Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1827-1853
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5320

Stability Analysis of a Diseased Prey - Predator - Scavenger System Incorporating Migration and Competition

Marwah Ali Yousif[a,*], Hassan Fadhil Al-Husseiny[a]

[a]*Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, a prey-predator-scavenger model is proposed and analyzed. It is assumed that the model considered the effect of disease on the prey. Firstly, the existence, uniqueness and boundedness of the solution of the model are discussed. Secondly, we studied the existence and local stability of all equilibrium points. Furthermore, some of the Sufficient conditions of the global stability of the positive equilibrium are established using suitable Lyapunov functions. Finally, those theoretical results are demonstrated with numerical simulations.

Keywords: prey-predetor-scavenger model, stability analysis, migrations.

1. Introduction

The eco-epidemiology is an important branch in mathematical biology which focus both the ecological and epidemiological situations simultaneously. In addition to other factors that affect prey such as harvesting, predation, migration, the impact of disease on the ecosystem is also an important factor from a mathematical and ecological perspective. In recent time many researchers were keen to explore the ecological system subject to epidemiological aspects such as Kermack and Mckendrick [16] illustrated a SIRS system in which the development of disease which become transmitted by direct contact was described. At first the disease factor effect in the predator-prey system were considered by Anderson and May [3]. In [7], Hadeler and Freedman studied a prey-predator model with disease in prey. Mohsen and Aaid introduced prey-predator model with SIS epidemic model in predator [17].

*Corresponding author

Email addresses: marwa.yousif1203@sc.uobaghdad.edu.iq (Marwah Ali Yousif), hassan.fadhil.r@sc.uobaghdad.edu.iq (Hassan Fadhil Al-Husseiny)

Received: March 2021 *Accepted:* July 2021

Naji and Ridha proposed and studied an ecological model consisting the dynamics of a prey-predator model incorporating SVIS-type of disease in prey [18], moreover, see [5, 8, 9, 10, 25, 11].

In addition, the functional response has an important role to study the dynamics of a predator–prey modeling. The predators and the preys carry a dynamic relationship among themselves. And for its universal existence and importance, this relationship is one of the dominant themes in theoretical ecology for example Kumar et al. [21] studied a two species prey predator eco system having Holing type III functional response under stochastic influence is investigated. Kang [13] formulated a Rosenzweig-MacArthur prey-predator two patch model with mobility only in predator and the assumption that predators move towards patches with more concentrated prey-predator interactions. Claudio [4] analysis a modified May-Holling-Tanner predator-prey model considering an Allee effect in the prey and alternative food sources for predator. Slimani et al. [23] studied a modified version of a prey-predator system with modified Leslie-Gower and Holling type II functional responses studied by Alaoui and Okiye. Suryanto and Darti [24] discuss a fractional order predator-prey model with ratio-dependent functional response.

Also, an important factor is migration. Migration usually occurs due to the loss of water or food or changes in climatic conditions. So, It is considered one of the important factors and there are many literatures on this field such as Kumar and Kharbanda [15] formulated and analyzed eco-epidemiological model consisting of susceptible prey, infected prey, vaccinated prey and predator. Abdulkadhim and Al-Husseiny [1] proposed and analyzed a predator-prey model with disease effects in predator as well the immigration. Kant and Kumar [14] formulated and studied a predator-prey system with migrating prey and disease infection in both species.

Many researchers in the field of prey- predator modeling have studied a new species called the scavenger, which is an animal that lives from consuming cadavers i.e. animals which starve naturally or are killed by other animals. There were several authors have studied the models with different assumptions in relation to the presence of the scavenger. Panja [20] developed a prey, predator and scavenger interaction dynamical model .Jansen and Gorder [12] studied an ecosystem model consisting predator-prey-quarry-resource-scavenger. Gupta and Chandra [6] proposed and analyzed an extended model for the prey-predator-scavenger in presence of harvesting to study the effects of harvesting of predator as well as scavenger. Abdul Satar and Naji [22], proposed and studied a prey-predator-scavenger food web model and considered the effect of harvesting and all the species are infected by some toxicants released by some other species. Ali and Mustafa [2] formulated the dynamics of scavenger species in a web food model incorporating time delay and prey harvesting mathematically. In this work, eco-epidemiological model consisting of susceptible prey, Infected prey, predator and scavenger is proposed and studied. So, this paper rest be order as following The next Section is concerned with the model formulation. Discusses the existence, uniqueness and boundedness of the solution in Section 3. The stability conditions are established about all equilibrium points in Section 4. Finally, to confirm the analytical results numerical simulation is carried out in Section 5, and the last section included the discussion and conclusions.

2. Model Formation

In this section an eco-epidemiological system consisting of prey, predator and scavenger incorporating infection disease in prey species is proposed. In order to formulate the dynamics of such system the following hypotheses are considered.

1. The existence of disease in prey species divided the prey population into two classes, namely susceptible prey that denotes by $X_1(T)$and infected prey denoted by $X_2(T)$. It is assumed

that in the absence of predator the susceptible prey logistically with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$.

2. The susceptible prey becomes infected by contact with infected prey at a rate $\alpha_1 > 0$.
3. The predator which denoted by $Y_1(T)$ consumes the prey according to lotka – volttera functional response with positive attack rates α_2 , β_1 for susceptible and infected prey respectively.
4. The scavenger which denoted by $Y_2(T)$ consumes the prey according to lotka – volttera functional response with positive attack rates α_3 , β_2 for susceptible and infected prey respectively.
5. It is assumed that there is enter-specific competition between the predator and scavenger with intensity of competition rates $\gamma_1 > 0$ and $\gamma_2 > 0$ respectively, and the completion is exploitive.
6. The food is up taken by the predator with up take rates $0 < e_2 < 1$ and $0 < e_2 < 1$.
7. The infected prey facing death with natural death rate $d_1 > 0$.
8. In the absence of the prey the predator and scavenger decays exponentially with natural death rate d_2, d_3 respectively.
9. Prey (susceptible and infection) may have out migration ,they can migrate to other geographical zone. Let m_1 and m_2 are the rate of migration of susceptible and infective populations, respectively. Also ecology suggested that $m_1 > m_2$. It is natural factor that susceptible (healthy/ sound). Prey are more strong as compared to infected one therefore the probability of migration of healthy prey is more than that of infected prey. Further the positive parameters m_3, m_4 represent the coefficients of migration of predator and scavenger, respectively.
10. $c_i, i = 1, 2$ represent the scavenger befits rates from naturally died predator to susceptible and infective prey, respectively.
11. $\sigma_i, i = 1, 2$ represent the scavenger befit rates from the killed prey by predator.

According to these hypotheses, when $X_1(T), X_2(T), Y_1(T)$ and $Y_2(T)$ represent the density of susceptible prey, infected prey, predator and scavenger at time T, the dynamics of the susceptible prey, infected prey. Predator and scavenger model can be described using the following set of differential equations:

$$
\begin{aligned}
\frac{dX_1}{\mathrm{dT}} &= rX_1\left(1 - \frac{X_1 + X_2}{k}\right) - \alpha_1 X_1 X_2 - \alpha_2 X_1 Y_1 - \alpha_3 X_1 Y_2 - m_1 X_1 \\
\frac{dX_2}{\mathrm{dT}} &= \alpha_1 X_1 X_2 - \beta_1 X_2 Y_1 - \beta_2 X_2 Y_2 - m_2 X_2 - d_1 X_2 \\
\frac{dY_1}{\mathrm{dT}} &= e_1 \alpha_2 X_1 Y_1 + e_2 \beta_1 X_2 Y_1 - \gamma_1 Y_1 Y_2 - m_3 Y_1 - d_2 Y_1 \\
\frac{dY_2}{\mathrm{dT}} &= c_1 \alpha_3 X_1 Y_2 + c_2 \beta_2 X_2 Y_2 + o_1 e_1 \alpha_2 X_1 Y_1 Y_2 + o_2 e_2 \beta_1 X_2 Y_1 Y_2 + \gamma_3 Y_1 Y_2 - \gamma_2 Y_1 Y_2 \\
&\quad - m_4 Y_2 - d_3 Y_2.
\end{aligned}
\tag{2.1}
$$

With initial conditions $X_1(0) \geq 0$, $\mathrm{X}_2(0) \geq 0$, $Y_1(0) \geq 0$, $Y_2(0) \geq 0$.
In order to study the above system of equations more generally, we drop all the units from it by using the following dimensionless variables and constants.

$$
\begin{aligned}
&X_1=p_1, \quad X_2=kZ_2, \quad Y_1=\frac{r}{\alpha_2}Z_3, \quad Y_2=\frac{r}{\alpha_3}Z_4, \quad t=rT, \quad u_1=k\frac{\alpha_1}{r},\\
&u_2=\frac{m_1}{r}, \quad u_3=\frac{\beta_1}{\alpha_2}, \quad u_4=\frac{\beta_2}{\alpha_3}, \quad u_5=\frac{m_2}{r}, \quad u_6=\frac{d_1}{r}, \quad u_7=ke_1\frac{\alpha_2}{r},\\
&u_8=ke_2\frac{\beta_1}{r}, \quad u_9=\frac{\gamma_1}{\alpha_3}, \quad u_{10}=\frac{m_3}{r}, \quad u_{11}=\frac{d_2}{r}, \quad u_{12}=k\frac{c_1}{r}\alpha_3, \quad u_{13}=k\frac{c_2}{r}\beta_2,\\
&u_{14}=k\ o_1e_1, \quad u_{15}=k\frac{o_2}{\alpha_2}e_2\beta_1\ , \quad u_{16}=\frac{\gamma_3}{\alpha_2}, u_{17}=\frac{\gamma_2}{\alpha_2}, \quad u_{18}=\frac{m_4}{r}, \quad u_{19}=\frac{d_3}{r}
\end{aligned}
\tag{2.2}
$$

The dimensionless of system (2.1) becomes

$$
\begin{aligned}
\frac{\mathrm{dz_1}}{\mathrm{dt}}&=z_1\left[1-(z_1+z_2)\right]-u_1z_1z_2-z_1z_3-z_1z_4-u_2z_1\\
\frac{\mathrm{dz_2}}{\mathrm{dt}}&=u_1z_1z_2-u_3z_2z_3-u_4z_2z_4-u_5z_2-u_6z_2\\
\frac{dz_3}{\mathrm{dt}}&=u_7z_1z_3+u_8z_2z_3-u_9z_3z_4-u_{10}z_3-u_{11}z_3\\
\frac{dz_4}{\mathrm{dt}}&=u_{12}z_1z_4+u_{13}z_2z_4+u_{14}z_1z_3z_4+u_{15}z_2z_3z_4+u_{16}z_3z_4-u_{17}z_3z_4-u_{18}z_4-u_{19}z_4
\end{aligned}
\tag{2.3}
$$

Therefore, system (2.3) has the following domain:

$$R^4_+=\left\{(z_1,z_2,z_3,z_4)\in R^4|z_1\geq 0,\ z_2\geq 0, z_3\geq 0, z_4\geq 0\right\}$$

Theorem 2.1. *All solutions of the system (2.3) with the initial condition belonging to R^4_+ are uniformly bounded.provided that the following sufficient condition holds.*

$$u_9+u_{17}\ >u_{16}+u_{14}+\frac{u_{15}}{L} \tag{2.4}$$

Where L is given in the proof

Proof . *Since the prey species consisting of two compartments, namely susceptible and infected population respectively. Then the total prey population is given by $N=z_1+z_2$, which is growing logistically in the absent of predation . Therefore, it is easy to verify that*

$$\frac{dN}{dt}=\frac{dz_1}{dt}+\frac{dz_2}{dt}\ \leq N(1-N)$$

Straightforward computation gives that

$$\limsup N(t)\leq 1, \qquad N(t)=z_1\left(t\right)+z_2\left(t\right)\leq 1, \quad t>0$$

Let $M\left(t\right)=z_1\left(t\right)+z_2\left(t\right)+z_3\left(t\right)+z_4\left(t\right)$, then from system (2.3) we obtain that

$$\frac{dM}{dt}\leq z_1-u_2z_1-\left(u_5+u_6\right)z_{2-}\left(u_{10}+u_{11}\right)z_3-\left(u_{18}+u_{19}\right)z_4-\left\{u_9+u_{17}-\left(u_{16}+u_{14}+\frac{u_{15}}{L}\right)\right\}z_3\ z_4$$

$$L=\{u_2\ ,\ u_5+u_6\ \}$$

$$\frac{dM}{dt}\leq\ z_1-u_2z_1-(u_5+u_6)z_{2-}(u_{10}+u_{11})z_3-(u_{18}+u_{19})z_4$$

$$\frac{dM}{dt}\leq\frac{1}{4}-qM, q=min\ \{u_{10}+u_{11}\ ,u_{18}+u_{19}\}$$

Thus $M\left(t\right)\leq\frac{1}{4q}$,$\forall\ t>0$ and hence the proof is complete . □

3. Existence of Equilibrium Points

It is observed that, system (2.3) has almost nine biologically feasible equilibrium points, namely $E_i, i = 0, 1, 2, 3, \ldots, 8$. the existence conditions for each of these equilibrium points are derived in the following. The vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ always exists. And prey free equilibrium point $E_1 = (\check{z_1}, 0, 0, 0)$ where

$$\check{z_1} = 1 - u_2 \tag{3.1a}$$

exists under the condition

$$u_2 < 1. \tag{3.1b}$$

The first two species equilibrium point $E_2 = (\underline{z}_1, \underline{z}_2, 0, 0)$, where

$$\underline{z}_1 = \frac{u_5 + u_6}{u_1} \quad and \quad \underline{z}_2 = \frac{u_1 - (u_1 u_2 + u_5 + u_6)}{u_1(1 + u_1)} \tag{3.2a}$$

Exist under the condition

$$u_1 > u_1 u_2 + u_5 + u_6 \tag{3.2b}$$

The second two species equilibrium point $E_3 = (\widehat{z}_1, 0\ , \widehat{z}_3, 0)$, with

$$\widehat{z}_1 = \frac{u_{10} + u_{11}}{u_7} \quad and \quad \widehat{z}_3 = \frac{u_7 - (u_2 u_7 + u_{10} + u_{11})}{u_7} \tag{3.3a}$$

Exist under the condition

$$(1 - u_2) > \frac{u_{10} + u_{11}}{u_7} \tag{3.3b}$$

The third two species equilibrium point $E_4 = (z_1^*, 0\ , 0, z_4^*\ )$. where

$$z_1^* = \frac{u_{18} + u_{19}}{u_{12}}\ , \quad and \quad z_4^* = \frac{u_{12} - (u_2 u_{12} + u_{18} + u_{19})}{u_{12}} \tag{3.4a}$$

Exits under the condition

$$(1 - u_2) > \frac{u_{18} + u_{19}}{u_{12}} \tag{3.4b}$$

Moreover, the first three species equilibrium point $E_5 = (\tilde{z}_1,\ \tilde{z}_2,\ \tilde{z}_3,\ 0)$ where

$$\begin{aligned}
\tilde{z}_1 &= \frac{u_3 \tilde{z}_3 + \ u_5 + \ u_6}{u_1} \\
\tilde{z}_2 &= \frac{u_1 u_{10} + u_1 u_{11} - u_7\ \ (u_3 z_3 + u_5 + u_6\ )}{u_1 u_8} \\
\tilde{z}_3 &= \frac{u_8\,(u_5 + \ u_6) + (1 + u_1)\,(u_1 u_{10} + u_1 u_{11}) + \ u_1 u_2 u_8 - \ u_7\ \ (1 + u_1)\,(u_5 + \ u_6) - \ u_1\ \mathrm{u}_8}{u_7\,(1 + u_1) - \ u_8\ \ (u_1 + u_3)}
\end{aligned} \tag{3.5a}$$

Exists under the following sets of conditions

$$\begin{aligned}&u_8(u_5+u_6)+(1+u_1)(u_1u_{10}+u_1u_{11})+u_1u_2u_8>u_7(1+u_1)(u_5+u_6)\\&+u_1u_8\ u_7(1+u_1)>u_8(u_1+u_3)\ \ u_1u_{10}\ +\ u_1u_{11}\ >\ u_7(u_3\ \tilde{z}_3+\ u_5\ +u_6\ )\end{aligned}\tag{3.5b}$$

The second three species equilibrium point $E_6=(\bar{\bar{z}}_1,\bar{\bar{z}}_2,0,\bar{\bar{z}}_4)$where

$$\begin{aligned}\bar{\bar{z}}_1&=\frac{u_4(1+u_1)(u_{18}+u_{19})-(u_4u_{13}+u_{13}(u_5+u_6))+u_2u_4u_{13}}{u_4(1+u_2)u_{12}-(u_4u_{13}+u_1u_{13})}\\\bar{\bar{z}}_2&=\frac{u_{18}+u_{19}-u_{12}[u_4(1+u_1)(u_{18}+u_{19})-u_4u_{13}+u_{13}(u_5+u_6)+u_2u_4u_{13}}{u_{13}[u_4(1+u_2)u_{12}-(u_4u_{13}+u_1u_{13})]}\\\bar{\bar{z}}_4&=\frac{u_1[u_4(1+u_1)(u_{18}+u_{19})-(u_4u_{13}+u_{13}(u_5+u_6))+u_2u_4u_{13}]-(u_5+u_6)}{u_4[u_4(1+u_2)u_{12}-(u_4u_{13}+u_1u_{13})]}\end{aligned}\tag{3.6a}$$

Exists under the following sets of conditions

$$\begin{aligned}&u_4(1+u_1)(u_{18}+u_{19})+u_2u_4u_{13}>u_4u_{13}+u_{13}(u_5+u_6)\ \ u_4(1+u_2)u_{12}\\&>u_4u_{13}+u_1u_{13}\ \ \frac{u_5+u_6}{u_1}<\bar{\bar{z}}_1<\ \frac{u_{18}+u_{19}}{u_{12}}\end{aligned}\tag{3.6b}$$

The third three species equilibrium point $E_7=(\tilde{\tilde{z}}_1,0,\tilde{\tilde{z}}_3,\tilde{\tilde{z}}_4)$

$$\tilde{\tilde{z}}_1=\frac{u_9\tilde{\tilde{z}}_4+(u_{10}+u_{11})}{u_7},\qquad\tilde{\tilde{z}}_3=\frac{u_7-\left\{u_{10}+u_{11}+\ u_2u_7+(u_9+u_7)\,\tilde{\tilde{z}}_4\right\}}{u_7}\tag{3.7a}$$

While $\tilde{\tilde{z}}_4$ represents a positive root of the following second order polynomial equation

$$A_1z_4^2+A_2z_4+A_3=0\tag{3.7b}$$

Here

$$\begin{aligned}A_1&=-u_9u_{14}(u_9+u_7)<0\\A_2&=\{u_9u_{12}+(1-u_2)u_7u_9u_{14}-2u_9u_{14}(u_{10}+u_{11})-(u_{10}+u_{11})u_7u_{14}\\&\quad-(u_{16}-u_{17})(u_9+u_7)\}\\A_3&=\{(1-u_2)u_7u_{10}u_{14}-\left(u_{10}^2+u_{11}^2\right)u_{14}-2u_{10}u_{11}u_{14}+(1-u_2)u_7u_{11}u_{14}\\&\quad+u_{12}(u_{10}+u_{11})u_7(u_{18}+u_{19})+(u_{16}-u_{17})(u_7-(u_{10}+u_{11}+u_2u_7))\}\end{aligned}\tag{3.7c}$$

Clearly, E_7 exists uniquely in interior of R^4, provided that the following conditions hold

$$A_3>0\tag{3.8a}$$

The positive equilibrium point $E_8=\left(z_1^\circ,z_2^\circ,z_3^\circ,z_4^\circ\right)$ of system (2.3) can be determined by equating the right hard side of system to the zero and save the resulting algebraic system. Straight forward computation gives second order polynomial equation

$$\begin{aligned}z_1^\circ&=\frac{Az_4^\circ+B}{C}\\z_2^\circ&=\frac{(Cu_9-\ u_7A)\ z_4^\circ+C(u_{10}+u_{11})-\ u_7\ \mathrm{B}}{Cu_8}\\z_3^\circ&=\frac{(u_1A-\ \mathrm{Cu}_4)\,z_4^\circ+(u_1B-C(u_5+u_6))}{Cu_3}\end{aligned}\tag{3.8b}$$

While z_4° represents a positive root of the following second order polynomial equation

$$D_1 z_4^2 + D_2 z_4 + D_3 = 0 \tag{3.8c}$$

Where

$$\begin{aligned}
D_1 &= (u_1 A - C u_4)\left(u_8 u_{14} A + u_{15}(C u_9 - u_7 A)\right) > 0 \\
D_2 &= C u_3 u_8 u_{12} A + C u_3 u_{13}\left(C u_9 - u_7 A\right) + u_8 u_{14} A\left(u_1 B - C\left(u_5 + u_6\right)\right) \\
&\quad + u_{15}\left(C u_9 - u_7 A\right)\left(u_1 B - C\left(u_5 + u_6\right)\right) + u_{15}\left(C(u_{10} + u_{11}) - u_7 B\right)\left(u_1 A - C u_4\right) \\
&\quad + C u_8\left(u_{16} - u_{17}\right)\left(u_1 A - C u_4\right) + u_8 u_{14} B\left(u_1 A - C u_4\right) \\
D_3 &= C u_3 u_8 u_{12} B + C u_3 u_{13}\left(C\left(u_{10} + u_{11}\right) - u_7 B\right) + u_8 u_{14} B\left(u_1 B - C\left(u_5 + u_6\right)\right) \\
&\quad + u_{15}\left(C\left(u_{10} + u_{11}\right) - u_7 B\right)\left(u_1 B - C\left(u_5 + u_6\right)\right)) - C^2 u_3 u_8\left(u_{18} + u_{19}\right) \\
&\quad + C u_8 (u_{16} - u_{17})(u_{11} B - C\left(u_5 + u_6\right)
\end{aligned}$$

Where

$$\begin{aligned}
A &= \left(1 + u_1\right) u_3 u_9 + u_3 u_8 - u_4 u_8 \\
B &= u_3\left(1 + u_1\right)\left(u_{10} + u_{11}\right) + u_1 u_3 u_8 - u_3 u_8 - u_8\left(u_5 + u_6\right) \\
C &= u_3 u_7\left(1 + u_1\right) - \left(u_3 u_8 + u_1 u_8\right)
\end{aligned}$$

Clearly, z_4° unique positive root if the following condition hold $D_3 < 0$

4. The Stability Analysis

In the section, the local stability of the equilibrium points of system (2.3) is investigated using the linearization method. It is easy to verify that the jacobian matrix of system (2.3). At the general point (z_1, z_2, z_3, z_4), can be written as

$$J = \left(a_{\mathrm{ij}}\right)_{4\times 4}, \qquad i, j = 1, 2, 3, 4 \tag{4.1a}$$

Where

$$\begin{aligned}
&a_{11} = \left[1 - z_1 - \left(1 + u_1\right) z_2 - z_3 - z_4 - u_2\right] - z_1, \quad a_{12} = -\left(1 + u_1\right) z_1, \quad a_{13} = -z_1, \quad a_{14} = -z_1, \\
&a_{21} = u_1 z_2, \quad a_{22} = u_1 z_1 - u_3 z_3 - u_4 z_4 - \left(u_5 + u_6\right), \quad a_{23} = -u_3 z_2, \quad a_{24} = -u_4 z_2, \\
&a_{31} = u_7 z_3, \quad a_{32} = u_8 z_3, \quad a_{33} = u_7 z_1 + u_8 z_2 - u_9 z_4 - \left(u_{10} + u_{11}\right), \quad a_{34} = -u_9 z_3, \\
&a_{41} = \left(u_{12} + u_{14} z_3\right) z_4, \quad a_{42} = \left(u_{13} + u_{15} z_3\right) z_4, \quad a_{43} = \left[u_{14} z_1 + u_{15} z_2 + \left(u_{16} - u_{17}\right)\right] z_4, \\
&a_{44} = \left[u_{12} z_1 + u_{13} z_2 + u_{14} z_1 z_3 + u_{15} z_2 z_3 + \left(u_{16} - u_{17}\right) z_3 - \left(u_{18} + u_{19}\right)\right]
\end{aligned}$$

Therefore, the Jacobian matrix of system (2.3) at the equilibrium point E_0 is

$$J\left(E_0\right) = \begin{pmatrix} 1 - u_2 & 0 & 0 & 0 \\ 0 & -u_5 - u_6 & 0 & 0 \\ 0 & 0 & -u_{10} - u_{11} & 0 \\ 0 & 0 & 0 & -u_{18} - u_{19} \end{pmatrix} \tag{4.1b}$$

Thus the eigenvalues of $J(E_0)$ are:

$$\lambda_1 = 1 - u_2 < 0, \qquad \lambda_2 = -u_5 - u_6 < 0,$$
$$\lambda_3 = -u_{10} - u_{11} < 0, \qquad \lambda_4 = -u_{18} - u_{19} < 0 \tag{4.1c}$$

All the eigenvalues of $J(E_0)$ have negative real parts if the following condition hold:

$$u_2 > 1 \tag{4.1d}$$

Therefore E_0 is locally asymptotically stable.

The jacobian matrix of system (2.3) at E_1 is written by

$$J(E_1) = \begin{pmatrix} -\check{z}_1 & -(1+u_1)\check{z}_1 & -\check{z}_1 & -\check{z}_1 \\ 0 & u_1\check{z}_1 - (u_5+u_6) & 0 & 0 \\ 0 & 0 & u_7\check{z}_1 - (u_{10}+u_{11}) & 0 \\ 0 & 0 & 0 & u_{12}\check{z}_1 - (u_{18}+u_{19}) \end{pmatrix} \tag{4.2a}$$

Thus the eigenvalues of $J(E_1)$ are:

$$\check{\lambda}_1 = -\check{z}_1, \quad \check{\lambda}_2 = u_1\check{z}_1 - (u_{5+}u_6),$$
$$\check{\lambda}_3 = u_7\check{z}_1 - (u_{10}+u_{11}), \quad \check{\lambda}_4 = u_{12}\check{z}_1 - (u_{18}+u_{19}) \tag{4.2b}$$

E_1 is locally asymptotically stable under the condition

$$\check{z}_1 < \left\{ \frac{u_{5+}u_6}{u_1}, \frac{u_{10+}u_{11}}{u_7}, \frac{u_{18+}u_{19}}{u_{12}} \right\} \tag{4.2c}$$

The jacobian matrix of system (2.3) at E_2 is written by

$$J(E_2) = \begin{pmatrix} -\bar{z}_1 & -(1+u_1)\bar{z}_1 & -\bar{z}_1 & -\bar{z}_1 \\ u_1\bar{z}_2 & 0 & -u_3\bar{z}_2 & -u_4\bar{z}_2 \\ 0 & 0 & u_7\bar{z}_1 + u_8\bar{z}_2 - (u_{10}+u_{11}) & 0 \\ 0 & 0 & 0 & u_{12}\bar{z}_1 + u_{13}\bar{z}_2 - (u_{18}+u_{19}) \end{pmatrix} \tag{4.3a}$$

Accordingly the characteristic equation of $J(E_2)$ can be written as

$$[(u_{12}\bar{z}_1 + u_{13}\bar{z}_2 - (u_{18}+u_{19})) - \lambda]\,[u_7\bar{z}_1 + u_8\bar{z}_2 - (u_{10}+u_{11})) - \lambda]\,\left[\lambda^2 + \bar{A}_1\lambda + \bar{A}_2\right] = 0 \tag{4.3b}$$

Here

$$\bar{A}_1 = -\{a_{11} + a_{22}\} = -a_{11} = -(-\bar{z}_1) = \bar{z}_1 > 0, \quad \bar{A}_2 = u_1(1+u_1)\bar{z}_1\bar{z}_2 > 0$$

So either

$$[(u_{12}\bar{z}_1 + u_{13}\bar{z}_2 - (u_{18}+u_{19})) - \lambda]\,[u_7\bar{z}_1 + u_8\bar{z}_2 - (u_{10}+u_{11})) - \lambda] = 0 \tag{4.3c}$$

Which gives two of Eigen values of $J(E_2)$ by

$$\bar{\lambda}_1 = u_{12}\bar{z}_1 + u_{13}\bar{z}_2 - (u_{18} + u_{19}), \qquad \bar{\lambda}_2 = u_7\bar{z}_1 + u_8\bar{z}_2 - (u_{10} + u_{11}) \tag{4.3d}$$

Or

$$\lambda^2 + \bar{A}_1\lambda + \bar{A}_2 = 0$$

Which gives the other two eigenvalues of $J(E_2)$

$$\bar{\lambda}_3 = \frac{-\bar{A}_1}{2} + \frac{1}{2}\sqrt{-\bar{A}_1^2 - 4\bar{A}_2}, \qquad \bar{\lambda}_4 = \frac{-\bar{A}_1}{2} - \frac{1}{2}\sqrt{-\bar{A}_1^2 - 4\bar{A}_2} \tag{4.3e}$$

Therefore, all the above eigenvalues have negative real parts if the following conditions hold

$$\begin{aligned} &u_{12}\bar{z}_1 + u_{13}\bar{z}_2 < u_{18} + u_{19} \\ &u_7\bar{z}_1 + u_8\bar{z}_2 < u_{10} + u_{11} \end{aligned} \tag{4.3f}$$

The jacobian matrix of system (2.3) at E_3 is given by

$$J(E_3) = \begin{pmatrix} -\widehat{z}_1 & -(1+u_1)\widehat{z}_1 & -\widehat{z}_1 & -\widehat{z}_1 \\ 0 & \widehat{a}_{22} & 0 & 0 \\ u_7\widehat{z}_3 & u_8\widehat{z}_3 & 0 & -u_9\widehat{z}_3 \\ 0 & 0 & 0 & \widehat{a}_{44} \end{pmatrix} \tag{4.4a}$$

$$\widehat{a}_{22} = u_1\widehat{z}_1 - u_3\widehat{z}_3 - (u_5 + u_6), \quad \widehat{a}_{44} = u_{12}\widehat{z}_1 + u_{14}\widehat{z}_1\widehat{z}_3 + (u_{16} - u_{17})\,\widehat{z}_3 - (u_{18} + u_{19})$$

Therefore the characteristic equation is

$$[\widehat{a}_{44} - \lambda]\,[\widehat{a}_{22} - \lambda]\left[\lambda^2 + \widehat{\beta}_1\lambda + \widehat{\beta}_2\right] = 0 \tag{4.4b}$$

Here $\widehat{\beta}_1 = \widehat{z}_1 > 0$ and $\widehat{\beta}_2 = u_7\widehat{z}_1\widehat{z}_3 > 0$

So either

$$[\widehat{a}_{44} - \lambda]\,[\widehat{a}_{22} - \lambda] = 0 \tag{4.4c}$$

Which give the eigenvalues of $J\,(E_3)$

$$\widehat{\lambda}_1 = u_{12}\widehat{z}_1 + u_{14}\widehat{z}_1\widehat{z}_3 + (u_{16} - u_{17})\,\widehat{z}_3 - (u_{18} + u_{19}), \quad \widehat{\lambda}_2 = u_1\widehat{z}_1 - u_3\widehat{z}_3 - (u_5 + u_6) \tag{4.4d}$$

Or

$$\lambda^2 + \widehat{\beta}_1\lambda + \widehat{\beta}_2 = 0 \tag{4.4e}$$

Which gives the other two eigenvalues of $J\left(E_3\right)$

$$\widehat{\lambda}_3 = \frac{-\widehat{\beta}_1}{2} + \frac{1}{2}\sqrt{\widehat{\beta}_1^2 - 4\widehat{\beta}_2}, \quad \widehat{\lambda}_4 = \frac{-\widehat{\beta}_1}{2} - \frac{1}{2}\sqrt{\widehat{\beta}_1^2 - 4\widehat{\beta}_2} \tag{4.4f}$$

Straightforward computation shows that all the eigenvalues of $J\left(E_3\right)$ have negative real parts if the following conditions hold:

$$\widehat{z}_1 < min\left\{\frac{u_{18} + u_{19} + \left(u_{17} - u_{16}\right)\widehat{z}_3}{u_{12} + u_{14}\widehat{z}_3}, \frac{u_5 + u_6 + u_3\widehat{z}_3}{u_1}\right\}, \qquad u_{16} < u_{17} \tag{4.4g}$$

Hence E_3 is locally asymptotically stable. However, it is a saddle point otherwise.

The jacobian matrix of system (2.3) at E_4 can be written as

$$J(E_4) = \begin{pmatrix} -z_1^* & -(1+u_1)z_1^* & -z_1^* & -z_1^* \\ 0 & c_{22}^* & 0 & 0 \\ 0 & 0 & c_{33}^* & 0 \\ u_{12}z_4^* & u_{13}z_4^* & c_{43}^* & 0 \end{pmatrix} \tag{4.5a}$$

Here

$$c_{22}^* = u_1 z_1^* - u_4 z_4^* - (u_5 + u_6), \quad c_{33}^* = u_7 z_1^* - u_9 z_4^* - \left(u_{10} + u_{11}\right), \quad c_{43}^* = u_{14} z_1^* z_4^* + \left(u_{16} - u_{17}\right) z_4^*$$

The characteristic equation of $J(E_4)$is given by

$$[(u_7 z_1^* - u_9 z_4^* - (u_{10} + u_{11})) - \lambda][(u_1 z_1^* - u_4 z_4^* - (u_5 + u_6)) - \lambda][\lambda^2 + C_1^*\lambda + C_2^*] = 0 \tag{4.5b}$$

Here

$$C_1^* = z_1^* > 0 \;\; and \;\; C_2^* = u_{12} z_1^* z_4^* > 0$$

So either

$$[\, c_{33}^* - \lambda] \, [\, c_{22}^* - \lambda] = 0 \tag{4.5c}$$

Which give the eigenvalues of $J(E_4)$

$$\lambda_1^* = u_7 z_1^* - u_9 z_4^* - \left(u_{10} + u_{11}\right), \quad \lambda_2^* = u_1 z_1^* - u_4 z_4^* - \left(u_5 + u_6\right) \tag{4.5d}$$

Or

$$\lambda_2^2 + c_1^*\lambda + c_2^* = 0 \tag{4.5e}$$

$$\lambda_3^*, \lambda_4^* = \frac{-c_1^*}{2} \pm \frac{1}{2}\sqrt{c_1^{*2} - 4c_2^*} \tag{4.5f}$$

Accordingly, it is easy to verify that all these eigenvalues have negative real parts if the following conditions are satisfied

$$z_1^* < min\left\{\frac{u_{10} + u_{11} + u_9 z_4^*}{u_7}, \frac{u_5 + u_6 + u_4 z_4^*}{u_1}\right\} \tag{4.5g}$$

Hence, E_4 is locally asymptotically stable. However, it is a saddle point otherwise.

The jacobian matrix of system (2.3) at E_5 can be written as

$$J(E_5) = \begin{pmatrix} -\tilde{z}_1 & -(1+u_1)\tilde{z}_1 & -\tilde{z}_1 & -\tilde{z}_1 \\ u_1\tilde{z}_2 & 0 & -u_3\tilde{z}_2 & -u_4\tilde{z}_2 \\ u_7\tilde{z}_3 & u_8\tilde{z}_3 & 0 & -u_9\tilde{z}_3 \\ 0 & 0 & 0 & \tilde{a}_{44} \end{pmatrix} \tag{4.6a}$$

Here

$$\tilde{a}_{44} = u_{12}\tilde{z}_1 + u_{13}\tilde{z}_2 + u_{14}\tilde{z}_1\tilde{z}_3 + u_{15}\tilde{z}_2\tilde{z}_3 + u_{16}\tilde{z}_3 - u_{17}\tilde{z}_3 - (u_{18} + u_{19})$$

The characteristic equation of $J(E_5)$ is given by

$$(\tilde{a}_{44} - \lambda)\left(\lambda^3 + \tilde{A}_1\lambda^2 + \ \tilde{A}_2\lambda + \tilde{A}_3\right) = 0 \tag{4.6b}$$

Where

$$\tilde{A}_1 = -(a_{11} + a_{22} + a_{33}) = -a_{11} = \tilde{z}_1 > 0, \quad \tilde{A}_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}$$
$$\tilde{A}_3 = -a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}$$

While

$$\Delta = \tilde{A}_1\tilde{A}_2 - \tilde{A}_3 \tag{4.6c}$$

So the eigenvalue in the fourth direction

$$\tilde{\lambda}z_4 = u_{12}\tilde{z}_1 + u_{13}\tilde{z}_2 + u_{14}\tilde{z}_1\tilde{z}_3 + u_{15}\tilde{z}_2\tilde{z}_3 + u_{16}\tilde{z}_3 - u_{17}\tilde{z}_3 - (u_{18} + u_{19}) \tag{4.6d}$$

Or

$$\lambda^3 + \tilde{A}_1\lambda^2 + \tilde{A}_2\lambda + \tilde{A}_3 = 0 \tag{4.6e}$$

However the other eigenvalues represent the roots of the third order polynomial which have negative real parts if the following conditions are satisfied

$$(1+u_1)u_3u_7 < u_1u_8$$
$$u_{12}\tilde{z}_1 + u_{13}\tilde{z}_2 + u_{14}\tilde{z}_1\tilde{z}_3 + u_{15}\tilde{z}_2\tilde{z}_3 + u_{16}\tilde{z}_3 < u_{17}\tilde{z}_3 + (u_{18} + u_{19}) \tag{4.6f}$$

So, E_5 is locally asymptotically stable, however, it is saddle point otherwise.

(4.6g)

The jacobian matrix of system (2.3) at E_6 can be written as

$$J(E_6)=\begin{pmatrix} -\bar{\bar{z}}_1 & -(1+u_1)\bar{\bar{z}}_1 & -\bar{\bar{z}}_1 & -\bar{\bar{z}}_1 \\ u_1\bar{\bar{z}}_2 & 0 & -u_3\bar{\bar{z}}_2 & -u_4\bar{\bar{z}}_2 \\ 0 & 0 & \bar{\bar{a}}_{33} & 0 \\ u_{12}\bar{\bar{z}}_4 & u_{13}\bar{\bar{z}}_4 & \bar{\bar{a}}_{43} & 0 \end{pmatrix} \tag{4.7a}$$

$$\bar{\bar{a}}_{33}=u_7\bar{\bar{z}}_1+u_8\bar{\bar{z}}_2-u_9\bar{\bar{z}}_4-(u_{10}+u_{11}), \qquad \bar{\bar{a}}_{43}=u_{14}\bar{\bar{z}}_1\bar{\bar{z}}_4+u_{15}\bar{\bar{z}}_2\bar{\bar{z}}_4+ \ (u_{16}-u_{17})\,\bar{\bar{z}}_4$$

The characteristic equation of $J(E_6)$ is written as:

$$\left(\bar{\bar{a}}_{33}-\lambda\right)\left(\lambda^3+\bar{\bar{a}}_1\lambda^2+\bar{\bar{a}}_2\lambda+\bar{\bar{a}}_3\right)=0 \tag{4.7b}$$

Where

$$\begin{aligned} \bar{\bar{A}}_1 &= -\left(a_{11}+a_{22}+a_{44}\right) = -\left(-\bar{\bar{z}}_1\right) = \bar{\bar{z}}_1 > 0, \\ \bar{\bar{A}}_2 &= a_{11}a_{22}-a_{12}a_{21}+a_{11}a_{44}-a_{14}a_{41}+a_{22}a_{44}-a_{24}a_{42} \\ \bar{\bar{A}}_3 &= -a_{11}a_{22}a_{44}-a_{12}a_{24}a_{41}-a_{14}a_{21}a_{42}+a_{14}a_{22}a_{41}+a_{11}a_{24}a_{42}+a_{12}a_{21}a_{44} \end{aligned}$$

While

$$\Delta=\bar{\bar{A}}_1\ \bar{\bar{A}}_2-\bar{\bar{A}}_3 \tag{4.7c}$$

Therefore the eigenvalue

$$\bar{\bar{\lambda}}=u_7\bar{\bar{z}}_1+u_8\bar{\bar{z}}_2-u_9\bar{\bar{z}}_4-(u_{10}+u_{11}) \tag{4.7d}$$

However the eigenvalue represent the roots of the third order polynomial which have negative real parts iff $\bar{\bar{a}}_1>0,\ \bar{\bar{a}}_2>0,\quad ,\bar{\bar{a}}_3>0\ and\ \Delta>0$. So straight forward computation shows that all the eigenvalues of $J(E_6)$ have negative real parts if the following conditions are satisfied:

$$(1+u_1)\,u_4u_{12}<u_1u_{13}, \quad u_7\bar{\bar{z}}_1+u_8\bar{\bar{z}}_2<u_9\bar{\bar{z}}_4+(u_{10}+u_{11}) \tag{4.7e}$$

So, E_6 is locally asymptotically stable, however, it is saddle point otherwise.

The jacobian matrix of system (2.3) at E_7 can be written as

$$J(E_7)=\begin{pmatrix} -\tilde{\tilde{z}}_1 & -(1+u_1)\tilde{\tilde{z}}_1 & -\tilde{\tilde{z}}_1 & -\tilde{\tilde{z}}_1 \\ 0 & u_1\tilde{\tilde{z}}_1-u_3\tilde{\tilde{z}}_3-u_4\tilde{\tilde{z}}_4-(u_5+u_6) & 0 & 0 \\ u_7\tilde{\tilde{z}}_3 & u_8\tilde{\tilde{z}}_3 & 0 & -u_9\tilde{\tilde{z}}_3 \\ u_{12}\tilde{\tilde{z}}_4+u_{14}\tilde{\tilde{z}}_3\tilde{\tilde{z}}_4 & u_{13}\tilde{\tilde{z}}_4+u_{15}\tilde{\tilde{z}}_3\tilde{\tilde{z}}_4 & \tilde{\tilde{a}}_{43} & 0 \end{pmatrix} \tag{4.8a}$$

Here

$$\tilde{\tilde{a}}_{43}=u_{14}\tilde{\tilde{z}}_1\tilde{\tilde{z}}_4+(u_{16}-u_{17})\,\tilde{\tilde{z}}_4$$

The characteristic equation of $J(E_7)$ is given by

$$\left(\tilde{\tilde{a}}_{22}-\lambda\right)\left(\lambda^3+\tilde{\tilde{A}}_1\lambda^2+\tilde{\tilde{A}}_2\lambda+\tilde{\tilde{A}}_3\right)=0 \tag{4.8b}$$

Where

$$\begin{aligned}
\tilde{\tilde{A}}_1 &= -\left(a_{11}+a_{33}+a_{44}\right) = -a_{11} = \tilde{\tilde{z}}_1 > 0,\\
\tilde{\tilde{A}}_2 &= a_{11}a_{33}-a_{13}a_{31}+a_{11}a_{44}-a_{14}a_{41}+a_{33}a_{44}-a_{34}a_{43},\\
\tilde{\tilde{A}}_3 &= -a_{11}a_{33}a_{44}-a_{13}a_{34}a_{41}-a_{14}a_{31}a_{42}+a_{14}a_{33}a_{41}
\end{aligned}$$

While

$$\Delta=\tilde{\tilde{A}}_1\tilde{\tilde{A}}_2-\tilde{\tilde{A}}_3 \tag{4.8c}$$

Therefore the eigenvalues

$$\tilde{\tilde{\lambda}}=u_1\tilde{\tilde{z}}_1-u_3\tilde{\tilde{z}}_3-u_4\tilde{\tilde{z}}_4-\left(u_5+u_6\right) \tag{4.8d}$$

However the eigenvalue represent the roots of the third order polynomial which have negative real parts iff

$$\tilde{\tilde{A}}_1>0,\quad \tilde{\tilde{A}}_2>0,\quad \tilde{\tilde{A}}_3>0\quad and\ \Delta>0$$

So straightforward computation shows that all the eigenvalues of $J(E_7)$ have negative real parts if the following are satisfied:

$$u_9u_{12}+u_9u_{14}\tilde{\tilde{z}}_3<u_7u_{13}+u_7u_{15}\tilde{\tilde{z}}_3,\quad u_1\tilde{\tilde{z}}_1<u_3\tilde{\tilde{z}}_3+u_4\tilde{\tilde{z}}_4+\left(u_5+u_6\right) \tag{4.8e}$$

So, E_7 is locally asymptotically stable, however, it is saddle point otherwise.

(4.8f)

The Jacobian matrix of system (2.3) at E_8 can be written as

$$J(E_8)=\begin{pmatrix} \overset{\circ}{a}_{11} & \overset{\circ}{a}_{12} & \overset{\circ}{a}_{13} & \overset{\circ}{a}_{14}\\ \overset{\circ}{a}_{21} & 0 & \overset{\circ}{a}_{23} & \overset{\circ}{a}_{24}\\ \overset{\circ}{a}_{31} & \overset{\circ}{a}_{32} & 0 & \overset{\circ}{a}_{34}\\ \overset{\circ}{a}_{41} & \overset{\circ}{a}_{42} & \overset{\circ}{a}_{43} & 0 \end{pmatrix} \tag{4.9a}$$

$\overset{\circ}{a}_{11}=\overset{\circ}{a}_{13}=\overset{\circ}{a}_{14}=-\overset{\circ}{z}_1,\quad \overset{\circ}{a}_{12}=-\left(1+u_1\right)\overset{\circ}{z}_1,\quad ,\ \overset{\circ}{a}_{21}=u_1\overset{\circ}{z}_2,\quad a_{22}=0,\quad \overset{\circ}{a}_{23}=-u_3\overset{\circ}{z}_2,$
$\overset{\circ}{a}_{24}=-u_4\overset{\circ}{z}_2,\quad \overset{\circ}{a}_{31}=u_7\overset{\circ}{z}_3,\quad \overset{\circ}{a}_{32}=u_8\overset{\circ}{z}_3,\quad \overset{\circ}{a}_{33}=0,\quad \overset{\circ}{a}_{34}=-u_9\overset{\circ}{z}_3,\quad \overset{\circ}{a}_{41}=\left(u_{12}+u_{14}\overset{\circ}{z}_3\right)\overset{\circ}{z}_4,$
$\overset{\circ}{a}_{42}=\left(u_{13}+u_{15}\overset{\circ}{z}_3\right)\overset{\circ}{z}_4,\quad \overset{\circ}{a}_{43}=\left(u_{14}\overset{\circ}{z}_1+u_{15}\overset{\circ}{z}_2+\left(u_{16}-u_{17}\right)\overset{\circ}{z}_4\right),\quad \overset{\circ}{a}_{44}=0$

The characteristic equation of $J(E_8)$ is given by

$$\lambda^4+G_1\lambda^3+G_2\lambda^2+G_3\lambda+G_4=0 \tag{4.9b}$$

Where

$$\begin{aligned}
&G_1 = -\left(\mathring{a}_{11} + \mathring{a}_{22} + \mathring{a}_{33} + \mathring{a}_{44}\right) = -\left(-\mathring{z}_1\right) = \mathring{z}_1 > 0,\\
&G_2 = -\mathring{a}_{12}\mathring{a}_{21} - \mathring{a}_{13}\mathring{a}_{31} - \mathring{a}_{14}\mathring{a}_{41} - \mathring{a}_{23}\mathring{a}_{32} - \mathring{a}_{24}\mathring{a}_{42} - \mathring{a}_{34}\mathring{a}_{43} > 0\\
&G_3 = \mathring{a}_{11}\mathring{a}_{23}\mathring{a}_{32} - \mathring{a}_{21}\mathring{a}_{13}\mathring{a}_{32} - \mathring{a}_{31}\mathring{a}_{12}\mathring{a}_{23} + \mathring{a}_{11}\mathring{a}_{34}\mathring{a}_{43} - \mathring{a}_{31}\mathring{a}_{14}\mathring{a}_{43} - \mathring{a}_{41}\mathring{a}_{13}\mathring{a}_{34} + \mathring{a}_{11}\mathring{a}_{24}\mathring{a}_{42}\\
&\quad - \mathring{a}_{21}\mathring{a}_{14}\mathring{a}_{42} - \mathring{a}_{41}\mathring{a}_{12}\mathring{a}_{24} - \mathring{a}_{32}\mathring{a}_{24}\mathring{a}_{43} - \mathring{a}_{42}\mathring{a}_{23}a_{34}\\
&G_4 = \mathring{a}_{11}\mathring{a}_{32}\mathring{a}_{24}\mathring{a}_{43} + \mathring{a}_{11}\mathring{a}_{42}\mathring{a}_{23}\mathring{a}_{34} + \mathring{a}_{21}\mathring{a}_{12}\mathring{a}_{34}\mathring{a}_{43} - \mathring{a}_{21}\mathring{a}_{32}\mathring{a}_{14}\mathring{a}_{43} - \mathring{a}_{21}\mathring{a}_{42}\mathring{a}_{13}\mathring{a}_{34} - \mathring{a}_{31}\mathring{a}_{12}\mathring{a}_{24}\mathring{a}_{43}\\
&\quad + \mathring{a}_{31}\mathring{a}_{42}\mathring{a}_{13}\mathring{a}_{24} - \mathring{a}_{31}\mathring{a}_{42}\mathring{a}_{14}\mathring{a}_{23} - \mathring{a}_{41}\mathring{a}_{12}\mathring{a}_{23}\mathring{a}_{34} - \mathring{a}_{41}\mathring{a}_{32}\mathring{a}_{13}\mathring{a}_{24} + \mathring{a}_{41}\mathring{a}_{32}\mathring{a}_{14}\mathring{a}_{23}\\
&\Delta = G_3\left(G_1G_2 - G_3\right) - G_1^2G_4 \quad = H_1\left(H_2 - H_1\right) + H_3
\end{aligned}$$

Here

$$\begin{aligned}
&H_1 = \mathring{a}_{11}\mathring{a}_{23}\mathring{a}_{32} - \mathring{a}_{21}\mathring{a}_{13}\mathring{a}_{32} - \mathring{a}_{31}\mathring{a}_{12}\mathring{a}_{23} + \mathring{a}_{11}\mathring{a}_{34}\mathring{a}_{43} - \mathring{a}_{31}\mathring{a}_{14}\mathring{a}_{43} - \mathring{a}_{41}\mathring{a}_{13}\mathring{a}_{34}\\
&\quad + \mathring{a}_{11}\mathring{a}_{24}\mathring{a}_{42} - \mathring{a}_{21}\mathring{a}_{14}\mathring{a}_{42} - \mathring{a}_{41}\mathring{a}_{12}\mathring{a}_{24} - \mathring{a}_{32}\mathring{a}_{24}\mathring{a}_{43} - \mathring{a}_{42}\mathring{a}_{23}\mathring{a}_{34}\\
&H_2 = \mathring{z}_1\left(-\mathring{a}_{12}\mathring{a}_{21} - \mathring{a}_{13}\mathring{a}_{31} - \mathring{a}_{14}\mathring{a}_{41} - \mathring{a}_{23}\mathring{a}_{32} - \mathring{a}_{24}\mathring{a}_{42} - \mathring{a}_{34}\mathring{a}_{43}\right)\\
&H_3 = \mathring{z}_1^{2}(\mathring{a}_{11}\mathring{a}_{32}\mathring{a}_{24}\mathring{a}_{43} + \mathring{a}_{11}\mathring{a}_{42}\mathring{a}_{23}\mathring{a}_{34} + \mathring{a}_{21}\mathring{a}_{12}\mathring{a}_{34}\mathring{a}_{43} - \mathring{a}_{21}\mathring{a}_{32}\mathring{a}_{14}\mathring{a}_{43} - \mathring{a}_{21}\mathring{a}_{42}\mathring{a}_{13}\mathring{a}_{34} - \mathring{a}_{31}\mathring{a}_{12}\mathring{a}_{24}\mathring{a}_{43}\\
&\quad + \mathring{a}_{31}\mathring{a}_{42}\mathring{a}_{13}\mathring{a}_{24} - \mathring{a}_{31}\mathring{a}_{42}\mathring{a}_{14}\mathring{a}_{23} - \mathring{a}_{41}\mathring{a}_{12}\mathring{a}_{23}\mathring{a}_{34} - \mathring{a}_{41}\mathring{a}_{32}\mathring{a}_{13}\mathring{a}_{24} + \mathring{a}_{41}\mathring{a}_{32}\mathring{a}_{14}\mathring{a}_{23})
\end{aligned} \tag{4.9c}$$

Now, the region of global stability (basin of attraction) of each equilibrium points of the system (2.3) is presented as shown in the following theorems

Theorem 4.1. *Assume that E_0 is locally asymptotically stable in R_+^4 and the following condition hold*

$$u_9 + u_{17} > u_{16} + u_{14} + \frac{u_{15}}{L} \tag{4.10}$$

Then the equilibrium point E_0 is globally asymptotically stable.

Proof . *consider the following function*

$$V_0\left(z_1, z_2, z_3, z_4\right) = z_1 + z_2 + z_3 + z_4$$

Clearly, $V_0 : R_+^4 \to R$ is a continuously differentiable function such that $V_0\left(0,0,0,0\right) = 0$ and

$$V_0\left(z_1, z_2, z_3, z_4\right) > 0,\ \forall\left(z_1, z_2, z_3, z_4\right) \neq \left(0,0,0,0\right).$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}
\frac{d\ V_0}{dt} = &\left\{z_1\left[1 - \left(z_1 + z_2\right)\right] - u_1z_1z_2 - z_1z_3 - z_1z_4 - u_2z_1\right\} + \left\{u_1z_1z_2 - u_3z_2z_3 - u_4z_2z_4 - \left(u_5 + u_6\right)z_2\right\}\\
&+ \left\{u_7z_1z_3 + u_8z_2z_3 - u_9z_3z_4 - \left(u_{10} + u_{11}\right)z_3\right\} + \left\{u_{12}z_1z_4 + u_{13}z_2z_4 + u_{14}z_1z_3z_4 + u_{14}z_2z_3z_4\right.\\
&\left. + u_{16}z_3z_4 - u_{17}z_3z_4 - \left(u_{18} + u_{19}\right)z_4\right\}
\end{aligned}$$

Now, by using the given condition we obtains that

$$\begin{aligned}
\frac{d\ V_0}{dt} \leq &-\left(u_2 - 1\right)z_1 - \left(u_5 + u_6\right)z_2 - \left(u_{10} + u_{11}\right)z_3 - \left(u_{18} + u_{19}\right)z_4\\
&- \left\{u_9 + u_{17} - \left(u_{16} + u_{14} + \frac{u_{15}}{L}\right)\right\}z_3z_4
\end{aligned}$$

Now, it's easy to verify that $\frac{d\ V_0}{dt}$ is negative definite.

Thus E_0 is a globally asymptotically stable and the proof is complete.□

Theorem 4.2. *Assume that E_1 is locally asymptotically stable in R_+^4 and the following condition hold.*

$$\check{z_1} < min\left\{\frac{u_5+u_6}{1+u_1}\ , u_{10}+u_{11}\ , u_{18}+u_{19}\ \right\} \tag{4.11}$$

Then the equilibrium point E_1 is globally asymptotically stable.

Proof . *consider the following function*

$$V_1\left(z_1,z_2,z_3,z_4\right)=\left(z_1-\check{z_1}-\check{z_1}\ ln\ \frac{z_1}{\check{z_1}}\right)+z_2+z_3+z_4$$

Obviously, $V_1 : R_+^4 \to R$ is a continuously differentiable function such that

$$\begin{aligned}&V_1\left(\check{z_1},0,0,0\right)=0,\quad while\\ &V_1\left(z_1,z_2,z_3,z_4\right)>0\ for\ all\ \left(z_1,z_2,z_3,z_4\right)\in R_+^4\quad and\quad\left(z_1,z_2,z_3,z_4\right)\neq\left(\check{z_1},0,0,0\right).\end{aligned}$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dV_1}{dt}=&\left(\frac{Z_1-Z_1}{Z_1}\right)\left[z_1\left(1-z_1-z_2-u_1z_2-z_3-z_4-u_2\right)\right]+\left[u_1z_1z_2-u_3z_2z_3-u_4z_2z_4-u_5z_2-u_6z_2\right]\\&+\left[u_7z_1z_3+u_8z_2z_3-u_9z_3z_4-\left(u_{10}+u_{11}\right)z_3\right]+\left[u_{12}z_1z_4+u_{13}z_2z_4+u_{14}z_1z_3z_4+u_{15}z_2z_3z_4\right.\\&\left.+\left(u_{16}-u_{17}\right)z_3z_4-\left(u_{18}+u_{19}\right)z_4\right]\end{aligned}$$

Therefore, by using the condition (4.11) *the derivative $\frac{dV_1}{dt}$ becomes*

$$\begin{aligned}\frac{dV_1}{dt}\leq&-\left(z_1-\check{z_1}\right)^2-\left\{\left(u_5+u_6\right)-\left(1+u_1\right)\check{z_1}\right\}z_2-\left\{\left(u_{10}+u_{11}\right)-\check{z_1}\right\}z_3\\&-\left\{\left(u_{18}+u_{19}\right)-\check{z_1}\right\}z_4-\left\{u_9+u_{17}-\left(u_{16}+u_{14}+\frac{u_{15}}{L}\right)\right\}z_3z_4\end{aligned}$$

Now, it's easy to verify that $\frac{dV_1}{dt}$ is negative definite.

Hence the solution of system (2.3) *will approach asymptotically to E_1 from any initial point satisfies the above condition and then the proof is complete.* □

Theorem 4.3. *Suppose the point of equilibrium E_2 is asymptotically locally. So, it is asymptotically globally stable in the sub region of R_+^4 if the following conditions hold:*

$$\underline{z}_1 < min\left\{z_1, u_{10}+u_{11}-u_{32}, u_{18}+u_{19}-u_{42}\right\} \tag{4.12a}$$

$$\underline{z}_2 < z_2 \tag{4.12b}$$

Proof . *considering the positive following definite being a faction*

$$V_2\left(z_1,z_2,z_3,z_4\right)=\left(z_1-\underline{z}_1-\underline{z}_1\ln\frac{z_1}{\underline{z}_1}\right)+\left(z_2-\underline{z}_2-\underline{z}_2\ln\frac{z_2}{\underline{z}_2}\right)+z_3+z_4$$

Obviously, $V_2 : R_+^4 \to R$ is a continuously differentiable function such that $V_2\left({}_1,{}_2,0,0\right)=0$ while

$$V_2\left(z_1,z_2,z_3,z_4\right)>0\ \forall\left(z_1,z_2,z_3,z_4\right)\in R_+^4\ and\ \left(z_1,z_2,z_3,z_4\right)\neq\left({}_1,{}_2,0,0\right).$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dV_2}{dt} = \left(\frac{z_1 - {}_1}{z_1}\right)[z_1(1 - z_1 - z_2 - u_1 z_2 - z_3 - z_4 - u_2)] + \left(\frac{z_2 - {}_2}{z_2}\right)[z_2(u_1 z_1 - u_3 z_3 - u_4 z_4 - (u_5 + u_6)]$$
$$+ [u_7 z_1 z_3 + u_8 z_2 z_3 - u_9 z_3 z_4 - (u_{10} + u_{11}) z_3] + [u_{12} z_1 z_4 + u_{13} z_2 z_4 + u_{14} z_1 z_3 z_4 + u_{15} z_2 z_3 z_4 + u_{16} z_3 z_4$$
$$- u_{17} z_3 z_4 - (u_{18} + u_{19}) z_4]$$

Consequently, due to conditions (4.12a)-(4.12b), *we have*

$$\frac{dV_2}{dt} \leq -(z_1 - {}_1)^2 - (z_1 - {}_1)(z_2 - {}_2) - \{u_{10} + u_{11} - u_{32} - {}_1\} z_3 - \{u_{18} + u_{19} - {}_1 - u_{42}\} z_4$$
$$- \left\{u_9 + u_{17} - \left(u_{16} + u_{14} + \frac{u_{15}}{L}\right)\right\} z_3 z_4$$

Obviously $\frac{dV_2}{dt}$ *is negative definite and hence* V_2 *is a Lyapunov function with respect to* E_2 *in the sub region in* R_+^4. *So* E_2 *is a globally asymptotically stable.* □

Theorem 4.4. *Suppose the point of equilibrium* E_3 *is asymptotically locally. So, it is asymptotically globally stable in the sub region of* R_+^4 *if the following conditions hold:*

$$\widehat{z}_1 < min\left\{z_1, \frac{u_5 + u_6 + u_8\widehat{z}_3}{1 + u_1}\right\} \ \widehat{z}_3 < z_3 \tag{4.13}$$

Proof .*considering the positive following definite being a faction*

$$V_3(z_1,\ z_2,\ z_3,\ z_4) = \left(z_1 - \widehat{z}_1 - \widehat{z}_1 \ln\ln\ \frac{z_1}{\widehat{z}_1}\right) + z_2 + \left(z_3 - \widehat{z}_3 - \widehat{z}_3 \ln\ln\ \frac{z_3}{\widehat{z}_3}\right) + z_4$$

Obviously, $V_3 : R_+^4 \to R$ *is a continuously differentiable function such that* $V_3(\widehat{z}_1, 0, \widehat{z}_3, 0) = 0$ *while*

$$V_3(z_1,\ z_2,\ z_3,\ z_4) > 0,\ \forall (z_1,\ z_2,\ z_3,\ z_4) \in R_+^4 \ and\ (z_1,\ z_2,\ z_3,\ z_4) \neq (\widehat{z}_1, 0, \widehat{z}_3, 0).$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we got that

$$\frac{dV_3}{dt} = \left(\frac{z_1 - \widehat{z}_1}{z_1}\right)[z_1(1 - z_1 - z_2 - u_1 z_2 - z_3 - z_4 - u_2)] + [u_1 z_1 z_2 - u_3 z_2 z_3 - u_4 z_2 z_4 - (u_5 + u_6) z_2]$$
$$+ \left(\frac{z_3 - \widehat{z}_3}{z_3}\right)[z_3(u_7 z_1 + u_8 z_2 - u_9 z_4 - (u_{10} + u_{11}))] + [u_{12} z_1 z_4 + u_{13} z_2 z_4 + u_{14} z_1 z_3 z_4 + u_{15} z_2 z_3 z_4$$
$$+ (u_{16} - u_{17}) z_3 z_4 - (u_{18} + u_{19}) z_4]$$

Now, by using the given condition we obtains that

$$\frac{dV_3}{dt} \leq -(z_1 - \widehat{z}_1)^2 - (1 - u_7)(z_1 - \widehat{z}_1)(z_3 - \widehat{z}_3) - \{(u_5 + u_6) + u_8\widehat{z}_3 - (1 + u_1)\widehat{z}_1\} z_2 - \{u_7\widehat{z}_1 - \widehat{z}_1\} z_3$$
$$- \{(u_{18} + u_{19}) - u_9\widehat{z}_3 - \widehat{z}_1\} z_4 - \{u_9 + u_{17} - (u_{16} + u_{14} + \frac{u_{15}}{L})\} z_3 z_4$$

Obviously $\frac{dV_3}{dt}$ *is negative definite and hence* V_3 *is a Lyapunov function with respect to* E_3 *in the sub region in* R_+^4. *So* E_3 *is a globally asymptotically stable.* □

Theorem 4.5. *Assume that $E_4$is locally asymptotically stable in R_+^4, then following condition hold.*

$$z_1^* < \{\frac{u_5 + u_6 + u_{13} z_4^*}{1 + u_1}\ ,\ u_{10} + u_{11} + (u_{16} - u_{17})\, z_4^*\ ,\ u_{18} + u_{19}\} \tag{4.14}$$

Then the equilibrium point E_4 is globally asymptotically stable.
Proof . *consider the following function*

$$V_4(z_1,\ z_2,\ z_3,\ z_4) = \left(z_1 - z_1^* - z_1^* \ln\ln\ \frac{z_1}{z_1^*}\right) + z_2 +\ z_3 + \left(z_4 - z_4^* - z_4^* \ln\ln\ \frac{z_4}{z_4^*}\right)$$

Obviously, $V_4 : R_+^4 \to R$ is a continuously differentiable function such that $V_4(z_1^*, 0, 0, z_4^*) = 0$ While

$$V_4(z_1,\ z_2,\ z_3,\ z_4) > 0\ for\ all\ \ (z_1,\ z_2,\ z_3,\ z_4) \in R_+^4\ and\ \ (z_1,\ z_2,\ z_3,\ z_4) \neq (z_1^*, 0, 0, z_4^*)\,.$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dV_4}{dt} &= \left(\frac{z_1 - z_1^*}{z_1}\right)[z_1(1 - z_1 - z_2 - u_1 z_2 - z_3 - z_4 - u_2)] + [u_1 z_1 z_2 - u_3 z_2 z_3 - u_4 z_2 z_4 - u_5 z_2 - u_6 z_2] \\ &+ [u_7 z_1 z_3 + u_8 z_2 z_3 - u_9 z_3 z_4 - (u_{10} + u_{11})\, z_3] + \left(\frac{z_4 - z_4^*}{z_4}\right)[z_4(u_{12} z_1 + u_{13} z_2 + u_{14} z_1 z_3 + u_{15} z_2 z_3 \\ &+ (u_{16} - u_{17})\, z_3 - (u_{18} + u_{19})]\end{aligned}$$

Therefore, by using the condition(4.14) *the derivative $\frac{dV_4}{dt}$ becomes*

$$\begin{aligned}\frac{dV_4}{dt} &\leq -(z_1 - z_1^*)^2 - \{u_{13} z_4^* - (1 + u_1)\, z_1^* + (u_5 + u_6)\} z_2 - \{(u_{16} - u_{17})\, z_4^* + (u_{10} + u_{11}) - z_1^*\}\, z_3 \\ &- \{(u_{18} + u_{19}) - z_1^*\}\, z_4 - \left\{u_9 + u_{17} - \left(u_{16} + u_{14} + \frac{u_{15}}{L}\right)\right\} z_3 z_4\end{aligned}$$

Now its easy to verify that $\frac{dV_4}{dt}$ is negative definite.
Hence the solution of system (2.3) *will approach asymptotically to E_4 from any initial point satisfies the above condition and then the proof is complete.* □

Theorem 4.6. *Suppose the point of equilibrium E_5 is asymptotically locally. So, it is asymptotically globally stable in the sub region of R_+^4 if the following conditions hold:*

$$z_3 > max\left\{\frac{\tilde{z}_2 + (1 - u_7)\,\tilde{z}_3}{(1 - u_7)}, \frac{\tilde{z}_1 + (u_3 - u_8)\,\tilde{z}_2}{(u_3 - u_8)}\right\} \tag{4.15a}$$

$$u_{18} + u_{19} > \tilde{z}_1 + u_4 \tilde{z}_2 + u_9 \tilde{z}_3 \tag{4.15b}$$

Proof . *consider the following function*

$$V_5(z_1, z_2, z_3, z_4) = \left(z_1 - \tilde{z}_1 - \tilde{z}_1 \ln\frac{z_1}{\tilde{z}_1}\right) + \left(z_2 - \tilde{z}_2 - \tilde{z}_2 \ln\frac{z_2}{\tilde{z}_2}\right) + \left(z_3 - \tilde{z}_3 - \tilde{z}_3 \ln\frac{z_3}{\tilde{z}_3}\right) + z_4$$

Obviously, $V_5 : R_+^4 \to R$ is a continuously differentiable function such that $V_5(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, 0) = 0$ While

$$V_5(z_1,\ z_2,\ z_3,\ z_4) > 0,\ \forall\ \ (z_1,\ z_2,\ z_3,\ z_4) \in R_+^4\ and\ \ (z_1,\ z_2,\ z_3,\ z_4) \neq (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, 0)$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms we get that:

$$\begin{aligned}\frac{dV_5}{dt} &= \left(\frac{z_1-\tilde{z}_1}{z_1}\right)\left[z_1\left(1-z_1-z_2-u_1z_2-z_3-z_4-u_2\right)\right]\\ &+\left(\frac{z_2-\tilde{z}_2}{z_2}\right)\left[z_2\left(u_1z_1-u_3z_3-u_4z_4-\left(u_5+u_6\right)\right)\right]\\ &+\left(\frac{z_3-\tilde{z}_3}{z_3}\right)\left[z_3\left(u_7z_1-u_8z_2-u_9z_4-\left(u_{10}+u_{11}\right)\right)\right]\\ &+\left[u_{12}z_1z_4+u_{13}z_2z_4+u_{14}z_1\ z_3\ z_4+u_{15}z_2\ z_3\ z_4\ +\left(u_{16}-u_{17}\right)z_3\ z_4-\left(u_{18}+u_{19}\right)z_4\right]\end{aligned}$$

Consequently, due to conditions (4.15a)-(4.15b), *we have*

$$\begin{aligned}\frac{dV_5}{dt} &\le -(z_1-\tilde{z}_1)^2-\left\{\left(1-u_7\right)z_3-\tilde{z}_2-\left(1-u_7\right)\tilde{z}_3\right\}z_1-\left\{\left(u_3-u_8\right)z_3-\tilde{z}_1-\left(u_3-u_8\right)\tilde{z}_3\right\}z_2-z_1z_2\\ &-\left\{u_9+u_{17}-\left(u_{16}+u_{14}+\frac{u_{15}}{L}\right)\right\}z_3z_4\\ \frac{dV_5}{dt} &\le -(z_1-\tilde{z}_1)^2-\left\{u_1\tilde{z}_2-\left(1+u_1\right)\tilde{z}_2-\tilde{z}_3+u_7\tilde{z}_3\right\}z_1-\left\{u_8\tilde{z}_3-\left(1+u_1\right)\tilde{z}_1+u_1\tilde{z}_1-u_3\tilde{z}_3\right\}z_2\\ &-\left\{u_7\tilde{z}_1-\tilde{z}_1-u_3\tilde{z}_2+u_8\tilde{z}_2\right\}z_3-\left\{u_{18}+u_{19}-\tilde{z}_1-u_4\tilde{z}_2-u_9\tilde{z}_3\right\}z_4\\ &-\left\{u_9+u_{17}-(u_{16}+u_{14}+\frac{u_{15}}{L})\right\}z_3z_4\end{aligned}$$

Now it is easy to verify that $\frac{dV_5}{dt}$ *is negative definite.*
Hence the solution of system (2.3) *will approach asymptotically to* E_5 *from every initial point satisfies the above condition and the proof is complete.* □

Theorem 4.7. *Assume that* E_6 *is locally asymptotically stable in* R^4_+ *, then it is globally asymptotically stable in the sub region of* R^4_+ *that satisfies the following conditions:*

$$z_3 > max\left\{\frac{\bar{\bar{z}}_2+\left(1-u_{12}\right)\bar{\bar{z}}_4}{\left(1+u_{14}-u_7\right)},\frac{\bar{\bar{z}}_1+\left(u_4-u_{13}\right)\bar{\bar{z}}_4}{\left(u_3+u_5\bar{\bar{z}}_4-u_8\right)}\right\} \tag{4.16a}$$

$$z_2 > \frac{\left(1-u_{12}\right)\bar{\bar{z}}_1+\left(u_4-u_{13}\right)\bar{\bar{z}}_2}{u_4-u_{13}} \tag{4.16b}$$

$$u_3+u_{10}+u_{11} > \bar{\bar{z}}_1 \tag{4.16c}$$

Proof . *considering the positive following definite being a faction*

$$V_6\left(z_1,z_2,z_3,z_4\right)=\left(z_1-\bar{\bar{z}}_1-\bar{\bar{z}}_1\ln\frac{z_1}{\bar{\bar{z}}_1}+\right)+\left(z_2-\bar{\bar{z}}_2-\bar{\bar{z}}_2\ln\frac{z_2}{\bar{\bar{z}}_2}\right)+z_3+\left(z_4-\bar{\bar{z}}_4-\bar{\bar{z}}_4\ln\frac{z_4}{\bar{\bar{z}}_4}\right)$$

Obviously, $V_6 : R^4_+ \to R$ *is a continuously differentiable function such that* $V_6\left(\bar{\bar{z}}_1,\ \bar{\bar{z}}_2,0,\bar{\bar{z}}_4\right)=0$ *while*

$$V_6\left(z_1,\ z_2,\ z_3,\ z_4\right)>0,\ \forall\ \left(z_1,\ z_2,\ z_3,\ z_4\right)\in R^4_+\ and\ \left(z_1,\ z_2,\ z_3,\ z_4\right)\ne\left(\bar{\bar{z}}_1,\bar{\bar{z}}_2,0,\bar{\bar{z}}_4\right)$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms we *get that*

$$\frac{dV_6}{dt} = \left(\frac{z_1 - \bar{\bar{z}}_1}{z_1}\right)\left[z_1\left(1 - z_1 - (1 + u_1)\, z_2 - z_3 - z_4 - u_2\right)\right]$$
$$+ \left(\frac{z_2 - \bar{\bar{z}}_2}{z_2}\right)\left[z_2\left(u_1 z_1 - u_3 z_3 - u_4 z_4 - (u_5 + u_6)\right)\right] + \left[u_7 z_1\ z_3 + u_8 z_2\ z_3 - u_9 z_3\ z_4 - (u_{10} + u_{11})\, z_3\right]$$
$$+ \left(\frac{z_4 - \bar{\bar{z}}_4}{z_4}\right)\left[z_4(u_{12} z_1 + u_{13} z_2 + u_{14} z_1 z_3 + u_{15} z_2 z_3 + (u_{16} - u_{17})\, z_3 - (u_{18} + u_{19})\right]$$

Therefore, by using the conditions (4.16a)-(4.16c) *the derivative* $\frac{dV_6}{dt}$ *becomes*

$$\frac{dV_6}{dt} \leq -(z_1 - \bar{\bar{z}}_1)^2 - \{u_1\bar{\bar{z}}_2 - (1 + u_1)\,\bar{\bar{z}}_2 - \bar{\bar{z}}_4 + u_{124}\}z_1 - \{u_1\bar{\bar{z}}_1 - (1 + u_1)\,\bar{\bar{z}}_1 - u_4\bar{\bar{z}}_4 + u_{13}\bar{\bar{z}}_4\}\, z_2$$
$$- \{u_{10} + u_{11} - \bar{\bar{z}}_1 - u_3\bar{\bar{z}}_2 + u_{16}\bar{\bar{z}}_4 - u_{17}\bar{\bar{z}}_4\}\, z_3 - \{u_{12}\bar{\bar{z}}_1 + u_{13}\bar{\bar{z}}_2 - u_4\bar{\bar{z}}_2 - \bar{\bar{z}}_1\}\, z_4$$
$$- \left\{u_9 + u_{17} - \left(u_{16} + u_{14} + \frac{u_{15}}{L}\right)\right\} z_3 z_4$$

Obviously $\frac{dV_6}{dt}$ *is negative definite and hence* V_6 *is a Lyapunov function with respect to* E_6 *in the sub region in* R^4_+. *So* E_6 *is a globally asymptotically stable.* □

Theorem 4.8. *Suppose the point of equilibrium* E_7 *is asymptotically locally .so, it is asymptotically globally stable in the sub region of* R^4_+ *if the following conditions hold:*

$$z_2 > (u_7 - 1)\,\tilde{\tilde{z}}_3 + (u_{12} - 1)\,\tilde{\tilde{z}}_4 \tag{4.17a}$$
$$u_5 + u_6 + u_8\tilde{\tilde{z}}_3 + u_{13}\tilde{\tilde{z}}_4 > \tilde{\tilde{z}}_1 + u_1\tilde{\tilde{z}}_1 \tag{4.17b}$$
$$u_7\tilde{\tilde{z}}_1 + u_{16}\tilde{\tilde{z}}_4 > \tilde{\tilde{z}}_1 + u_9\tilde{\tilde{z}}_4 + u_{17}\tilde{\tilde{z}}_4 \tag{4.17c}$$
$$u_{12}\tilde{\tilde{z}}_1 + u_{14}\tilde{\tilde{z}}_1\tilde{\tilde{z}}_3 + u_{16}\tilde{\tilde{z}}_3 > \tilde{\tilde{z}}_1 + u_9\tilde{\tilde{z}}_3 + u_{17}\tilde{\tilde{z}}_3 \tag{4.17d}$$

Proof . *considering the positive following definite being a faction*

$$V_7\,(z_1,\ z_2,\ z_3,\ z_4) = \left(z_1 - \tilde{\tilde{z}}_1 - \tilde{\tilde{z}}_1 \ln\frac{z_1}{\tilde{\tilde{z}}_1}\right) + z_2 + \left(z_3 - \tilde{\tilde{z}}_3 - \tilde{\tilde{z}}_3 \ln\frac{z_3}{\tilde{\tilde{z}}_3}\ \right) + \left(z_4 - \widehat{\widehat{z}}_4 - \tilde{\tilde{z}}_4 \ln\frac{z_4}{\tilde{\tilde{z}}_4}\ \right)$$

Obviously, $V_7 : R^4_+ \to R$ *is a continuously differentiable function such that* $V_7\left(\tilde{\tilde{z}}_1, 0, \tilde{\tilde{z}}_3, \tilde{\tilde{z}}_4\right) = 0$ *while*

$$V_7\,(z_1, z_2, z_3, z_4) > 0,\ \forall\ (z_1,\ z_2,\ z_3,\ z_4) \in R^4_+ \text{ and } (z_1,\ z_2,\ z_3,\ z_4) \neq \left(\tilde{\tilde{z}}_1, 0, \tilde{\tilde{z}}_3, \tilde{\tilde{z}}_4\right).$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms we get that

$$\frac{dV_7}{dt} = \left(\frac{z_1 - \tilde{\tilde{z}}_1}{z_1}\right)\left[z_1\left(1 - z_1 - (1 + u_1)\, z_2 - z_3 - z_4 - u_2\right)\right]\ + u_1 z_1 z_2 - u_3 z_2 z_3 - u_4 z_2 z_4 - (u_5 + u_6)\, z_2$$
$$+ \left(\frac{z_3 - \tilde{\tilde{z}}_3}{z_3}\right)\left[z_3\left(u_7 z_1 + u_8 z_2 - u_9 z_4 - (u_{10} + u_{11})\right)\right]$$
$$+ \left(\frac{z_4 - \tilde{\tilde{z}}_4}{z_4}\right)\left[z_4\left(u_{12} z_1 + u_{13} z_2 + u_{14} z_1 z_3 + u_{15} z_2 z_3 + (u_{16} - u_{17})\, z_3 - (u_{18} + u_{19})\right)\right]$$

Consequently, due to conditions (4.17a)-(4.17d)*, we have*

$$\frac{dV_7}{dt} \le -(z_1-\tilde{\tilde{z}}_1)^2 - \left\{z_2 - \left\{(u_7-1)\,\tilde{\tilde{z}}_3 + (u_{12}-1)\,\tilde{\tilde{z}}_4\right\}\right\} z_1$$
$$- \left\{(u_5+u_6) - \tilde{\tilde{z}}_1 - u_1\tilde{\tilde{z}}_1 + u_8\tilde{\tilde{z}}_3 + u_{13}\tilde{\tilde{z}}_4\right\} z_2$$
$$- \left\{u_7\tilde{\tilde{z}}_1 - \tilde{\tilde{z}}_1 - u_9\tilde{\tilde{z}}_4 + u_{16}\tilde{\tilde{z}}_4 - u_{17}\tilde{\tilde{z}}_n\right\} z_3 - \left\{u_{12}\tilde{\tilde{z}}_1 - \tilde{\tilde{z}}_1 - u_9\tilde{\tilde{z}}_3 + u_{14}\tilde{\tilde{z}}_1\tilde{\tilde{z}}_3 + u_{16}\tilde{\tilde{z}}_3 - u_{17}\tilde{\tilde{z}}_3\right\} z_4$$
$$- \left\{u_9 + u_{17} - \left(u_{16} + u_{14} + \frac{u_{15}}{L}\right)\right\} z_3 z_4$$

Obviously $\frac{dV_7}{dt}$ *is negative definite and hence* V_7 *is a Lyapunov function with respect to* E_7 *in the sub region in* R_+^4*. So* E_7 *is a globally asymptotically stable.* □

Theorem 4.9. *Suppose the point of equilibrium* E_8 *is asymptotically locally .so, it is asymptotically globally stable in the sub region of* R_+^4 *if the following conditions hold:*

$$z_i > \overset{\circ}{z}_i, \qquad i = 1,2,3,4 \tag{4.18a}$$
$$z_3 < min\left\{\frac{1-u_{12}}{u_{14}}, \frac{u_4-u_{13}}{u_{15}}, \frac{u_9+u_{17}-u_{16}-u_{15}\overset{\circ}{z}_2}{u_{15}}\right\} \tag{4.18b}$$

Proof . *considering the positive following definite being a faction*
$V_8\,(z_1,z_2,z_3,z_4) = \left(z_1 - \overset{\circ}{z}_1 - \overset{\circ}{z}_1 \ln\frac{z_1}{\overset{\circ}{z}_1}\right) + \left(z_1 - \overset{\circ}{z}_2 - \overset{\circ}{z}_2 \ln\frac{z_2}{\overset{\circ}{z}_2}\right) + \left(z_3 - \overset{\circ}{z}_3 - \overset{\circ}{z}_3 \ln\frac{z_3}{\overset{\circ}{z}_3}\right) + \left(z_4 - \overset{\circ}{z}_4 - \overset{\circ}{z}_4 \ln\frac{z_4}{\overset{\circ}{z}_4}\right)$

Obviously, $V_8 : R_+^4 \to R$ *is a continuously differentiable function such that* $V_8\left(\overset{\circ}{z}_1, \overset{\circ}{z}_2, \overset{\circ}{z}_3, \overset{\circ}{z}_4\right) = 0$ *while* $V_8\,(z_1,z_2,z_3,z_4) > 0$ *for all* $(z_1,z_2,z_3,z_4) \in R_1^4$ *and* $(z_1,z_2,z_3,z_4) \neq \left(\overset{\circ}{z}_1, \overset{\circ}{z}_2, \overset{\circ}{z}_3, \overset{\circ}{z}_4\right)$*. Furthermore by taking the derivative with respect to the and simplifying the reselling terms, we yet that*

$$\frac{dV_8}{dt} = \left(\frac{z_1-\overset{\circ}{z}_1}{z_1}\right)\left[z_1\,(1-z_1-z_2-u_1z_2-z_3-z_4-u_2)\right]$$
$$+ \left(\frac{z_2-\overset{\circ}{z}_2}{z_2}\right)\left[z_2\,(u_1z_1-u_3z_3-u_4z_4-(u_5+u_6))\right]$$
$$+ \left(\frac{z_3-\overset{\circ}{z}_3}{z_3}\right)\left[z_3\,(u_7z_1+u_8z_2-u_9z_4-(u_{10}+u_{11}))\right]$$
$$+ \left(\frac{z_4-\overset{\circ}{z}_4}{z_4}\right)\left[z_4\,(u_{12}z_1+u_{13}z_2+u_{14}z_1z_3+u_{15}z_2z_3+(u_{16}-u_{17})\,z_3-(u_{18}+u_{19})\right]$$

Now, by using the given condition we obtains that

$$\frac{dV_8}{dt} \le -(z_1-\overset{\circ}{z}_1)^2 - \left(z_1-\overset{\circ}{z}_1\right)\left(z_2-\overset{\circ}{z}_2\right) - (1-u_7)\left(z_1-\overset{\circ}{z}_1\right)\left(z_3-\overset{\circ}{z}_3\right)$$
$$- (1-u_{12}-u_{14}z_3)\left(z_1-\overset{\circ}{z}_1\right)\left(z_4-\overset{\circ}{z}_4\right) - (u_3-u_8)\left(z_2-\overset{\circ}{z}_2\right)\left(z_3-\overset{\circ}{z}_3\right)$$
$$- (u_4-u_{13}-u_{15}z_3)\left(z_2-\overset{\circ}{z}_2\right)\left(z_4-\overset{\circ}{z}_4\right) - \left(u_9-u_{16}+u_{17}-u_{15}\overset{\circ}{z}_2-u_{15}z_3\right)\left(z_3-\overset{\circ}{z}_3\right)\left(z_4-\overset{\circ}{z}_4\right)$$

Obviously $\frac{dV_8}{dt}$ *is negative definite and hence* V_8 *is a Lyapunov function with respect to* E_8 *in the sub region in* R_+^4*. So* E_8 *is a globally asymptotically stable.* □

5. Numerical Simulation

In this section, the global dynamics of system (2.3) is investigated numerically for different sets of initial values and different sets of parameters values. The objectives of such investigation are determine the effect of varying the parameters values and confirm our obtained results. It is observed that, for the following biologically feasible set of hypothetical parameters values:

$$\begin{aligned} &u_1 = 0.1, \quad u_2 = 0.03, \quad u_3 = 0.07, \quad u_4 = 0.05, \quad u_5 = 0.01, \\ &u_6 = 0.01, \quad u_7 = 0.03, \quad u_8 = 0.08\ , \quad u_9 = 0.04, \quad u_{10} = 0.03, \\ &u_{11} = 0.01, \quad u_{12} = 0.02, \quad u_{13} = 0.05, \quad u_{14} = 0.02, \quad u_{15} = 0.06, \\ &u_{16} = 0.03, \quad u_{17} = 0.01, \quad u_{18} = 0.02, \quad u_{19} = 0.01 \end{aligned} \tag{5.1}$$

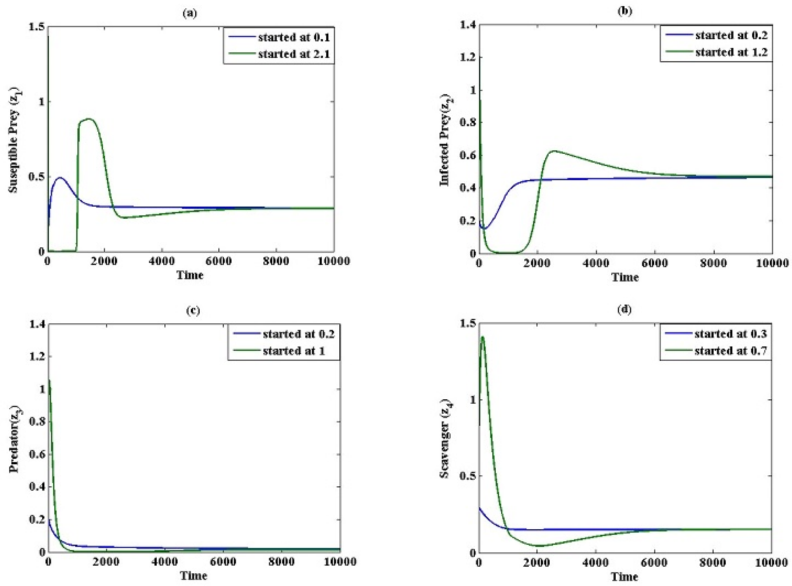

Figure 1: Time series of the trajectory of system (2.3) for the data (5.1). (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

Obviously, Figure 1. shows the existence of a globally asymptotically stable positive equilibrium point $E_8 = (0.289, 0.462, 0.019, 0.152)$ for system (2.3).

However, for the data given by Eq.(5.1) with varying the parameter u_2 in the range $u_2 > 1$,then the trajectory of system (2.3), starting from different sets of initial data, is approaching asymptotically to the vanishing equilibrium point $E_0 = (0, 0,\ 0,\ 0)$ as shown in the typical figure represented by Figure 2.

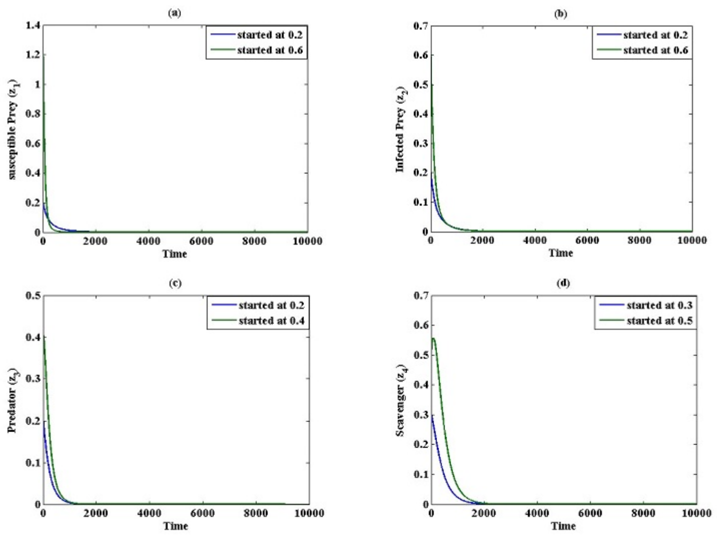

Figure 2: Time series of the trajectory of system (2.3) for the data (5.1) for $u_2 = 1.5$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

It is easy to verify that for the data , we have, and the solution approaches to $E_0 = (0, 0,\ 0,\ 0)$.Now in order to investigate the effect of verifying one parameter value at a time on the dynamical behavior of system (2.3) the following result are observed. According to the Figure 2. it is clear that the solution of system(2.3) approaches asymptotically to the prey free equilibrium point.
Moreover, for the parameters values given in Eq.(5.1) with $u_1 = 0.01$ the solution of system(2.3) approaches asymptotically to $E_1 = (0.9, 0, 0, 0)$ as shown in the typical figure that given by Figure 3.

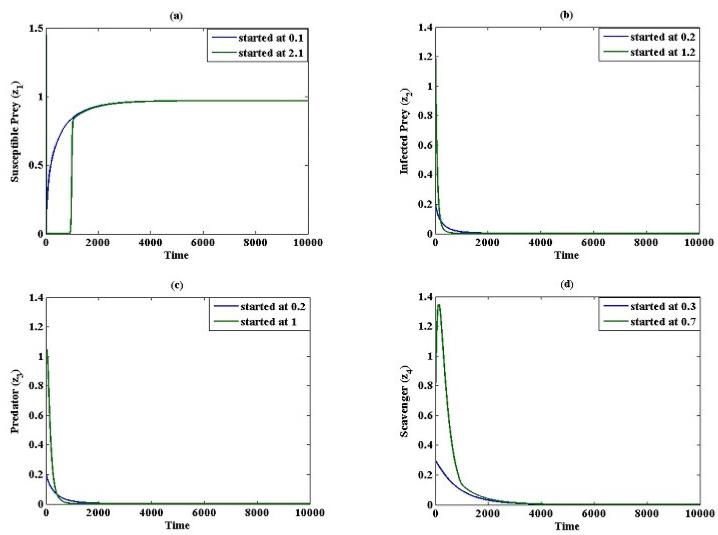

Figure 3: Time series of the trajectory of system(2.3) for the data(5.1) for $u_1 = 0.01$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Figure 3., it's clear that the solution of system(2.3) approaches asymptotically to the prey free equilibrium point.
Moreover, for the parameters values given in Eq.(5.1) with $u_8 = 0.008$ *and* $u_{13} = 0.005$ the solution of system(2.3) approaches asymptotically to the first two species $E_2 = (0.2,\ 0.7,\ 0,\ 0)$ in the typical figure that given by Figure 4.

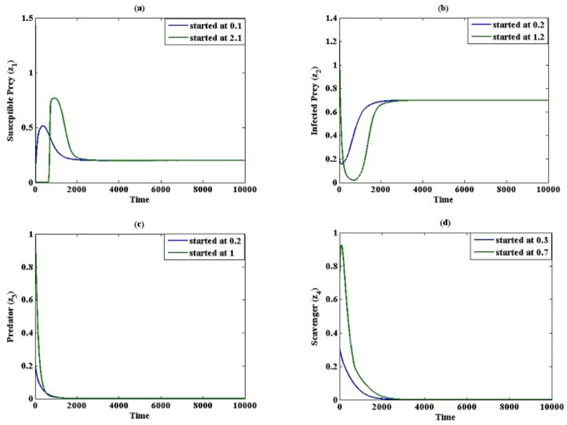

Figure 4: Time series of the trajectory of system (2.3) for the data (5.1) for $u_8 = 0.008$ and $u_{13} = 0.005$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Figure 4, it's clear that the solution of system(2.3) approaches asymptotically to the first two species equilibrium point.
Moreover, for the parameters values given in Eq.(5.1) with $u_1 = 0.01$ *and* $u_7 = 0.5$ the solution of system(2.3) approaches asymptotically to the second two species $E_3 = (\ 0.080,\ 0,\ 0.889,\ 0)$ as shown in the typical figure that given by Figure 5.

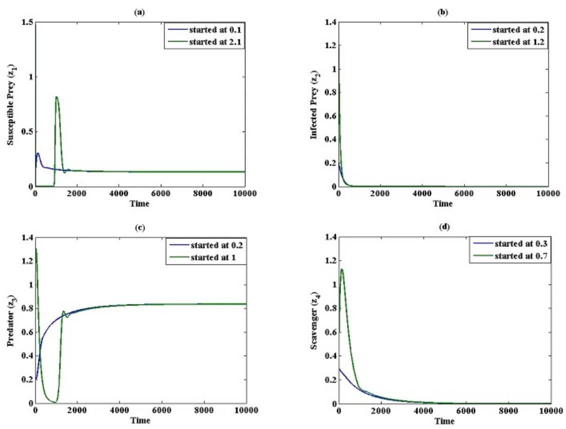

Figure 5: Time series of the trajectory of system(2.3) for the data(5.1) for $u_1 = 0.01$ and $u_7 = 0.5$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Figure 5., it's clear that the solution of system(2.3) approaches asymptotically to the E_3.

Moreover, for the parameters values given in Eq.(5.1) with $u_1 = 0.01$ *and* $u_{12} = 0.2$ the solution of system(2.3) approaches asymptotically to third two species $E_4 = (\ 0.15,\ 0,\ 0,\ 0.82)$ as shown in the typical figure that given by Figure 6.

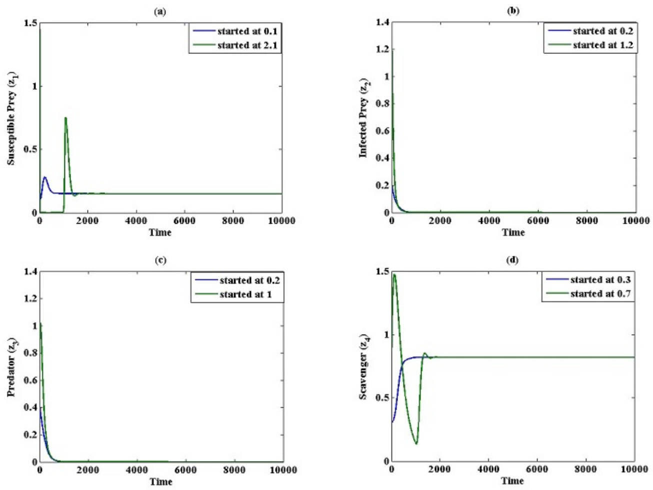

Figure 6: Time series of the trajectory of system(2.3) for the data(5.1) for $u_1 = 0.01$ and $u_{12} = 0.2$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Figure 6., it's clear that the solution of system(2.3) approaches asymptotically to the E_4.

Moreover, for the parameters values given in Eq.(5.1) with $u_{13} = 0.005$ the solution of system(2.3) approaches asymptotically to the first three species $E_5 = (0.350,\ 0.368,\ 0.214,\ 0)$ as shown in the typical figure that given by Figure 7.

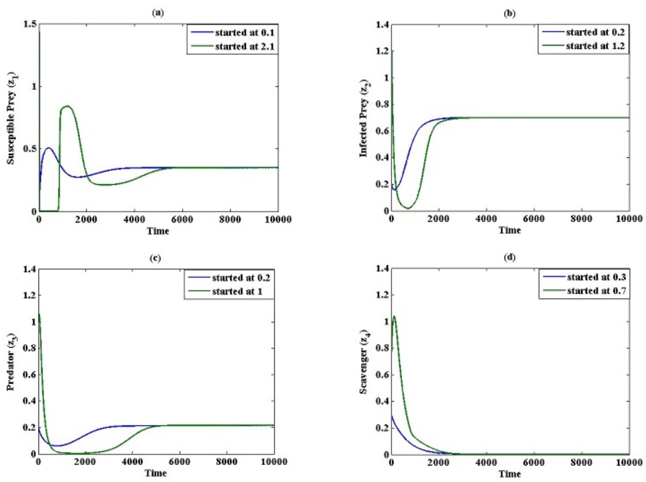

Figure 7: Time series of the trajectory of system(2.3) for the data(5.1) for $u_{13} = 0.005$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Fig .(7), it's clear that the solution of system(2.3) approaches asymptotically to the E_5.
Moreover, for the parameters values given in Eq.(5.1) with $u_7 = 0.003$ the solution of system (2.3) approaches asymptotically to the second three species $E_6 = (0.277,\ 0.489,\ 0,\ 0.154)$ as show in the typical figure that given by Figure 8.

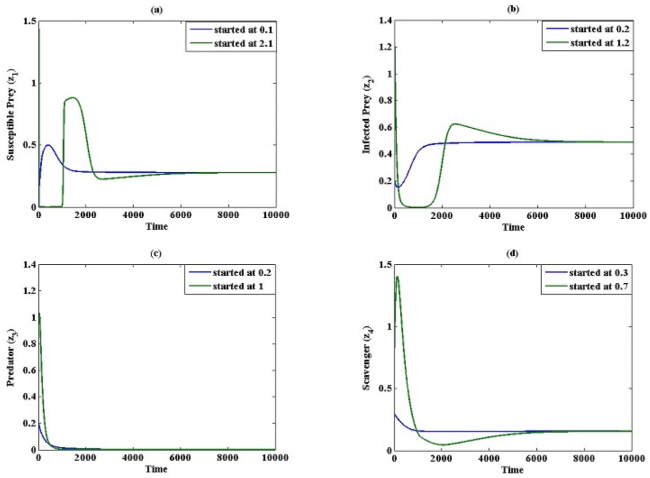

Figure 8: Time series of the trajectory of system (2.3) for the data (5.1) for $u_7 = 0.003$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

According to the Figure 8., it's clear that the solution of system(2.3) approaches asymptotically to the E_6.
Moreover, for the parameters values given in Eq.(5.1) with $u_1 = 0.01, u_7 = 0.3,\ and\ u_{14} = 0.3$ the solution of system(2.3) approaches asymptotically to the third three species equilibrium point $E_7 = (0.192,\ 0,\ 0.336,\ 0.440)$ as shown in the typical figure that given by Figure 9.

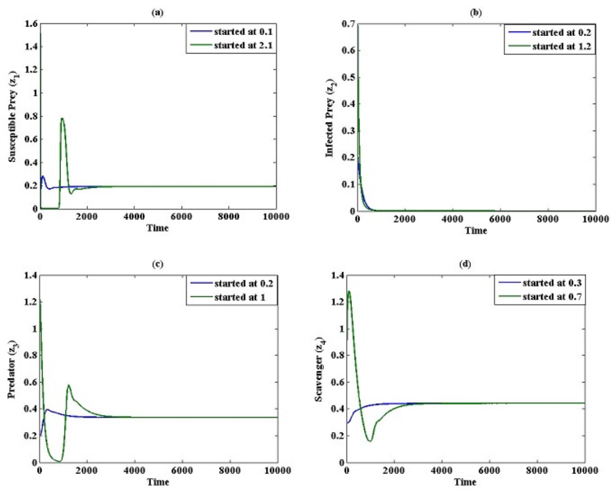

Figure 9: Time series of the trajectory of system(2.3) for the data (5.1) for $u_1 = 0.01$, $u_7 = 0.3$, $u_{14} = 0.3$ (a) Trajectories of susceptible prey (b) Trajectories of infected prey (c) Trajectories of predator, (d) Trajectories of scavenger.

6. Conclusions and Discussion

In this paper, we proposed and analyzed an Eco-epidemiological model that described the Prey, predator and scavenger with disease in prey. The model included four non-linear autonomous ordinary differential equations that describe the dynamics of four different species, namely Susceptible prey (X_1), Infected prey(X_2), Predator (Y_1) and (Y_2) which is represent the scavenger. The boundedness of system (2.3) has been discussed. The existence conditions of all possible equilibrium points are obtained. The local as well as global stability analyses of these points are carried out. Finally, numerical simulation is used to specific the control set of parameters that affect the dynamics of the system and confirm our obtained analytical results. Therefore system (2.3) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothetical set of data given by Eq. (5.1), and the following observations are obtained.

1. system (2.3) do not has periodic dynamic, instead of that the solution of system (2.3) approaches asymptotically to one of its equilibrium point.
2. As the value u_2 increasing and keeping the rest of parameters as in eq.(5.1) the solution of system (2.3) approaches asymptotically to the vanishing equilibrium point E_0.
3. Decreasing the value of, u_1 below the value 0.01 in eq.(5.1) caused destabilizing to the positive equilibrium point E_8 and the trajectories of system (2.3) approached asymptotically to the prey free equilibrium point E_1.
4. It is observed that, in case of Decreasing the values u_8 and u_{13} the positive equilibrium point E_8 becomes unstable and the trajectory of system (2.3) approaches asymptotically to the first two species E_2.
5. If we take $u_1 = 0.01$ and $u_7 = 0.5$ and keeping all the value in eq.(5.1),the positive equilibrium point E_8 becames unstable and the trajectory of system (2.3) approaches asymptotically to the second two species E_3.
6. If we choose the values $u_1 = 0.01$ and $u_{12} = 0.2$ respectively, keeping other parameters fixed as given in eq.(5.1) the positive equilibrium point E_8 will be unstable and the solution of system (2.3) approaches asymptotically to the third two species E_4.
7. Decreasing the value of, u_{13} , below the value 0.05 in eq.(5.1) caused destabilizing to the positive equilibrium point E_8 and the trajectories of system (2.3) approached asymptotically to the prey free equilibrium point E_5.
8. Decreasing the value of, u_7 , below the value 0.03 in eq.(5.1) caused destabilizing to the positive equilibrium point E_8 and the trajactories of system (2.3) approached asymptotically to the prey free equilibrium point E_6.
9. If we take $u_1 = 0.01$, $u_7 = 0.3$ and $u_{14} = 0.3$ and keeping all the value in eq.(5.1),the positive equilibrium point E_8 becames unstable and the trajectory of system (2.3) approaches asymptotically to the second two species E_7.

References

[1] M. M. Abdulkadhim and H. F. Al-Husseiny, *The stability analysis of the diseased predator-prey model incorporating migration in the contaminated environment*, J. Phys. Conf. Ser. 2021 pp.1–23.

[2] H Ali and A. N. Mustafa, *Analysis of a mathematical model in a food web system containing scavenger species*, J. Univ. Babylon Pure Appl. Sci. 27 (2019) 1–15.

[3] R.M. Anderson and R.M. May, *Regulation and stability of host-parasite population interactions: I. Regulatory processes*, J. Animal Eco. 47 (1978) 219–247.

[4] C. Arancibia-Ibarra, *The basins of attraction in a modified may-holling-tanner predator-prey model with allee effect*, Nonlinear Analysis, 185 (2019) 15-28.

[5] S. Gakkhar and R.K. Naji., *On a food web consisting of a specialist and a generalist predator*, J. Bio. Syst. 11 (2003) 365–376.

[6] R.P. Gupta and P. Chandra, *Dynamical properties of a prey-predator-scavenger model with quadratic harvesting*, Commun. Nonlinear Sci. Numerical Sim. 49 (2017) 202–214.

[7] K.P. Hadeler and H. I. Frerdman, *Predator-prey populations with parasitic infection*, J. Math. Biol. 27 (1989) 609-631.

[8] M. Haque and E. Venturino, *The effect of communicable disease in Leslie-Gower predator-prey model*, J. Bio. Syst. 163 (2008) 425–444.

[9] M. Haque, J. Zhen and E. Venturino, *Rich dynamics of Lotka-Volterra type predator prey model system with viral disease in prey species*, Math. Meth. Appl. Sci. 32 (2009) 875–898.

[10] M. Haque, *A predator-prey model with disease in the predator species only*, Nonlinear Anal. Real World Appl. 11 (2010) 2224–2236.

[11] M. Haque, S. Sarwardi, S. Preston and E. Venturino, *Effect of delay in a Lotka Volterra type predator-prey model with a transmissible disease in the predator species*, Math. Biosci. 234 (2011) 47-57.

[12] E.J. Jansen and A.R. Van Gorder, *Dynamics from a predator-prey-quarry-resource-scavenger model*, Theo. Eco. J. 11 (2018) 19–38.

[13] Y. Kang, *A two-patch prey-predator model with predator dispersal driven by the predation strength*, Math. Biosci. 14 (2017) 843–880.

[14] S. Kant, V. Kumar, *Stability analysis of predator-prey system with migrating prey and disease infection in both species*, Appl. Math. Model. 42 (2017) 509–539.

[15] S. Kumar and H. Kharbanda, *Stability analysis of prey-predator model with infection, migration and vaccination in prey*, arXiv preprint, arXiv:1709.10319, (2017) 1–27.

[16] W.O. Kermack and A.G. McKendrick, *Contributions to the mathematical theory of epidemics, part i.*, Proc. Royal Soc. Edinburgh Section A. Math. 115 (1927) 700-721.

[17] A. A. Mohsen and I. A. Aaid, *Stability of a prey-predator model with SIS epidemic disease in predator involving holling type II function response*, IOSR J. Math. 11 (2015) 38-53.

[18] R. K. Naji and H. F. Ridha, *The dynamics of a prey-predator model incorporating svis-type of disease in prey*, J. Math. 12 (2016) 90-101.

[19] R.K. Naji and A. N. Mustafa, *The dynamics of an eco-epidemiological model with nonlinear incidence rate*, J. Appl. Math. 2012 (2012) 1–25.

[20] P. Panja, *Prey-predator-scavenger model with Monod-Haldane type functional response*, Rend. del Circ. Mate. Palermo Ser. 2, 2 (2020) 1205-1219.

[21] C. V. Pavan Kumar , K. Shiva Reddy and M. A. S. Srinivas, *Dynamics of prey predator with Holling interactions and stochastic influences*, 57 (2018) 1079-1086.

[22] H. A. Satar and R. K. Naji, *Stability and bifurcation of a prey-predator-scavenger model in the existence of toxicant*, Int. J. Math. Math. Sci. 2019 (2019) 1–17.

[23] S. Slimani, P. Raynaud and I. Boussaada, *Dynamics of a prey-predator system with modified Leslie-Gower and Holling type II schemes incorporating a prey refuge*, Amer. Inst. Math. Sci. 24 (2019) 5003–5039.

[24] A. Suryanto and I. Darti, *Stability analysis and nonstandard grunwald-letnikov scheme for a fractional order predator-prey model with ratio-dependent functional response*, AIP Conference Proceedings, 2017, pp. 1–6.

[25] Y. Xiao and L. Chen, *A ratio-dependent preydator-prey model with disease in the prey*, Appl. Math. Comput. 131 (2002) 397–414.