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A qualitative study of an eco-toxicant model with anti-predator behavior

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Abstract

In this study, a mathematical model consisting of four species: first prey and second prey with stage structure and predator in the presence of toxicity and anti-predator has been proposed and studied using the functional response Holling's type IV and Lotka Volttra. The solution's existence, uniqueness, and boundedness have all been studied. All possible equilibrium points have been identified. The stability of this model has been studied. Finally, numerical simulations have been used to verify our analytical results.

Keywords: Prey-Predator, Function Response, Stage-Structure, Toxicity, Anti-Predator Behavior.

1. Introduction

In population dynamics, a mathematical model that used understand certain occurrences predation interactions are represented mathematically by interactions between predatory and prey animal species living in the same environment. The prey predator model featured prey density-dependent growth and functional responses.
When a population biologist starts evaluating a population of organisms, they employ a variety of tools to collect data. Experiments and observations are used to build mathematical formulas and models, which are then utilized to make forecasts. Essentially, the researchers must consider aspects that influence the population.
Some types of prey have already been studied that are capable of fighting predators, whether with chemicals, through community defines, or by excreting harmful substances. Many animals can escape by fleeing quickly, defeating or outnumbering their attacker. Some species are able to escape even

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when they catch them by sacrificing certain body parts: crabs can get rid of their paws, while lizards can shed their tails. Predators are often distracted long enough to allow prey to escape.

The anti-predator behaviour always influences more than one predator and also make the predator's competition become more complex [15].A response based on the density of prey only was considered. In (1989), Arditi and Ginzburg [1] propose a ratio – dependent function response which is a particular type of a predator dependence. Banerjee [3] constructed a prey and predator model, and there are some ratio-dependent mathematical models [16] .There are very few mathematical model [17] in which anti-predator behaviours have been.

The anti-predator behaviour property of first prey population has been introduced in our proposed mathematical model. Here, Holling's type IV functional response has been used on the basis of ratio-dependency of prey and predator [17] [2].

One of the most important problems that face the dynamic of the ecosystem is the effects of toxic substances. Defining a toxic substance as any human toxic substance released into an environment through human activities, for example, are the rodents in poultry farms, causing the presence of rodents in poultry breeding facilities. Large economic losses, so the farmer uses toxic pesticides for rodents and carefully follows the instructions for use.

It is necessary to assess the risk to living organisms exposed to toxins and to find relevant factors that determine the persistence of the organisms. Hallam and Deluna [8] discussed the effects of the toxin through a population food chain. Hallam and Clark [9] studied the effect of a toxic substance on populations, while Friedman and Shukla [6] developed a prey and predator pattern based on the toxicity of one species. Chattopadhyay [4] studied the effect of toxic substances on two competing species. And montoya et al [14] two types of factors were considered, such as (anti-predator behaviour and collective defines in the stage structure model), some researchers in mathematics have looked at prey and predator models in the effect of toxicity [10], [11], [12].

In recent years, many prey and predator models have been studied on the basis of the age structure [5], [13]. In many cases, the lifestyle of different species passes through two stages of life (mature and immature), the fully immature prey depends on its feeding on the mature prey in order to describe the interactions.

In this paper, mathematical model of four species with stage- structure and anti-predator behaviour have been proposed to study.

2. Model formulation

In this section, an ecological model consists of four species have been proposed : the first prey and second prey which have a stage- structure with only one predator , which are denoted to their populations sizes at time $E_1(t),\ E_2(t),\ E_3(t)\ and\ E_4(t)$, respectively .

1. The first prey grows logistically with intrinsic growth rate $S_1 > 0,$ and carrying capacity $L_1 > 0$, the second immature prey arose to mature with a growth rate $D > 0$, respectively and immature prey depends entirely on its feeding on mature prey that grows logistically with intrinsic growth rate $S_2 > 0$, and carrying capacity $L_2 > 0,$ in the absence of predator.
2. The predator also consumes first prey according to the response of the Holling type- IV with maximum attack rates $C_1 > 0$, and measure the extent to which the environment provides protection to the prey and predator $m > 0$, a portion of this food contributes to a conversion rate$A_1 > 0$, with a normal mortality rate $k_3 > 0$, the predator faces death when deprived of food, at the same time, mature and immature prey are consumed, depending on the predator. to the response of the Lotka Volttra functional with consumption rates on the $C_i > 0, \quad i = 2, 3,$

respectively, a portion of this food contributes to a conversion rates $A_i > 0, \quad i = 2,3$, it is referred to as a natural death for mature and immature prey is denoted by $k_i > 0, \quad i = 1,2$.

3. It assumes that the function response to the predation of the first prey is taken as Holling type-IV response function because it describes a group defense phenomenon where it represents $n > 0$, the rate of anti- predator behavior of first prey to predator.
4. Finally, $\alpha_i > 0, \quad i = 1,2,3$, the toxicity represents the mature and second immature prey and predator respectively.

$$
\begin{aligned}
\frac{dE_1}{dt} &= S_1E_1\left(1-\frac{E_1}{L_1}\right)-\frac{C_1E_1E_4}{m+E_1^2},\\
\frac{dE_2}{dt} &= S_2E_3\left(1-\frac{E_3}{L_2}\right)-DE_2-C_2E_2E_4-\alpha_1{E_2}^2-K_1E_2,\\
\frac{dE_3}{dt} &= DE_2-C_2E_3E_4-\alpha_2E_3^2-K_2E_3,\\
\frac{dE_4}{dt} &= \frac{A_1E_1E_4}{m+E_1^2}+A_2E_2E_4+A_3E_3E_4-nE_1E_4-\alpha_3E_4-K_3E_4.
\end{aligned}
\tag{2.1}
$$

With initial condition $E_1(0) \geq 0,\ E_2(0) \geq 0,\ E_3(0) \geq 0,\ E_4(0) \geq 0$. Therefore these functions are Lipschitzian on R_+^4, and therefore the solution of the system (2.1) exists and is unique.

Theorem 2.1. *All the solutions of system(2.1) with initial condition belonging to R_+^4 are uniformly bounded.*

Proof . *Let $E_1(t), E_2(t), E_3(t), E_4(t)$ be a solution of system(2.1) with an initial non-negative condition $(E_1(0), E_2(0), E_3(0), E_4(0)) \in R_+^4$.*
Now according to the first equation of system(2.1) we have:

$$
\frac{dE_1}{dT} \leq S_1E_1\left(1-\frac{E_1}{l_1}\right).
$$

By comparison theorem [7] for solving this differential inequality, we get:

$$
\lim_{n\to\infty}\sup E_1(t) \leq L_1.
$$

Now consider a function:

$$
N(t) = E_1(t)+E_2(t)+E_3(t)+E_4(t),
$$

then after take the function's time derivative along with the system (2.1) solution, we get:

$$
\begin{aligned}
\frac{dN}{dt} =& S_1E_1\left(1-\frac{E_1}{L_1}\right)+S_2E_3\left(1-\frac{E_3}{L_2}\right)-K_1E_2-K_2E_3-K_3E_4-(C_2-A_2)E_2E_4\\
&-(C_1-A_1)\frac{E_1E_4}{m+E_1^2}-(C_3-A_3)E_3E_4-\left(\alpha_1E_2^2+\alpha_2E_3^2+\ \alpha_3E_4+nE_1E_2\right).
\end{aligned}
$$

So according to the biological facts always $C_i > A_i$, $i = 1,2,3$, we get:

$$
\frac{dN}{dt} < 2S_1E_1+S_2E_3\left(1-\frac{E_3}{L_2}\right)-(S_1E_1+K_1E_2+K_2E_3+K_3E_4).
$$

Now since the function $f(E_3) = S_2E_3\left(1 - \frac{E_3}{L_2}\right)$ *in the second term represents a logistic function with respect to* E_3 *and hence it is bounded above by the constant* $\frac{S_2L_2}{4}$. *So,*

$$\frac{dN}{dt} \leq 2S_1L_1 + \frac{S_2L_2}{4} - (S_1 + K_1 + k_2 + k_3)\,N.$$
$$\frac{dN}{dt} + SN \leq 2S_1L_1 + \frac{S_2L_2}{4}\ , \qquad where \quad N = \min\{S_1, K_1, K_2, K_3\}\,.$$
$$\frac{dN}{dt} + SN \leq H, \qquad where \quad H = \left(2S_1L_1 + \frac{S_2L_2}{4}\right).$$

Again, by comparison theorem to solving this differential inequality for the initial value $N(0) = N_0$, *we get:*

$$N(t) \leq \frac{H}{S} + \left(N_0 - \frac{H}{S}\right)\ e^{-st},$$

Then, $\lim_{t\to\infty} \leq \dfrac{H}{S}$. *So,* $\quad 0 \leq N(t) \leq \frac{H}{S}\ , \quad \forall\ t > 0.$
So, all solutions of system (2.1) *are uniformly bounded.* □

3. Existence of Equilibrium Points

In this section, we see that model (2.1) has discussed all the points of equilibrium that can check the conditions of existence , as shown below:

1. The equilibrium point $Q_0 = (\ 0\ ,0\ ,0\ ,0\ )$, which known as trivial point always exists.
2. The equilibrium point $Q_1 = (L_1, 0, 0, 0)$ always exists, as the prey population grows to the carrying capacity in the absence of predator.
3. The free second prey equilibrium point $Q_2 = (\bar{E}_1, 0, 0, \bar{E}_4)$ exists uniquely in Int. R^4_+ of E_1E_4-plane if there is positive solution to the following of equations

$$S_1 - \frac{S_1E_1}{L_1} - \frac{C_1E_4}{(m + E_1^2)} = 0, \tag{3.1a}$$

$$\frac{A_1E_1}{m + E_1^2} - nE_1 - \alpha_3 - K_3 = 0, \tag{3.1b}$$

From equation (3.1a) we have,

$$E_4 = \frac{s_1\ (m + E_1^2)(L_1 - E_1)}{C_1L_1}, \tag{3.1c}$$

From equation (3.1b) we have,

$$f(x) = \beta_1E_1^3 + \beta_2E_1^2 + \beta_3E_1 + \beta_4 = 0, \tag{3.1d}$$

where: $\beta_1 = -n \quad < 0, \quad \beta_2 = -(\alpha_3 + k_3)\ < 0\ , \quad \beta_3 = A_1 - nm, \quad \beta_4 = -(\alpha_3 m + k_3 m)\ < 0,$

By discarte rule Eq.(3.1d) either has no positive root or it has two positive root, denoted by $Q_2 = \left(\bar{E}_1, 0, 0, \bar{E}_4\right)$ and $Q_3 = \left(\bar{E}_1'\ , 0, 0, \bar{E}_4'\ \right)$, depending on the following conditions:

$$L_1 > E_1, \tag{3.1e}$$
$$A_1 > nm \tag{3.1f}$$

4. The equilibrium point $Q_4 = (0, \widehat{E}_2, \widehat{E}_3, 0)$ exists uniquely in Int. R^4_+ of E_2E_3- plane if there is positive solution to the following equations:

$$S_2E_3\left(1-\frac{E_3}{L_2}\right) - DE_2 - \alpha_1 E_2^2 - K_1E_2 = 0 \tag{3.2a}$$
$$D(E_2) - \alpha_2 E_3^2 - K_2E_3 = 0 \tag{3.2b}$$

From equation (3.2b) we have

$$E_2 = \frac{(\alpha_2 E_3 + K_2)E_3}{D}, \tag{3.2c}$$

By substituting (3.2c) in (3.2a) and then simplifying the resulting term we obtain that:

$$f(x) = R_1E_3^3 + R_2E_3^2 + R_3E_3 + R_4 = 0, \tag{3.2d}$$

where

$$R_1 = -\alpha_1\alpha_2^2 L_2 < 0,$$
$$R_2 = -2\alpha_1\alpha_2 K_2 L_2 \ < 0,$$
$$R_3 = -(D^2S_2 + D^2L_2\alpha_2 + \alpha_1 L_2 K_2^2 + K_1DL_2\alpha_2) < 0,$$
$$R_4 = L_2D\left(D\left(S_2 - K_2\right) - K_2K_1\right).$$

By discarte rule Eq. (3.2d) unique positive root, namely $\widehat{E}_3$ provided that:

$$S_2 > K_2, \tag{3.2e}$$
$$D\left(S_2 - K_2\right) > K_2K_1. \tag{3.2f}$$

So, $Q_4 = (0, \widehat{E}_2, \widehat{E}_3, 0)$ where $\widehat{E}_2 = E_2(\widehat{E}_3)$ exists, provided that the above conditions hold.

5. The free predator equilibrium point $Q_5 = (\dot{E}_1, \dot{E}_2, \dot{E}_3, 0)$ exists uniquely in Int.R^4_+ of $E_1E_2E_3-$ *space* if there is positive solution to the following set of equations:

$$S_1 - \frac{S_1E_1}{L_1} = 0, \tag{3.3a}$$
$$S_2E_3\left(1-\frac{E_3}{L_2}\right) - DE_2 - \alpha_1 E_2^2 - K_1E_2 = 0, \tag{3.3b}$$
$$DE_2 - \alpha_2 E_3^2 - K_2E_3 = 0, \tag{3.3c}$$

From equation (3.3a) we have

$$\dot{E}_1 = L_1.$$

From equation (3.3c) we have

$$E_2 = \frac{(\alpha_2 E_3 + K_2)E_3}{D}, \tag{3.3d}$$

By substituting (3.3d) in (3.3b) and then simplifying the resulting term we obtain that:

$$f(x) = \delta_1 E_3^3 + \delta_2 E_3^2 + \delta_3 E_3 + \delta_4 = 0, \tag{3.3e}$$

where

$$\begin{aligned}
\delta_1 &= -\alpha_1 \alpha_2^2 L_2 < 0,\\
\delta_2 &= -2\alpha_1 \alpha_2 K_2 L_2 \ < 0,\\
\delta_3 &= -(D^2 S_2 + D^2 L_2 \alpha_2 + \alpha_1 L_2 K_2^2 + K_1 D L_2 \alpha_2) < 0,\\
\delta_4 &= L_2 D\left(D\left(S_2 - K_2\right) - K_2 K_1\right)
\end{aligned}$$

By discarte rule Eq. (3.3e) has unique positive root, namely $\dot{E}_3$, provided that (3.2e) and (3.2f) hold. So, $Q_5 = (\dot{E}_1, \dot{E}_2, \dot{E}_3, 0)$ where $\dot{E}_2 = E_2\left(\dot{E}_3\right)$, exists under the above conditions.

6. The free first prey equilibrium point $Q_6 = (0, \bar{\bar{E}}_2, \bar{\bar{E}}_3, \bar{\bar{E}}_4)$ exists uniquely in Int.R_+^4 of $E_2E_3E_4-$*plane* if there is positive solution to the following set of equations:

$$S_2 E_3\left(1 - \frac{E_3}{L_2}\right) - DE_2 - C_2 E_2 E_4 - \alpha_1 E_3^2 - K_1 E_2 = 0, \tag{3.4a}$$

$$DE_2 - C_3 E_3 E_4 - \alpha_2 E_3^2 - K_2 E_3 = 0, \tag{3.4b}$$

$$A_2 E_2 + A_3 E_3 - \alpha_3 - K_3 = 0, \tag{3.4c}$$

From equation (3.4c) we have

$$E_2 = \frac{\alpha_3 + K_3 - A_3 E_3}{A_2}, \tag{3.4d}$$

Now by substituting (3.4d) in (3.4b) we get

$$E_4 = \frac{\frac{D}{A_2}(\alpha_3 + K_3 - A_3 E_3) - (\alpha_2 E_3 + K_2) E_3}{C_3 E_3}, \tag{3.4e}$$

By substituting (3.4e) and (3.4d) in (3.4a) and then simplifying the resulting term we obtain that

$$f(x) = B_1 E_3^3 + B_2 E_3^2 + B_3 E_3 + B_4 = 0, \tag{3.4f}$$

where

$$\begin{aligned}
B_1 &= -(S_2 A_2^2 C_3 + \alpha_2 A_2 A_3 C_2 L_2 + \alpha_1 L_2 C_3 A_3^2) < 0,\\
B_2 &= L_2\left(C_2\left(\left(\alpha_3 + K_3\right)\alpha_2 A_2 - A_3\left(A_3 + A_2 K_2\right)\right) + C_3\left(A_2 A_3\left(D + K_1\right) + S_2 A_2^2 + 2(\alpha_3 + K_3)\alpha_1 A_3\right)\right)\\
B_3 &= L_2\left(\alpha_3 + K_3\right)\left(A_3 C_2\right)\left(2 + k_2\right) - C_3\left(A_2\left(D + k_1\right) + \alpha_1\left(\alpha_3 + K_3\right)\right),\\
B_3 &= -C_2 L_2 D\left(\alpha_3 + k_3\right)^2 < 0.
\end{aligned}$$

By discarte rule Eq.(3.4f) either has no positive root or it has two positive root, denoted by $Q_6 = \left(0, \bar{\bar{E}}_2, \bar{\bar{E}}_3, \bar{\bar{E}}_4\right)$ and $Q_7 = \left(0, \bar{\bar{E}}_2^{'}, \bar{\bar{E}}_3^{'}, \bar{\bar{E}}_4^{'}\right)$, depending on the following conditions:

$$\alpha_3 + K_3 > A_3E_3, \tag{3.4g}$$

$$\frac{D}{A_2}\left(\alpha_3 + k_3 - A_3E_3\right) > \left(\alpha_2E_3 + K_2\right)E_3\ , \tag{3.4h}$$

$$\left(\alpha_3 + K_3\right)\alpha_2A_2 > A_3\left(A_3 + A_2K_2\right), \tag{3.4i}$$

$$A_3C_2\left(2 + k_2\right) > C_3\left(\ A_2\left(D + k_1\right) + \alpha_1\left(\alpha_3 + K_3\right)\right). \tag{3.4j}$$

7. Finally the positive equilibrium point $Q_8 = \left(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\right)$ exists in the $Int.R^4_+$ if and only if there is appositive solution of the following set of equations:

$$S_1 - \frac{S_1E_1}{L_1} - \frac{C_1E_4}{(m + E_1^2)} = 0, \tag{3.5a}$$

$$S_2E_3\left(1 - \frac{E_3}{L_2}\right) - DE_2 - C_2E_2E_4 - \alpha_1E_2^2 - K_1E_2 = 0, \tag{3.5b}$$

$$DE_2 - C_3E_3E_4 - \alpha_2E_3^2 - K_2E_3 = 0, \tag{3.5c}$$

$$\frac{A_1E_1}{(m + E_1^2)} + A_2E_2 + A_3E_3 - nE_1 - \alpha_3 - K_3 = 0, \tag{3.5d}$$

From equation (3.5a) we have

$$E_4 = \frac{s_1\left(L_1 - E_1\right)\left(m + E_1^2\right)}{C_1L_1}, \tag{3.5e}$$

From equation (3.5d) we have

$$E_3 = \frac{E_1\left(n - \frac{A_1}{\left(m + E_1^2\right)}\right) + \alpha_3 + K_3 - A_2E_2}{A_3} \tag{3.5f}$$

Subtitling (3.5e) and (3.5f) in (3.5b) and then simplifying the resulting term we obtain that:

$$\begin{aligned} F_1\left(E_1, E_2\right) &= A_3l_2S_2\left(E_1(n - \frac{A_1}{(m + E_1^2)}) + (\alpha_3 + K_3) - A_2E_2\right) \\ &- S_2\left(E_1(n - \frac{A_1}{(m + E_1^2)}) + (\alpha_3 + K_3) - A_2E_2\right) \\ &- E_2\left(D + \alpha_1E_2 + K_1 + \frac{C_2}{C_1L_1}\left(S_1L_1m + S_1L_1E_1^2 - S_1mE_1 - S_1E_1^3\right)\right) = 0, \end{aligned} \tag{3.5g}$$

Now, by subtitling (3.5e) and (3.5f) in (3.5c) and them simplifying the resulting term we obtain that:

$$\begin{aligned} F_2\left(E_1, E_2\right) &= DE_2 - C_3A_3\left(E_1(n - \frac{A_1}{(m + E_1^2)}) + (\alpha_3 + K_3) - A_2E_2\right)(S_1L_1m + S_1L_1E_1^2 \\ &- S_1mE_1 - S_1E_1^3) - \alpha_2C_1L_1\left(E_1(n - \frac{A_1}{(m + E_1^2)}) + (\alpha_3 + K_3) - A_2E_2\right) \\ &- C_1A_3L_1K_2\left(E_1(n - \frac{A_1}{(m + E_1^2)}) + (\alpha_3 + K_3) - A_2E_2\right) = 0, \end{aligned} \tag{3.5h}$$

Now from (3.5g) we notice that, when $E_2 \to 0$, $E_1 \to \tilde{E}_1$, where $\tilde{E}_1$is the unique positive root of the equation.

$$f(E_1) = \gamma_1 E_1^6 + \gamma_2 E_1^5 + \gamma_3 E_1^4 + \gamma_4 E_1^3 + \gamma_5 E_1^2 + \gamma_6 E_1 + \gamma_7 = 0, \tag{3.5i}$$

where

$$\begin{aligned}
&\gamma_1 = -n^2 < 0,\\
&\gamma_2 = n\left(L_2 A_3 - 2\left(\alpha_3 + K_3\right)\right),\\
&\gamma_3 = \left(\alpha_3 + K_3\right)\left(L_2 A_3 - \left(\alpha_3 + K_3\right)\right) + 2n\left(A_1 - nm\right),\\
&\gamma_4 = \left(L_2 A_3 - 2\left(\alpha_3 + K_3\right)\right)\left(2nm - A_1\right),\\
&\gamma_5 = 2m\left(\alpha_3 + K_3\right)\left(L_2 A_3 - \left(\alpha_3 + K_3\right)\right) + nm\left(A_1 - nm\right) + A_1^2,\\
&\gamma_6 = \left(A_1 - nm\right)\left(m\left(L_2 A_3 - 2\left(\alpha_3 + K_3\right)\right)\right),\\
&\gamma_7 = m^2\left(\alpha_3 + K_3\right)\left(L_2 A_3 - \left(\alpha_3 + K_3\right)\right).
\end{aligned}$$

If in addition to condition (3.1f), the following conditions hold:

$$L_2 A_3 > \max\left\{\left(\alpha_3 + K_3\right),\ 2\left(\alpha_3 + K_3\right)\right\}, \tag{3.5j}$$
$$A_1 < 2nm. \tag{3.5k}$$

Moreover from Eq.(3.5g) we have $\frac{dE_1}{dE_2} = -\left(\frac{\frac{\partial F_1}{\partial E_2}}{\frac{\partial F_1}{\partial E_1}}\right)$. So, $\frac{dE_1}{dE_2} > 0$ if one set of the following set of conditions holds.

$$\frac{\partial F_1}{\partial E_2} > 0, \frac{\partial F_1}{\partial E_1} < 0 \quad Or \quad \frac{\partial F_1}{\partial E_2} < 0, \frac{\partial F_1}{\partial E_1} > 0 \tag{3.5l}$$

Further, from Eq. (3.5h) we notice that, when $E_2 \to 0$, $E_1 \to \tilde{E}_1'$, where $\tilde{E}_1'$ is the unique positive root of the equation.

$$f(E_1) = \rho_1 E_1^8 + \rho_2 E_1^7 + \rho_3 E_1^6 + \rho_4 E_1^5 + \rho_5 E_1^4 + \rho_6 E_1^3 + \rho_7 E_1^2 + \rho_8 E_1 + \rho_9 = 0, \tag{3.5m}$$

where

$$\begin{aligned}
&\rho_1 = S_1 A_3 n C_3 \ > 0,\\
&\rho_2 = S_1 C_3 A_3\left(\left(\alpha_3 + K_3\right) - nL_1\right),\\
&\rho_3 = S_1 C_3 A_3\left(3nm - \left(\alpha_3 + K_3\right)L_1\right) - \alpha_2 C_1 L_1 n^2,\\
&\rho_4 = 3S_1 C_3 A_3 m\left(\left(\alpha_3 + K_3\right) - nL_1\right) - A_3\left(S_1 C_3 A_3 + K_2 C_1 L_1\right) - 2\alpha_2 C_1 L_1\left(\alpha_3 + K_3\right),\\
&\rho_5 = \alpha_2 C_1 L_1\left(2n\left(\mathrm{nm} - A_1\right)\right) + \left(K_2 A_3 - \alpha_2\left(\alpha_3 + K_3\right)\right)\\
&C_1 L_1\left(\alpha_3 + K_3\right) + S_1 C_3 A_3\left(m\left(3nm - \left(\alpha_3 + K_3\right)L_1\right) + A_1 L_1\right),\\
&\rho_6 = C_3 A_3 m S_1\left(3m\left(\left(\alpha_3 + K_3\right) - nL_1\right) - 2A_1\right) + 2\alpha_2 C_1 L_1\left(\alpha_3 + \mathrm{K}_3\right) + K_2 C_1 L_1 A_3\right)\left(nm - A_1\right),\\
&\rho_7 = 2mC_1 L_1\left(\alpha_3 + \mathrm{K}_3\right)\left(K_2 A_3 - \alpha_2\left(\alpha_3 + K_3\right) + \alpha_2 C_1 L_1\left(\left(A_1 - 2nm\right) - n^2 m\right)\right.\\
&\quad + 3C_3 A_3 m^2 S_1\left(A_1 - \left(\alpha_3 + K_3\right)\right),\\
&\rho_8 = C_3 A_3 m^3 S_1\left(\left(\alpha_3 + K_3\right) - nL_1\right) + 2\alpha_2 C_1 L_1\left(\alpha_3 + \mathrm{K}_3\right) + K_2 C_1 L_1 A_3\right)\left(\mathrm{nm} - A_1\right) - C_3 A_3 m^2 S_1 A_1,\\
&\rho_9 = C_3 A_3 m^2 S_1\left(A_1 - \left(\alpha_3 + K_3\right)\right) - C_1 L_1\left(\alpha_3 + K_3\right)m^2\left(K_2 A_3 + \alpha_2\left(\alpha_3 + K_3\right)\right).
\end{aligned}$$

If in addition to conditions (3.1f) and (3.5k), the following conditions hold:

$$\alpha_3 + K_3 < nL_1, \tag{3.5n}$$
$$3nm < (\alpha_3 + K_3)\, L_1, \tag{3.5o}$$
$$K_2 A_3 < \alpha_2 (\alpha_3 + K_3)\,, \tag{3.5p}$$
$$A_1 (\alpha_3 + K_3)\,, \tag{3.5q}$$

Moreover from (3.5h) we have $\frac{dE_1}{dE_2} = -\frac{\partial F_2}{\partial E_2} \,/\, \frac{\partial F_2}{\partial E_1}$. *So,* $\frac{dE_1}{dE_2} < 0$ if one set of the following set of conditions holds.

$$\frac{\partial F_2}{\partial E_2} < 0\ ,\ \frac{\partial F_2}{\partial E_1} < 0 \quad Or \quad \frac{\partial F_2}{\partial E_2} > 0\ ,\ \frac{\partial F_1}{\partial E_1} > 0 \tag{3.5r}$$

Then the two isoclines (3.5g) and (3.5h) intersect at a unique positive point $\left(\tilde{E}_1, \tilde{E}_2\right)$ in the Int.R_4^+ of E_1E_2- space. If in addition to condition (3.1e), the following conditions hold

$$n > \frac{A_1 E_1}{(m + E_1^2)}, \tag{3.5s}$$
$$E_1\left(n - \frac{A_1 E_1}{(m + E_1^2)}\right) + \alpha_3 + K_3 > A_2 E_2, \tag{3.5t}$$
$$\tilde{E}_1 < \tilde{E}_1' \tag{3.5u}$$

So, $Q_8 = \left(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\right)$ where $\tilde{E}_3 = E_3\left(\tilde{E}_1, \tilde{E}_2\right)$ and $\tilde{E}_4 = E_4\left(\tilde{E}_1, \tilde{E}_2\right)$ exists under above conditions.

4. Local Stability Analysis

In this section discusses the local stability analysis of the system (2.1) for each of the previous equilibrium points have been discussed by computing the Jacobean matrix $J(E_1, E_2, E_3, E_4)$ of the system (2.1) as follows:

$$J = \begin{pmatrix} \frac{S_1(L_1-2E_1)}{L_1} - \frac{C_1E_4\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2} & 0 & 0 & -\frac{C_1E_1}{\left(m+E_1^2\right)} & \\ 0 & -D-C_2E_4-2\alpha_1E_2-K_1 & \frac{S_2(L_2-2E_3)}{L_2} & -C_2E_2 & 0 \\ 0 & D & -C_3E_4-2\alpha_2E_3-K_2 & -C_3E_3 & \\ \frac{A_1E_4\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2} - nE_4 & A_2E_4 & A_3E_4 & \frac{A_1E_1}{\left(m+E_1^2\right)} + A_2E_2 + A_3E_3 - nE_1 - \alpha_3 - K_3. & \end{pmatrix} \tag{4.1}$$

4.1. Local Stability Analysis of Q_0

The Jacobean matrix at $Q_0 = (\ 0\ ,0\ ,0\ ,0\ )$ is given by:

$$J_0 = J\left(Q_0\right) = \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & -D-K_1 & S_2 & 0 \\ 0 & D & -K_2 & 0 \\ 0 & 0 & 0 & -\alpha_3 - K_3 \end{pmatrix} \tag{4.2}$$

Then the characteristic equation of J_0 is given by:

$$(S_1-\lambda)\left[\lambda^2+(K_1+K_2+D)\lambda+(D(K_2-S_2)+K_2K_1)\right](-\alpha_3-K_3-\lambda)=0,$$

So, either $(S_1-\lambda)(-\alpha_3-K_3-\lambda)=0$, which gives

$$\begin{aligned}\lambda_{0E_1}&=S_1>0,\\ \lambda_{0E_4}&=-(\alpha_3+K_3)<0,\end{aligned}$$

Or $[\lambda^2+(K_1+K_2+D)\lambda+(D(K_2-S_2)+K_2K_1)]=0$, wich gives

$$\begin{aligned}\lambda_{0E_2}+\lambda_{0E_3}&=-(K_1+K_2+D)<0,\\ \lambda_{0E_2}\bullet\lambda_{0E_3}&=D(K_2-S_2)+K_2K_1\end{aligned}$$

Hence Q_0 is saddle point, and it is (unstable).

4.2. Local Stability Analysis of Q_1

The Jacobean matrix *at* $Q_1=(L_1,0,0,0)$ is given by:

$$J_1=J(Q_1)=\begin{pmatrix}-S_1 & 0 & 0 & -\frac{C_1L_1}{(m+L_1^2)}\\ 0 & -D-K_1 & S_2 & 0\\ 0 & D & -K_2 & 0\\ 0 & 0 & 0 & \frac{A_1L_1}{(m+L_1^2)}-nL_1-\alpha_3-K_3\end{pmatrix} \tag{4.3}$$

Then the characteristic equation of J_1 is given by

$$(-S_1-\lambda)\left[\lambda^2+(K_1+K_2+D)\lambda+(D(K_2-S_2)+K_2K_1)\right]\left(-\alpha_3-K_3+L_1(\frac{A_1}{(m+L_1^2)}-n)-\lambda\right)=0,$$

So, either $(-S_1-\lambda)\left(-\alpha_3-K_3+L_1\left(\frac{A_1}{(m+L_1^2)}-n\right)-\lambda\right)=0$, which gives

$$\begin{aligned}\lambda_{1E_1}&=-S_1<0,\\ \lambda_{1E_4}&=L_1\left(\frac{A_1}{(m+L_1^2)}-n\right)-\alpha_3-K_3\end{aligned}$$

Or $[\lambda^2+(K_1+K_2+D)\lambda+(D(K_2-S_2)+K_2K_1)]=0$, which gives

$$\begin{aligned}\lambda_{1E_2}+\lambda_{1E_3}&=-(K_1+K_2+D)<0,\\ \lambda_{1E_2}\bullet\lambda_{1E_3}&=D(K_2-S_2)+K_1K_2\end{aligned}$$

Hence Q_1 is locally asymptotically stable if in addition to conditions (3.2e) the following conditions hold.

$$D(S_2-K_2)<K_1K_2, \tag{4.4}$$

$$\frac{A_1L_1}{(m+L_1^2)}>n, \tag{4.5}$$

$$\frac{A_1L_1}{(m+L_1^2)}-n<\alpha_3+K_3, \tag{4.6}$$

Otherwise it is unstable.

4.3. Local Stability Analysis of Q_2

The Jacobean matrix of $Q_2 = \left(\bar{E}_1, 0, 0, \bar{E}_4\right)$, similarly for $Q_3 = \left(\bar{E}_1' \ , 0, 0, \bar{E}_4' \ \right)$, is given by

$$J_2 = J(Q_2) = \begin{pmatrix} \frac{S_1(l_1-2\bar{E}_1)}{l_1} - \frac{C_1\bar{E}_4\left(m-\bar{E}_1^2\right)}{\left(m+\bar{E}_1^2\right)^2} & 0 & 0 & -\frac{C_1\bar{E}_1}{\left(m+\bar{E}_1^2\right)} \\ 0 & -D-C_2\bar{E}_4-K_1 & S_2 & 0 \\ 0 & D & -C_3\bar{E}_4-K_2 & 0 \\ \left(\frac{A_1(m-\bar{E}_1^2)}{\left(m+\bar{E}_1^2\right)^2}-n\right)\bar{E}_4 & A_2\bar{E}_4 & A_3\bar{E}_4 & \frac{A_1\bar{E}_1}{\left(m+\bar{E}_1^2\right)}-n\bar{E}_1-\alpha_3-K_3 \end{pmatrix} \tag{4.7}$$

Then the characteristic equation of J_2 is given by

$$\left[\lambda^2 + (v_{11}+v_{44})\lambda + (v_{11})(v_{44}) - (v_{14})(v_{41})\right]\left[\lambda^2 + (v_{22}+v_{33})\lambda + (v_{22})(v_{33}) - (v_{23})(v_{32})\right] = 0,$$

So, either $[\lambda^2 + (v_{11}+v_{44})\lambda + (v_{11})(v_{44}) - (v_{14})(v_{41})] = 0$, which gives

$$\lambda_{2E_1} + \lambda_{2E_4} = \left(\frac{S_1\left(L_1 - 2\bar{E}_1\right)}{L_1} - \frac{C_1\bar{E}_4\left(m - \bar{E}_1^2\right)}{\left(m+\bar{E}_1^2\right)}\right) + \left(\left(\frac{A_1}{\left(m+\bar{E}_1^2\right)} - n\right)\bar{E}_1 - \alpha_3 - K_3\right)$$

$$\lambda_{2E_1} \bullet \lambda_{2E_4} = \left(\frac{S_1\left(L_1 - 2\bar{E}_1\right)}{L_1} - \frac{C_1\bar{E}_4\left(m - \bar{E}_1^2\right)}{\left(m+\bar{E}_1^2\right)}\right)\left(\left(\frac{A_1}{\left(m+\bar{E}_1^2\right)} - n\right)\bar{E}_1 - \alpha_3 - K_3\right)$$
$$+ \left(\frac{A_1\left(m-\bar{E}_1^2\right)}{\left(m+\bar{E}_1^2\right)^2} - n\right)E_4\left(\frac{C_1\bar{E}_1}{\left(m+\bar{E}_1^2\right)}\right)$$

Or $[\lambda^2 + (v_{22}+v_{33})\lambda + (v_{22})(v_{33}) - (v_{23})(v_{32})] = 0$, which gives

$$\lambda_{2E_2} + \lambda_{2E_3} = -(D + C_2\bar{E}_4 + K_1 + C_3\bar{E}_4 + K_2) < 0,$$
$$\lambda_{2E_2} \bullet \lambda_{2E_3} = \left(D + C_2\bar{E}_4 + K_1\right)\left(C_3\bar{E}_4 + K_2\right) - DS_2$$

So, all the eigenvalues of J_2 have negative and hence Q_2 and Q_3 is locally asymptotically stable provided that the following conditions hold.

$$\left(D + C_2\bar{E}_4 + K_1\right)\left(C_3\bar{E}_4 + K_2\right) > DS_2, \tag{4.8a}$$
$$L_1 < 2\bar{E}_1, \tag{4.8b}$$
$$m > \bar{E}_1^2, \tag{4.8c}$$
$$\left(\frac{A_1\bar{E}_1}{\left(m+\bar{E}_1^2\right)} - n\right) < \alpha_3 + K_3, \tag{4.8d}$$
$$n < \min\left\{\frac{A_1\bar{E}_1}{\left(m+\bar{E}_1^2\right)}, \ \frac{A_1\left(m-\bar{E}_1^2\right)}{\left(m+\bar{E}_1^2\right)^2}\right\}, \tag{4.8e}$$

Otherwise it is saddle point.

4.4. Local Stability Analysis of Q_4

The Jacobean matrix at $Q_4 = (0, \widehat{E}_2, \widehat{E}_3, 0)$ is given by

$$J_4 = J(Q_4) = \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & -D - 2\alpha_1\widehat{E}_2 - K_1 & \frac{S_2(L_2-2\widehat{E}_3)}{L_2} & -C_2\widehat{E}_2 \\ 0 & D & -2\alpha_2\widehat{E}_3 - K_2 & -C_3\widehat{E}_3 \\ 0 & 0 & 0 & A_2\widehat{E}_2 + A_3\widehat{E}_3 - \alpha_3 - K_3 \end{pmatrix} \tag{4.9}$$

Then the characteristic equation J_4 is given by

$$(d_{11}-\lambda)\left[\lambda^2+(d_{22}+d_{33})\lambda+(d_{22})(d_{33})-(d_{23})(d_{32})\right](d_{44}-\lambda)=0,$$

So, either $(d_{11}-\lambda)(d_{44}-\lambda)=0$, which gives

$$\begin{aligned}\lambda_{4E_1}&=S_1>0,\\ \lambda_{4E_4}&=A_2\widehat{E}_2+A_3\widehat{E}_3-\alpha_3-K_3,\end{aligned}$$

Or $[\lambda^2+(d_{22}+d_{33})\lambda+(d_{22})(d_{33})-(d_{23})(d_{32})]=0$, which gives

$$\begin{aligned}\lambda_{4E_2}+\lambda_{3E_3}&=-(D+2\alpha_1\widehat{E}_2+K_1+2\alpha_2\widehat{E}_3+K_2)<0,\\ \lambda_{4E_2}\bullet\lambda_{3E_3}&=\left(D+2\alpha_1\widehat{E}_2+K_1\right)\left(2\alpha_2\widehat{E}_3+K_2\right)-\frac{DS_2\left(L_2-2\widehat{E}_3\right)}{L_2},\end{aligned}$$

Hence Q_4 is saddle point, and it is (unstable).

4.5. Local Stability Analysis of Q_5

The Jacobean matrix at $Q_5=(\dot{E}_1,\dot{E}_2,\dot{E}_3,0)$ is given by:

$$J_5=J(Q_5)=\begin{pmatrix}\frac{S_1(L_1-2\dot{E}_1)}{L_1} & 0 & 0 & -\frac{C_1\dot{E}_1}{\left(m+\dot{E}_1^2\right)}\\ 0 & -D-2\alpha_1\dot{E}_2-K_1 & \frac{S_2(L_2-2\dot{E}_3)}{L_2} & -C_2\dot{E}_2\\ 0 & D & -2\alpha_2\dot{E}_3-K_2 & -C_3\dot{E}_3\\ 0 & 0 & 0 & \frac{A_1\dot{E}_1}{\left(m+\dot{E}_1^2\right)}+A_2\dot{E}_2+A_3\dot{E}_3-n\dot{E}_1-\alpha_3-K_3\end{pmatrix} \tag{4.10}$$

Then the characteristic equation of J_5 is given by

$$(u_{11}-\lambda)\left[\lambda^2+(u_{22}+u_{33})\lambda+(u_{22})(u_{33})-(u_{23})(u_{32})\right](u_{44}-\lambda)=0,$$

So, either $(u_{11}-\lambda)(u_{44}-\lambda)=0$, which gives

$$\begin{aligned}\lambda_{5E_1}&=\frac{S_1(L_1-2\dot{E}_1)}{L_1},\\ \lambda_{5E_4}&=\left(\frac{A_1}{\left(m+\dot{E}_1^2\right)}-n\right)\dot{E}_1+A_2\dot{E}_2+A_3\dot{E}_3-\alpha_3-K_3,\end{aligned}$$

Or $[\lambda^2+(u_{22}+u_{33})\lambda+(u_{22})(u_{33})-(u_{23})(u_{32})]=0$, which gives

$$\begin{aligned}\lambda_{5E_2}+\lambda_{4E_3}&=-(D+2\alpha_1\dot{E}_2+K_1+2\alpha_2\dot{E}_3+K_2)<0,\\ \lambda_{5E_2}\bullet\lambda_{4E_3}&=\left(D+2\alpha_1\dot{E}_2+K_1\right)\left(2\alpha_2\dot{E}_3+K_2\right)-\frac{DS_2\left(L_2-2\dot{E}_3\right)}{L_2},\end{aligned}$$

Hence Q_5 is locally asymptotically stable provided that the following conditions hold.

$$\frac{A_1\dot{E}_1}{\left(m+\dot{E}_1^2\right)} > n \ , \tag{4.11a}$$

$$\dot{E}_1\left(\frac{A_1\dot{E}_1}{\left(m+\dot{E}_1^2\right)} - n\right) + A_2\dot{E}_2 + A_3\dot{E}_3 < \alpha_3 + K_3, \tag{4.11b}$$

$$L_2 < 2\dot{E}_3, \tag{4.11c}$$

$$\left(D + 2\alpha_1\dot{E}_2 + K_1\right)\left(2\alpha_2\dot{E}_3 + K_2\right) > \frac{DS_2(L_2 - 2\dot{E}_3)}{L_2}, \tag{4.11d}$$

Otherwise it is saddle point.

4.6. Local Stability Analysis of Q_6

The Jacobean matrix of $Q_6 = \left(0, \bar{\bar{E}}_2, \bar{\bar{E}}_3, \bar{\bar{E}}_4\right)$, similarly for $Q_7 = \left(0, , \bar{\bar{E}}_2' , \bar{\bar{E}}_3' , \bar{\bar{E}}_4' \right)$, is given by

$$J_6 = J(Q_6) = \begin{pmatrix} S_1 - \frac{C_1\bar{\bar{E}}_4}{m} & 0 & 0 & 0 \\ 0 & -D-2\alpha_1\bar{\bar{E}}_2-C_2\bar{\bar{E}}_4-K_1 & \frac{S_2(L_2-2\bar{\bar{E}}_3)}{L_2} & -C_2\bar{\bar{E}}_2 \\ 0 & D & -2\alpha_2\bar{\bar{E}}_3-C_3\bar{\bar{E}}_4-K_2 & -C_3\bar{\bar{E}}_3 \\ \frac{A_1\bar{\bar{E}}_4}{m}-n\bar{\bar{E}}_4 & A_2\bar{\bar{E}}_4 & A_3\bar{\bar{E}}_4 & A_2\bar{\bar{E}}_2+A_3\bar{\bar{E}}_3-\alpha_3-K_3 \end{pmatrix} \tag{4.12}$$

Then the characteristic equation of J_6 is given by

$$(e_{11}-\lambda)\left[\lambda^3 + \bar{\bar{B}}_1\lambda^2 + \bar{\bar{B}}_2\lambda^1 + \bar{\bar{B}}_3\right] = 0 \tag{4.13a}$$

So, either $(e_{11}-\lambda) = 0$, which give $\lambda_{6E_1} = S_1 - \frac{C_1\bar{\bar{E}}_4}{m}$, *which is negative provided that* :

$$S_1 < \frac{C_1\bar{\bar{E}}_4}{m}, \tag{4.13b}$$

Or

$$\left[\lambda^3 + \bar{\bar{B}}_1\lambda^2 + \bar{\bar{B}}_2\lambda^1 + \bar{\bar{B}}_3\right] = 0, \tag{4.13c}$$

where

$$\begin{aligned}
&\bar{\bar{B}}_1 = -(e_{22}+e_{33}+e_{44}) > 0 \\
&\bar{\bar{B}}_2 = (e_{22}+e_{33})\,e_{44} + (e_{22})(e_{33}) - (e_{34})(e_{43}) - (e_{23})(e_{32}) - (e_{24})(e_{42}) \\
&\bar{\bar{B}}_3 = e_{44}\left[(e_{23})(e_{32}) - (e_{22})(e_{33})\right] + (e_{22})(e_{34})(e_{43}) + (e_{24})(e_{42})\left[(e_{33}) - (e_{32})\right] \\
&-\left[(e_{23})(e_{34})(e_{42}) + (e_{32})(e_{24})(e_{43})\right] > 0,
\end{aligned}$$

By the Routh-Hawirtiz criterion, equation (1.13c) has real negative parts, if $\bar{\bar{B}}_i > 0$, i = 1, 3 and $\Delta = \left(\bar{\bar{B}}_1\bar{\bar{B}}_2 - \bar{\bar{B}}_3\right)\bar{\bar{B}}_3 > 0$. Clearly, $\bar{\bar{B}}_i > 0$ if the following conditions hold

$$A_2\bar{\bar{E}}_2 + A_3\bar{\bar{E}}_3 < \alpha_3 + K_3, \tag{4.13d}$$

$$L_2 < 2\bar{\bar{E}}_3, \tag{4.13e}$$

$$\frac{S_2C_3A_2(L_2 - 2\bar{\bar{E}}_3)\bar{\bar{E}}_3\bar{\bar{E}}_4}{L_2} < DC_2A_3\bar{\bar{E}}_2\bar{\bar{E}}_4, \tag{4.13f}$$

Straightforward computation shows that $\bar{\bar{\bar{\Delta}}} = P_1 - P_2$, where

$$P_1 = (e_{44} + e_{22} + e_{33})\left[(e_{24})(e_{42}) + (e_{43})(e_{34}) - (e_{22} + e_{33})e_{44}\right] + (e_{22} + e_{33})\left[(e_{23})(e_{32}) - (e_{22})(e_{33})\right],$$

and

$$P_2 = \left[(e_{23})(e_{34})(e_{42}) + (e_{32})(e_{24})(e_{43})\right] - (e_{34})(e_{43})(e_{22}) - (e_{24})(e_{42})\left[e_{33} - e_{32}\right],$$

Hence, Δ will be positive if in addition of conditions and (4.13d)-(4.13f) the following condition holds

$$P_1 > P_2, \tag{4.13g}$$

So, all the eigenvalues of J_6 have negative real parts under the above conditions, hence Q_6 and Q_7 are locally asymptotically stable. It's unstable otherwise.

4.7. Local stability Analysis of Q_8

The Jacobean matrix at $Q_8 = \left(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\right)$ is given by

$$J_8 = J(Q_8) = \begin{pmatrix} \frac{S_1(L_1-2\tilde{E}_1)}{L_1} - \frac{C_1\tilde{E}_4\left(m-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)} & 0 & 0 & -\frac{C_1\tilde{E}_1}{\left(m+\tilde{E}_1^2\right)} \\ 0 & -D-C_2\tilde{E}_4-2\alpha_1\tilde{E}_2-K_1 & \frac{S_2(L_2-2\tilde{E}_3)}{L_2} & -C_2\tilde{E}_2 \\ 0 & D & -C_3\tilde{E}_4-2\alpha_2\tilde{E}_3-K_2 & -C_3\tilde{E}_3 \\ \frac{A_1\tilde{E}_4\left(m-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)^2} - n\tilde{E}_4 & A_2\tilde{E}_4 & A_3\tilde{E}_4 & \left(\frac{A_1}{\left(m+\tilde{E}_1^2\right)} - n\right)\tilde{E}_1 + A_2\tilde{E}_2 + A_3\tilde{E}_3 - \alpha_3 - K_3 \end{pmatrix} \tag{4.14}$$

Then the characteristic equation of J_8 is given by

$$\left[\lambda^4 + \tilde{\rho}_1\lambda^3 + \tilde{\rho}_2\lambda^2 + \tilde{\rho}_3\lambda + \tilde{\rho}_4\right] = 0, \tag{4.15a}$$
$$\tilde{\rho}_1 = -(\mu_0 + \mu_1) > 0,$$
$$\tilde{\rho}_2 = \mu_0\mu_1 + \mu_2 + \mu_7 - \mu_3 - \mu_4 - \mu_5 - \mu_6,$$
$$\tilde{\rho}_3 = \mu_1\left[\mu_4 - \mu_7\right] + \mu_0\left[\mu_6 - \mu_2\right] + \mu_3\mu_8 + \mu_5\mu_9 - \mu_{12} - \mu_{13} > 0,$$
$$\tilde{\rho}_4 = \mu_{10}\left[\mu_7 - \mu_3\right] + \mu_4\left[\mu_6 - \mu_2\right] - \mu_6\mu_7 + (h_{11})\left[\mu_{13} + \mu_{12}\right]$$
$$- \mu_{11}\mu_5 > 0.$$

with

$\mu_0 = h_{22} + h_{33} < 0$, $\mu_1 = h_{11} + h_{44}$, $\mu_2 = h_{11}h_{44}$, $\mu_3 = h_{34}h_{43} < 0$, $\mu_4 = h_{23}h_{32}$, $\mu_5 = h_{24}h_{42} < 0$, $\mu_6 = h_{14}h_{41}$, $\mu_7 = h_{22}h_{33} > 0$, $\mu_8 = h_{11} + h_{22}$, $\mu_9 = h_{11} + h_{33}$, $\mu_{10} = h_{11}h_{22}$, $\mu_{11} = h_{11}h_{33}$, $\mu_{12} = h_{23}h_{34}h_{42}$, $\mu_{13} = h_{24}h_{32}h_{43} < 0$.

By the Routh-Hawirtiz criterion, equation (4.15a) has real negative parts, if $\tilde{\rho}_i > 0, \quad i = 1, 3 \; and \; 4$ and $\Delta = (\tilde{\rho}_1\tilde{\rho}_2 - \tilde{\rho}_3)\,\tilde{\rho}_3 - \tilde{\rho}_1^2\tilde{\rho}_4 > 0$. Evidently, $\tilde{\rho}_i > 0, \; i = 1, \; 3 \; and \; 4$ if the following conditions hold

$$m > \tilde{E}_1^2, \tag{4.15b}$$

$$L_2 < \min\{2\tilde{E}_1, 2\tilde{E}_3\}, \tag{4.15c}$$

$$\tilde{E}_1\left(\frac{A_1\tilde{E}_1}{\left(m+\tilde{E}_1^2\right)} - n\right) + A_2\tilde{E}_2 + A_3\tilde{E}_3 < +\alpha_3 + K_3, \tag{4.15d}$$

$$n < \min\left\{\frac{A_1\tilde{E}_1}{\left(m+\tilde{E}_1^2\right)}, \; \frac{A_1\left(m-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)^2}\right\}, \tag{4.15e}$$

$$\frac{S_2C_3A_2(L_2-2\tilde{E}_3)\tilde{E}_3\tilde{E}_4}{L_2} < DC_2A_3\tilde{E}_2\tilde{E}_4, \tag{4.15f}$$

Straightforward computation shows that $\Delta = H_1 - H_2$, where

$$H_1 = \tilde{\rho}_3\left(\mu_0+\mu_1\right)\left(\mu_3-\mu_7+\mu_5-\mu_0\mu_1\right) - \mu_0\mu_{11}\mu_5\left(\mu_0+2\mu_1\right) + \tilde{\rho}_3[\mu_0\mu_4+\mu_1\left(\mu_6-\mu_2\right)], \quad and$$
$$H_2 = [(\mu_{12}+\mu_{13}) - \mu_3\mu_8 - \mu_5\mu_9]\,\tilde{\rho}_3 + \mu_1^2\left(\mu_6\mu_7+\mu_{11}\mu_5\right) - \left(\mu_0+\mu_1\right)^2$$
$$[\mu_{10}\left(\mu_7-\mu_3\right) + \mu_4\left(\mu_6-\mu_2\right) + (h_{11})\left(\mu_{12}+\mu_{13}\right)] + \mu_7\left(\mu_0^2\mu_6+\mu_1\tilde{\rho}_3\right)$$

$\Delta > 0$ if in addition to the condition (4.15b)-(4.15f) the following conditions hold

$$\mu_{13} > \mu_{12}, \tag{4.15g}$$

$$H_1 > H_2, \tag{4.15h}$$

So, all the eigenvalues of J_8 have negative real part under the given conditions hence Q_8 is locally asymptotically stable. However, it is unstable otherwise.

5. Global Stability Analysis

In this section, the global stability of the equilibrium points of system (2.1) is investigated by using the lyapunov function as shown in the following theorems.

Theorem 5.1. *The (EP) Q_1 is a globally asymptotically stable on any subregion $\Omega_1 \subset R_+^4$ that satisfies the next condition*

$$\frac{C_1E_4E_1}{m+E_1^2} + \frac{S_2L_2}{4} < \left[\frac{S_1}{L_1}\left(E_1-L_1\right)^2 + K_1E_2 + K_2E_3 + K_3E_4\right]. \tag{5.1a}$$

Proof . *Consider the following function*

$$N_1\left(E_1, E_2, E_3, E_4\right) = \left(E_1 - L_1 \; - L_1\ln\frac{E_1}{L_1}\right) + E_2 + E_3 + E_4.$$

Clearly $N_1 : R_+^4 \to R$ *is a* $N_1 \in C^1$ *positive definite function.*
Now, by differentiating N_1 *with regard to time t and some algebraic manipulation, gives the following*

$$\frac{dN_1}{dt} = -\frac{S_1}{L_1}(E_1 - L_1)^2 + S_2E_3\left(1 - \frac{E_3}{L_2}\right) + \frac{C_1E_4L_1}{m + E_1^2} - K_1E_2 - K_2E_3 - K_3E_4$$
$$- E_2E_4(C_2 - A_2) - E_3E_4(C_3 - A_3) - \frac{E_1E_4}{m + E_1^2}(C_1 - A_1).$$

Now since the function $f(E_3) = S_2E_3\left(1 - \frac{E_3}{L_2}\right)$ *in the second term represents a logistic function with respect to* E_3 *and hence it is bounded above by the constant* $\frac{S_2L_2}{4}$ *, then according to the biological facts,* $C_i > A_i, \quad i = 1, 2, 3$. *Hence,*

$$\frac{dN_1}{dt} < \frac{S_2L_2}{4} + \frac{C_1E_1E_4}{m + E_1^2} - \left[\frac{S_1}{L_1}(E_1 - L_1)^2 + K_1E_2 + K_2E_3 + K_3E_4\right].$$

Hence N_1 *is strictly Lyapunov function. So, by condition* (5.1) N_1 *is negative definite on the subregion* ω_1. *Thus* Q_1 *is a globally asymptotically stable.* □

Moreover since there are two equilibrium point $Q_2 = (\bar{E}_1, 0, 0, \bar{E}_4)$ and $Q_3 = (\bar{E}_1' \ , 0, 0, \bar{E}_4' \ )$ in the interior of R_+^4 having exactly the same conditions of local stability but with various neighborhoods of starting points then it is impossible to study the global stability of them using Lyapunove function. So we will study it numerically instead of analytically as shown in the next section.

Theorem 5.2. *The (EP)* Q_5 *is a globally asymptotically stable on any subregion* $\Omega_2 \subset R_+^4$ *that satisfies the next conditions:*

$$\left(\frac{S_2}{E_2} + \frac{D}{E_3}\right) \leq 2\sqrt{\left(\alpha_1 + \frac{S_2\dot{E}_3}{E_2\dot{E}_2}\right)\left(\alpha_2 + \frac{D\dot{E}_2}{E_3\dot{E}_3}\right)}, \tag{5.2a}$$

$$\dot{H}_1 > \dot{H}_2 \ . \tag{5.2b}$$

where

$$\dot{H}_1 = \left[\sqrt{\left(\alpha_1 + \frac{S_2\dot{E}_3}{E_2\dot{E}_2}\right)}\left(E_2 - \dot{E}_2\right) - \sqrt{\left(\alpha_2 + \frac{D\dot{E}_2}{E_3\dot{E}_3}\right)}\ \left(E_3 - \dot{E}_3\right)\right]^2 + \frac{S_1}{L_1}\left(E_1 - \dot{E}_1\right)^2 + K_3E_4,$$
$$\dot{H}_2 = \frac{C_1E_4\dot{E}_1}{m + E_1^2} + \left(C_3\dot{E}_3 + C_2\dot{E}_2\right)E_4 + \left(\frac{E_2\dot{E}_3^2}{\dot{E}_2} + \frac{\dot{E}_2E_3^2}{L_2E_3}\right)S_2. \tag{5.2c}$$

Proof . *Consider the following function*

$$N_2(E_1, E_2, E_3, E_4) = \left(E_1 - \dot{E}_1 - \dot{E}_1 \ln\frac{E_1}{\dot{E}_1}\right) + \left(E_2 - \dot{E}_2 - \dot{E}_2 \ln\frac{E_2}{\dot{E}_2}\right) + \left(E_3 - \dot{E}_3 - \dot{E}_3 \ln\frac{E_3}{\dot{E}_3}\right) + E_4.$$

Clearly $N_2 : R_+^4 \to R$ *is a* $N_2 \in C^1$ *positive definite function.*

Now, by differentiating N_2 with regard to time t and some algebraic manipulation, gives the following

$$\frac{dN_2}{dt} = -\frac{S_1}{L_1}\left(E_1-\dot{E}_1\right)^2+\frac{C_1E_4\dot{E}_1}{m+E_1^2}-\left(\alpha_1+\frac{S_2\dot{E}_3}{E_2\dot{E}_2}\right)\left(E_2-\dot{E}_2\right)^2+(\frac{S_2}{E_2}+\frac{D}{E_3})\left(E_2-\dot{E}_2\right)$$
$$\left(E_3-\dot{E}_3\right)-\left(\alpha_2+\frac{D\dot{E}_2}{E_3\dot{E}_3}\right)\left(E_3-\dot{E}_3\right)^2+C_3\dot{E}_3E_4+C_2\dot{E}_2E_4-\frac{E_1E_4}{m+E_1^2}\left(C_1-A_1\right)$$
$$-E_2E_4\left(C_2-A_2\right)-E_3E_4\left(C_3-A_3\right)-K_3E_4+\left(\frac{E_2\dot{E}_3^2}{\dot{E}_2}+\frac{\dot{E}_2E_3^2}{L_2E_3}\right)S_2.$$

So, according to condition (5.2a) *with the biological facts in Theorem 2.1, always* $C_i > A_i,\ i=1,2,3.$

$$\frac{dN_2}{dt} < -\left[\sqrt{\left(\alpha_1+\frac{S_2\dot{E}_3}{E_2\dot{E}_2}\right)}\left(E_2-\dot{E}_2\right)-\sqrt{\left(\alpha_2+\frac{D\dot{E}_2}{E_3\dot{E}_3}\right)}\left(E_3-\dot{E}_3\right)\right]^2-\frac{S_1}{L_1}\left(E_1-\dot{E}_1\right)^2$$
$$-K_3E_4+\frac{C_1E_4\dot{E}_1}{m+E_1^2}+\left(C_3\dot{E}_3+\ C_2\dot{E}_2\right)E_4+\left(\frac{E_2\dot{E}_3^2}{\dot{E}_2}+\frac{\dot{E}_2E_3^2}{L_2E_3}\right)S_2.$$

Then, $\frac{dN_2}{dt} < -\dot{H}_1+\dot{H}_2$ *. Hence* N_2 *is strictly Lyapunov function. So, by condition* (5.2b) N_2 *is negative definite on the subregion* Ω_2*. Thus* Q_5 *is a globally asymptotically stable.* □

Moreover since there are two equilibrium point $Q_6=\left(0,\bar{\bar{E}}_2,\bar{\bar{E}}_3,\bar{\bar{E}}_4\right)$ and $Q_7=\left(0,,\bar{\bar{E}}_2',\bar{\bar{E}}_3',\bar{\bar{E}}_4'\right)$ in the interior of R_+^4 having exactly the same conditions of local stability but with various neighborhoods of starting points then it is impossible to study the global stability of them using Lyapunove function. So we will study it numerically instead of analytically as shown in the next section.

Theorem 5.3. *The (EP)* Q_8 *is a globally asymptotically stable on any subregion* $\Omega_3 \subset R_+^4$ *that satisfies the next conditions*

$$\left(\frac{S_2}{E_2}+\frac{D}{E_3}\right) \le 2\sqrt{\left(\alpha_1+\frac{S_2\tilde{E}_3}{E_2\tilde{E}_2}\right)\left(\alpha_2+\frac{D\tilde{E}_2}{E_3\tilde{E}_3}\right)}, \tag{5.3a}$$
$$\tilde{E}_1 < E_1, \tag{5.3b}$$
$$\tilde{u}_2 < \tilde{u}_1, \tag{5.3c}$$
$$\tilde{u}_1=\left[\sqrt{\left(\alpha_1+\frac{S_2\tilde{E}_3}{E_2\tilde{E}_2}\right)}\left(E_2-\tilde{E}_2\right)-\sqrt{\left(\alpha_2+\frac{D\tilde{E}_2}{E_3\tilde{E}_3}\right)}\left(E_3-\tilde{E}_3\right)\right]^2+\frac{S_1}{L_1}\left(E_1-\tilde{E}_1\right)^2,$$
$$\tilde{u}_2=\frac{C_1E_4\left(E_1-\tilde{E}_1\right)\left(E_1^2-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)\left(m+E_1^2\right)}+\left(E_1\tilde{E}_4+\tilde{E}_1E_4\right)n+\left(E_2\tilde{E}_4+\tilde{E}_2E_4\right)C_2+\left(E_3\tilde{E}_4+\tilde{E}_3E_4\right)C_3.$$

Proof . *Consider the following function:*

$$N_3\left(E_1,E_2,E_3,E_4\right)=\left(E_1-\tilde{E}_1\ -\tilde{E}_1\ln\frac{E_1}{\tilde{E}_1}\right)+\left(E_2-\tilde{E}_2\ -\tilde{E}_2\ln\frac{E_2}{\tilde{E}_2}\right)+\left(E_3-\tilde{E}_3-\tilde{E}_3\ln\frac{E_3}{\tilde{E}_3}\right)$$
$$+\left(E_4-\tilde{E}_4\ -\tilde{E}_4\ln\frac{E_4}{\tilde{E}_4}\right).$$

Clearly $N_3: R_+^4 \to R$ *is a* $N_3 \in C_1$ *positive definite function.*
Now, by differentiating N_3 *with regard to time t and some algebraic manipulation, gives the following*

$$\begin{aligned}\frac{dN_3}{dt} =& -\frac{S_1}{L_1}\left(E_1-\tilde{E}_1\right)^2-\left(\alpha_1+\frac{S_2\tilde{E}_3}{E_2\tilde{E}_2}\right)\left(E_2-\tilde{E}_2\right)^2+\left(\frac{S_2}{E_2}+\frac{D}{E_3}\right)\left(E_2-\tilde{E}_2\right)\left(E_3-\tilde{E}_3\right)\\ &-\left(\alpha_2+\frac{D\tilde{E}_2}{E_3\tilde{E}_3}\right)\left(E_3-\tilde{E}_3\right)^2-\frac{C_1E_4\left(E_1-\tilde{E}_1\right)\left(E_4-\tilde{E}_4\right)}{\left(m+\tilde{E}_1^2\right)}(C_1-A_1)-E_2E_4\left(C_2-A_2\right)\\ &-E_3E_4\left(C_3-A_3\right)+\frac{C_1E_4\left(E_1-\tilde{E}_1\right)\left(E_1^2-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)(m+E_1^2)}+\left(E_1\tilde{E}_4+\tilde{E}_1E_4\right)n+\left(E_2\tilde{E}_4+\tilde{E}_2E_4\right)C_2\\ &+\left(E_3\tilde{E}_4+\tilde{E}_3E_4\right)C_3.\end{aligned}$$

So, according to by conditions (5.3a) *and* (5.3b) *with the biological facts,* $C_i > A_i, \ \ i = 1, 2, 3.$

$$\begin{aligned}\frac{dN_3}{dt} <& -\left[\sqrt{\left(\alpha_1+\frac{S_2\tilde{E}_3}{E_2\tilde{E}_2}\right)}\left(E_2-\tilde{E}_2\right)-\sqrt{\left(\alpha_2+\frac{D\tilde{E}_2}{E_3\tilde{E}_3}\right)}\left(E_3-\tilde{E}_3\right)\right]^2-\frac{S_1}{L_1}\left(E_1-\tilde{E}_1\right)^2\\ &+\frac{C_1E_4\left(E_1-\tilde{E}_1\right)\left(E_1^2-\tilde{E}_1^2\right)}{\left(m+\tilde{E}_1^2\right)(m+E_1^2)}+n\left(E_1\tilde{E}_4+\tilde{E}_1E_4\right)+C_2\left(E_2\tilde{E}_4+\tilde{E}_2E_4\right)\\ &+C_3\left(E_3\tilde{E}_4+\tilde{E}_3E_4\right)\end{aligned}$$

Then, $\frac{dN_3}{dt} = -\tilde{u}_1 + \tilde{u}_2$. *Hence* N_3 *is strictly Lyapunov function. So, by condition* (5.3c) N_3 *is negative definite on the subregion* ω_3. *Thus* Q_8 *is a globally asymptotically stable.* □

6. Numerical Simulation

In this section, numerical simulations have been used dynamic behavior of system (2.1). For one set of parameters and different set of initial points. The aim of this study:

1. the effects of parameters on the dynamics of our model.
2. Confirm the analytic results.

Figure 1. (a-d) it appears that the system (2.1) at the hypothetical set of parameters (6.1) has global positive equilibrium point.

$$\begin{aligned}&S_1=0.5,\quad L_1=0.5,\quad C_1=0.5,\quad m=0.5,\quad S_2=0.5,\quad L_2=0.5,\quad D=0.5,\\ &C_2=0.5, \alpha_1=0.1,\quad K_1=0.1,\quad C_3=0.5,\quad \alpha_2=0.1,\quad K_2=0.1,\quad A_1=0.3,\\ &A_2=0.3,\quad A_3=0.3,\quad n=0.1,\quad \alpha_3=0.1,\quad K_3=0.1\end{aligned} \tag{6.1}$$

Now, in order to discuss the effect of the parameters values of system (2.1) on the dynamical behavior of system, the system (2.1) solved numerically for the data given in (6.1) with change one parameter at each time the obtained results.
The effect of the following parameters summarized in table (1).

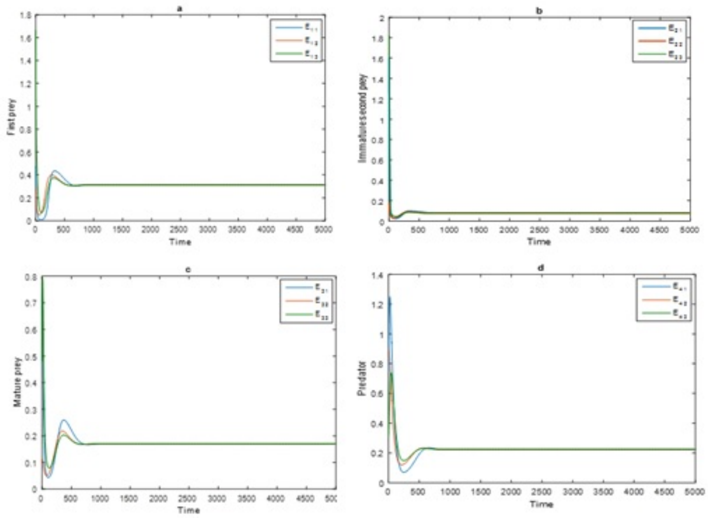

Figure 1: Time series of the solution of system (2.1) beginning with different initial points (0.5 , 1.8 , 0.6 , 0.7), (0.3, 0.2 , 0.1, 0.9), and (1.9 , 2 , 0.4, 0.3). (*a*) Trajectory of E_1 as a function of time, (*b*) Trajectory of E_2 as a function of time, (*c*) Trajectory of E_3 as a function of time, (*d*) Trajectory of E_4 as a function of time.

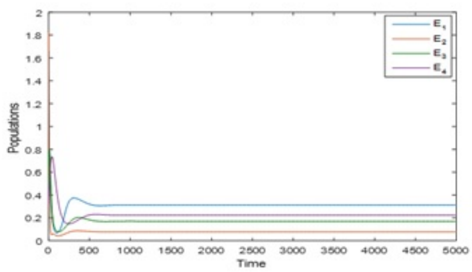

Figure 2: Graphical representation of the solution which approaches $Q_8 = (0.310,\ 0.062,\ 0.133,\ 0.500)$.

Table 1:

Range of parameter	*The stable point*	*Range of parameter*	*The stable point*
$0.1 \leq L_1 < 0.16$	Q_5	$0.1 \leq K_2 < 0.32$	Q_8
$0.16 \leq L_1 < 1$	Q_8	$0.32 \leq K_2 < 0.41$	Q_5
		$0.41 \leq K_2 < 1$	Q_1
$0.1 \leq S_1 < 2$	Q_8	$0.1 \leq D < 1$	Q_8
$0.1 \leq m < 0.98$	Q_8	$0.1 \leq \alpha_i < 1.5,\ i = 1, 2$	Q_8
$0.98 \leq m \leq 1.5$	Q_5		
$0.1 \leq S_2 < 0.14$	Q_1	$0.1 \leq A_i \leq 0.4,\ i = 1, 2, 3$	Q_8
$0.14 \leq S_2 < 0.20$	Q_5		
$0.20 \leq S_2 < 2$	Q_8		
$0.3 \leq C_i < 2, i = 1, 2, 3$	Q_8	$0.1 \leq n < 0.261$	Q_8
		$0.261 \leq n \leq 1.5$	Q_5
$0.1 \leq L_2 < 1.5$	Q_8	$0.1 \leq K_3, \alpha_3 < 0.179$	Q_8
$0.1 \leq K_1 < 1$	Q_8	$0.179 \leq K_3, \alpha_3 < 1$	Q_5

The effect of varying the parameter S_2 in the range $0.1 \leq S_2 < 0.14$ the solution approaches to Q_1, as shown in Figure 3 (a) , for model value $S_2 = 0.1$, increasing further in the range $0.14 \leq S_2 < 0.21$

the solution approaches to Q_5, as shown in Figure 3 (b), for model value $S_2 = 0.18$, but in the $0.21 \le S_2 < 2$ the solution approaches to Q_8 , as shown in Figure 3 (c), for model value $S_2 = 0.25$.

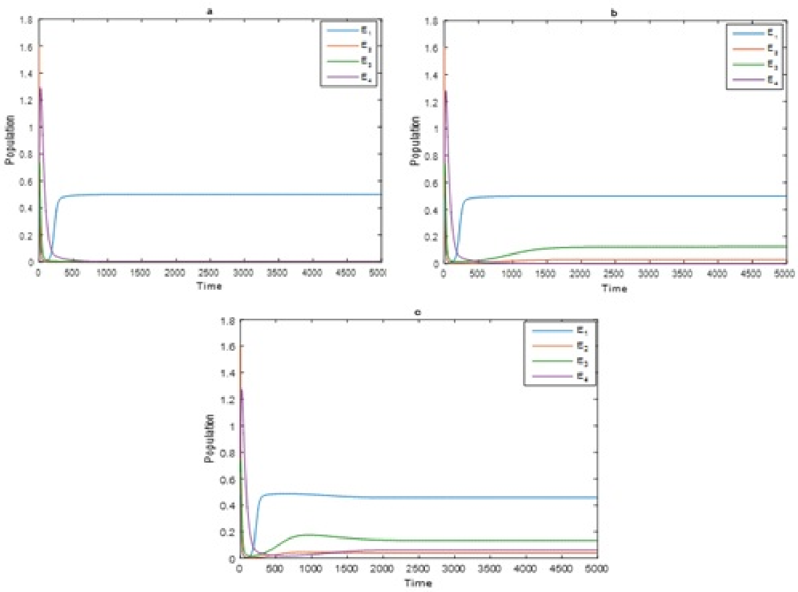

Figure 3: (a) Time series of the solution of system (2.1) with $S_2 = 0.1$, which approaches to $Q_1 = (0.5, 0, 0, 0)$, and (b) time series of the solution of system (2.1) with $S_2 = 0.15$, which approaches to $Q_5 = (0.410,\ 0.032, 0.123, 0)$, and (c) time series of the solution of system (2.1) with $S_2 = 0.25$, which approaches to $Q_8 = (0.410, 0.310, 0.140, 0.060)$.

For the parameter K_2 in the range $0.1 \le K_2 < 0.32$ the solution approaches to Q_8, as shown in Figure 4 (a), for model value $K_2 = 0.1$, increasing further in the range $0.32 \le K_2 < 0.41$ the solution approaches to Q_5, as shown in Figure 4 (b), for model value $K_2 = 0.33$, but in the $0.41 \le k_2 < 1$ the solution approaches to Q_1 , as shown in Figure 4 (c), for model value $K_2 = 0.5$.

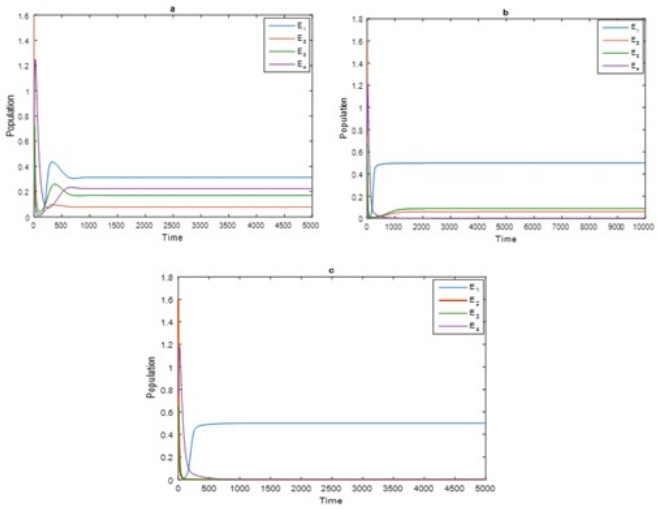

Figure 4: (a) Time series of the solution of system (2.1) with $K_2 = 0.1$, which approaches to $Q_8 = (0.424, 0.050, 0.140, 0.103)$, and (b) time series of the solution of system (2.1) with $\mathrm{K}_2 = 0.33$, which approaches to $Q_5 = (0.410,\ 0.032, 0.140, 0)$, and (c) time series of the solution of system (2.1) with $\mathrm{k}_2 = 0.5$, which approaches to $Q_1 = (0.5, 0, 0, 0)$.

Finally, change the parameter $\alpha_3, K_3, A_1,$ with the rest of parameter as given (6.1) in the range $0.01 \leq K_3, \alpha_3 < 0.015,\ \ 0.09 \leq A_1 < 0.043$, the solution of system (2.1) approaches to Q_8, as shown in Figure 5, for model values $\alpha_3 = 0.01, K_3 = 0.01$ and $A_1 = 0.09$.

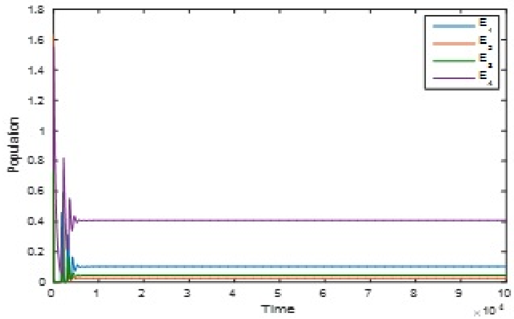

Figure 5: Time series of the solution of system (2.1) with $K_3, \alpha_3 = 0.01,\quad \mathrm{A}_1 = 0.09$, which approaches to $Q_8 = (0.012, 0.013, 0.023, 0.392)$.

7. Conclusions and Discussions

In this study, a mathematical model that consisting of four species: first prey and second prey with stage structure and predator in the presence of toxicity and anti -predator has been proposed and studied by using the functional response Holling's type IV and Lotka Volttra. The solution's existence, uniqueness, and boundedness have all been studied. All possible equilibrium points have been identified. They stability of this model have been studied. Finally, numerical simulation have been used to verify our analytical results. With data given in Eq. (6.1). Which are summarized as follow:

1. There is no periodic dynamics for system (2.1).
2. The parameters $L_1, m, S_2, K_2, n, \alpha_3$ and K_3 play an important role on the dynamics of system (2.1), while at others parameters $S_i, \alpha_i, i = 1, 2,\ \ \mathrm{A}_i, C_i, i = 1, 2, 3,\ \ L_2, K_1, D$, the solution still approaches to positive equilibrium point.

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