}{\to} \mathcal{M}$ and then consider $j_1 \circ f \circ p_1 \in End(\mathcal{M})$, where j_1 is the inclusion homomorphism of $\mathcal{M}_1$ into $\mathcal{M}$ and p_1 is the projection of $\mathcal{M}_1$, since $\mathcal{N} \leq_g \mathcal{M}$,hence $\mathcal{N} \leq_g \mathcal{M}_1$ and $f(\mathcal{N}) = j_1(f(p_1(\mathcal{N}))) \leq \mathcal{N}$.* □

Recall that A module M is said to be multiplication, if every submodule $\mathcal{N}$of $\mathcal{M}$ can be written as $\mathcal{N}$= A$\mathcal{M}$ for some idea A of $\mathfrak{R}$[2].

Remark 4.6. *Notice that every $g-$ small multiplication module is $g-$ small duo module , to see that let $\mathcal{N}$ be a $g-$ small submodule of $\mathcal{M}$, then $\mathcal{N} = A\mathcal{M}$ for some ideal A of $\mathfrak{R}$. Now $f(\mathcal{N}) = f(A\mathcal{M}) = Af(\mathcal{M}) \leq A\mathcal{M} = \mathcal{N}$, thus $\mathcal{M}$ is $g-$ small duo.*

We introduce the property P^*. If $\mathcal{N} \cap \mathcal{K} = 0$ of each $g-$ small submodule $\mathcal{K}$ of $\mathcal{M}$ means $\mathcal{K} = (0)$, then the submodule $\mathcal{N}$ of $\mathcal{M}$ is said to satisfy the attribute P^*.

Proposition 4.7. *if $\mathcal{M}$ is an $g-$ small duo module, then each monomorphism $f \in End(\mathcal{M})$, $f(\mathcal{M})$ is $g-$ small and satisfies the property P^*.*
Proof . *Let $\mathcal{N} \neq 0$, and $\mathcal{N}$ is $g-$ small in $\mathcal{M}$, if $\mathcal{N} \cap f(\mathcal{M}) = 0$, then by assumption $f(\mathcal{N}) \leq \mathcal{N}$ hence $f(\mathcal{N}) \leq \mathcal{N} \cap f(\mathcal{M}) = 0$, so $f(\mathcal{N}) = 0$ and so $\mathcal{N} = 0$, because f is a monomorphism. A contradiction* □

The following are well known [2].

Theorem 4.8. *Let $\mathcal{M} = \bigoplus_{i\in\Lambda} \mathcal{N}_i, \mathcal{N}_i \leq \mathcal{M}, \forall i \in \Lambda$ then for each $i \in \Lambda \mathcal{N}_i$ is fully invariant in $\mathcal{M}$ if and only if $Hom(\mathcal{N}_i, \mathcal{N}_\mathcal{K}) = 0. \forall\ i \neq \mathcal{K}$. .*

Theorem 4.9. *Let $\mathcal{M} = \bigoplus_{i\in N} \mathcal{N}_i, \mathcal{N}_i \leq \mathcal{M}$ the following statements are equivalent :*

(i) M is a $g-$ small duo module.
(ii) For every $i \in \Lambda, \mathcal{N}_i$ is $g-$ small duo module and $Hom(\mathcal{N}_i\ ,\ \mathcal{N}_j) = 0\ \ \forall\, i \neq\ j$.

Proof .
(i) $\Longrightarrow$ (ii) each $i \in \Lambda$,Let $\mathcal{N}_k$ be a $g-$ small, and let $\mathcal{N}_i$ be a fully invariant in $\mathcal{M}$. for $k \neq 0$.
Let $f \in Hom(\mathcal{N}_i, \mathcal{N}_1)$, then $j_1 \circ f \circ p_1 = g : \mathcal{M} \to \mathcal{M}$,where P_i is the projection of $\mathcal{M}$ into $\mathcal{N}_i$ and j_k , is the inclusion of $\mathcal{N}_k$ into $\mathcal{M}$, since $\mathcal{M}$ is $g-$ small duo module, then $g(\mathcal{N}_i) \subseteq \mathcal{N}_i, \forall\ i\ \in I$, but $g(\mathcal{N}_i) = j_1 \circ f \circ p_1(\mathcal{N}_i) = f(\mathcal{N}_i) \leq \mathcal{N}_k$. Thus $f(\mathcal{N}_i) \in \mathcal{N}_i \cap \mathcal{N}_k = 0$, hence $f(\mathcal{N}_i) = 0$, therefore $f \equiv 0$.

(ii) $\Longleftarrow$ *(i) Let* $\mathcal{K} \ll_g \mathcal{M}$, *let* $f \in End_R(\mathcal{M})$, *then* $\forall\ \mathcal{K} \in \Lambda$
consider $j_{\mathcal{K}} \circ p_{\mathcal{K}} : \mathcal{M} = \bigoplus_{i\in\Lambda} \mathcal{N}_i \overset{\mathcal{P}\mathcal{K}}{\to} N_{\mathcal{K}} \overset{jk}{\to} \bigoplus_{i\in\Lambda} \mathcal{N}_i$, *then* $j_{\mathcal{K}} \circ p_{\mathcal{K}} \in End_R(\mathcal{M}).P_{\mathcal{K}} \circ f \circ J_{\mathcal{K}}$.

Now $\mathcal{N}_{\mathcal{K}} \subseteq_{\oplus} \mathcal{M}$, *and* $\mathcal{K}\cap\mathcal{N}_{\mathcal{K}} \leq \mathcal{N}_{\mathcal{K}} \leq \mathcal{M}$, *because* $\mathcal{K} \ll_g \mathcal{M}$, *this implies that* $\mathcal{N}_{\mathcal{K}} \cap g-$ *small, but* $\mathcal{N}_{\mathcal{K}}$ *is* $g-$ *small duo module in* $\mathcal{M}$, *then* $(P_{\mathcal{K}}\circ f\circ J_{\mathcal{K}})(\mathcal{K}\cap\mathcal{N}_{\mathcal{K}}) \leq \mathcal{K}\cap\mathcal{N}_{\mathcal{K}}$. *By (2)* $P_{\mathcal{K}}\circ f\circ J_{\mathcal{K}}(\mathcal{K}\cap\mathcal{N}_{\mathcal{K}}) = 0$, *Hence* $(P_{\mathcal{K}}\circ f\circ J_i)(\mathcal{K}\cap\mathcal{N}_{\mathcal{K}}) = 0.\forall\ i \neq \mathcal{K}$, *because* $\mathcal{K} = \sum_{\mathcal{K}\in\wedge}(\mathcal{K}\cap\mathcal{N}_{\mathcal{K}}), f(\mathcal{K}) \subseteq \sum_{\mathcal{K}\in\wedge} f(\mathcal{K}\cap\mathcal{N}_{\mathcal{K}}) \subseteq_{\mathcal{K}\in\wedge}$ $(P_{\mathcal{K}} \circ f \circ J_{\mathcal{K}})(\sum \mathcal{K} \cap \mathcal{N}_{\mathcal{K}} \subseteq\ \sum_{\mathcal{K}\in\wedge}(\mathcal{K} \cap \mathcal{N}_{\mathcal{K}}) = \mathcal{K}$. *Thus is fully invariant so* $\mathcal{M}$ *is* $g-$ *small duo module.* $\square$

Proposition 4.10. *Let* M *be a supplemented* $\mathfrak{R}-$*module* in where each countable generated submodule is $g-$ *small* duo module, then M is $g-$ *small* duo module.
Proof . *Let* $f \in End(\mathcal{M})$ *and let* $x \in Rad_g(\mathcal{M})$ *put* $\mathcal{N}=\sum_{n=0}^{\infty} \mathrm{Ra^n(x)}$, then $\mathcal{N}$ is a countable generated submodule of $\mathcal{M}$, because $\mathcal{M}$ *is supplemented then by Lemma (2.4)* $Rad_g\mathcal{N} = Rad_g(\mathcal{M} \cap \mathcal{N})$, *therefore* $x \in Rad_g\mathcal{N}$ *and* $f(\mathcal{N}) \leq \mathcal{N}$, *so by proportion there is an element* $r \in R$ *such that* $f(x) =\ rx$ this meanse $\mathcal{M}$ *is* $g-$ *small duo.* $\square$

Definition 4.11. $\mathcal{N}$ is called $g-$ *small* dual stable ,if $\mathcal{N}$ is $g-$ *small in and for each* $\mathfrak{R}-$ *homomorphism* $f : \mathcal{M} \to \mathcal{M}/\mathcal{N},\ \ \mathcal{N} \leq \ker(f)$.

An $\mathfrak{R}-$module *is called* $g-$ small *fully dua stable, if each submodule* $\mathcal{N}$ *of* $\mathcal{M}$ *is* $g-$ small *dual stable.*

Example 4.12.

- $\mathcal{Z}$ as $\mathcal{Z}-$ *module* is $g-$ *small* fully dual stable.
- $\mathcal{Z}_p^{\infty}$as $\mathcal{Z}-$ *module* is not $g-$ *small* fully dual stable , since every submodule of $\mathcal{Z}_p^{\infty}$ is $g-$ *small and* $\mathcal{Z}_p^{\infty}/G_t\forall t \in N$, *Where* $G_0 \subset G_1 \subset G_2 \ldots \subset G_n \subset \ldots$

  $\mathcal{Z}_p^{\infty}/G_t \cong \mathcal{Z}_p^{\infty}$ *and* $G_t = \mathcal{Z}_p^t$ *proper submodule of* $\mathcal{Z}_p^{\infty}$ *as* $\mathcal{Z}-$ *module notice that* $f : \mathcal{Z}_p^{\infty} \to \mathcal{Z}_p^{\infty}/\mathcal{Z}_p^t$ *is an isomorphism hence* $\ker f = 0$ *so* $\mathcal{Z}_p^t \nsubseteq \ker f$.

Proposition 4.13. *Each* $g-$ *small fully dual stable is a* $g-$ *small duo module.*
Proof . *Let* $\mathcal{N} \leq \mathcal{M}$ *such that* $\mathcal{N} \ll_g \mathcal{M}$ *and* g: $\mathcal{M} \to \mathcal{M}$ *and* $g \in End\mathcal{M}$, *let* $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{N}$ *be the natura projection. let* $f = \pi \circ g : \mathcal{M} \to \mathcal{M} \to \mathcal{M}/\mathcal{N}$, *then by assumption* $\mathcal{N} \subseteq \ker f$, *but* $\ker f = \ker(\pi \circ g) = g^{-1}\ker \pi = g^{-1}(\mathcal{N})$, *thus* $g(\mathcal{N}) \subseteq \mathcal{N}$. $\square$

On the contrary for example. $\mathcal{Z}_p^{\infty}$ as $\mathcal{Z}-$ module is $g-$ small duo module but not $g-$ small fully dual stable.

Proposition 4.14. *A homomorphic image by* $g-$ *small submodule of a* $g-$ *small fully dual stable module is a gain* $g-$ *small fully dual stable.*
Proof . *Let* $\mathcal{N} \ll_g \mathcal{M}$ *and* $\mathcal{M}$ *is a* $g-$ *small fully dual stable module, Let* k *be a submodule of* $\mathcal{M}$ *containing* $\mathcal{N}$ *such that* $\mathcal{K}/\mathcal{N} \ll_g \mathcal{M}/\mathcal{N}$ *implies that* $\mathcal{K} \ll_g \mathcal{M}$ $\big($*by Lemma (2.1)(4)*$\big)$.
Let $\alpha : \mathcal{M}\backslash\mathcal{N} \to (\mathcal{M}\backslash\mathcal{N})\backslash(\mathcal{K}\backslash\mathcal{N})$ *be an* $\mathfrak{R}$ *-homomorphism. Let* $\pi : \mathcal{M} \to \mathcal{M}\backslash\mathcal{N}$ *be the natural projection and* $g : (\mathcal{M}\backslash\mathcal{N})\backslash(\mathcal{K}\backslash\mathcal{N} \to \mathcal{M}\backslash\mathcal{K}$ *be isomorphism (by third isomorphism theorem) , but* $\varphi = g \circ \alpha \circ \pi$.
Now $\mathcal{K} \leq \ker\varphi$ *(because* $\mathcal{M}$ *is fully small dual stable), but* $\ker\varphi = \ker(g \circ \alpha \circ \pi) = \pi^{-1}\ker(g \circ \alpha) = \pi^{-1}(\alpha^{-1}(\ker g)) = \pi^{-1}(\alpha^{-1}(0))$ *which meanse* $\mathcal{K} \leq \ker\varphi = \pi^{-1}(\ker\varphi)$, *thus* $\pi(K) \leq \pi\ (\ \pi^{-1}(\ker\alpha))$, *therefore* $\mathcal{K}/\mathcal{N} \leq \ker\alpha$. $\square$

5. Conclusions

In this work we have

- generalization of fully stable modules.
- generalization small duo modules

References

[1] M.A. Abbas, *On fully stable modules*, Ph. D. Thesis. University of Baghdad, 1991.
[2] Z.A.EL. Bast and P.F. smith, *Multiplication modules*, Commun. Alg. 16 (1988) 755–779.
[3] J. F. Kasch *Modules and Ring*, Academic Press Inc. London, 1982.
[4] W. Khalid and A. S. Wadi, *Generalized radical lifting modules*, Int. J. Sci. Res. 10 (2017) 1383–1385.
[5] B. Kosar, C. Nebiyer and N. Sokmez, *G-supplemented modules* Ukrain. Math. ZJ. 6 (2015) 861–864.
[6] M.A. Inaam Hadi, *e-hollow modules*, In. J. Adv. Sci. Tech. Res. 3 (1997) 2249–9954.
[7] A. Ozcan, H. Harmanc and P.F. Smith, *Duo modules*, Glasgow Math. J. 48 (2006) 533–545.
[8] M. Tamer kosan, *Generalized cofinitely semiperfect modules*, Int. Elect. J. Alg. 5 (2009) 58–69.
[9] C. Vebiyev, *G-Radical Supplemented Modules*, Dep. Math. Ondokuz Mayıs University, 2016.
[10] R. Wisbauer, *Foundations of Modules and Ring Theory*, Gordonand Breach, Reading, 1991.
[11] D. X. Zhou and X. R. Zhang, *small-essential submodules and Morita duality*, Southeast Asian Bull. Math. 35 (2011) 1051–1062.