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Generalized dynamic process for generalized (ψ, S, F)-contraction with applications in b-Metric Spaces

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Abstract

In this paper, we develop the notion of (ψ, F)-contraction mappings introduced in [49] in b-metric spaces. To achieve this, we introduce the notion of generalized multi-valued (ψ, S, F)-contraction type I mapping with respect to generalized dynamic process $D(S, T, x_0)$, generalized multi-valued (ψ, S, F)-contraction type II mapping with respect to generalized dynamic process $D(S, T, x_0)$, and establish common fixed point results for these classes of mappings in complete b-metric spaces. As an application, we obtain the existence of solutions of dynamic programming and integral equations. The results presented in this paper extends and complements some related results in the literature.

Keywords: fixed point, dynamic process, generalized multi-valued (ψ, S, F)-contraction type, b-metric space; integral equations, dynamic programming.
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1. Introduction and Preliminaries

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [12] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful

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applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [7, 26, 27, 28, 29, 37, 38, 41, 51] and the references therein). Also, over the years, several authors have developed several iterative schemes for solving fixed point problem for different operators in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, (see [1, 2, 3, 4, 5, 21, 23, 30, 32, 33, 34, 35, 46, 47] and the references therein). For example, Berinde [15, 16] introduced and studied a class of contractive mappings, which is defined as follows:

Definition 1.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [44] introduced a class of mappings satisfying condition (C), known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.2. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.3. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

In 2012, Wardowski [50] introduced the notion of F-contractions, which is defined as follows:

Definition 1.4. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F_1) *F is strictly increasing;*
(F_2) *for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
(F_3) *there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

He established the following result:

Theorem 1.5. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.*

Remark 1.6. *[50] If we suppose that $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction mapping.*

In [38], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

Theorem 1.7. *Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

In 2013, Secelean in [43] replaced the condition (F_2) in the definition of F-contraction with the following condition.
(F_*) $\inf F = -\infty$
or, also by
(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$. He also established the following result:

Lemma 1.8. *[43] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:*

1. *if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$;*
2. *if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

In the same year, Turinici in [48] observed that the condition (F_2) in the definition of F-contraction can be replaced with
(F_2') $\lim_{n\to\infty} F(\alpha_n) = -\infty$. Then, the implication is as follows
(F_2'') $\lim_{n\to\infty} F(\alpha_n) = -\infty \Rightarrow \alpha_n \to 0$, can be derived from (F_1).
Motivated by the work of Turinici [48], Wardowski [49] introduced a modified F-contraction called (ψ, F)-contraction in the setting of a metric space. He gave the following definition:

Definition 1.9. *Let (X, d) be a metric space. A mapping $T : X \to X$ is called (ψ, F)-contraction if there are $\psi : [0, \infty) \to [0, \infty)$ and $F : [0, \infty) \to \mathbb{R}$ such that*

1. *F satisfies (F_1) and (F_2');*
2. *$\liminf_{s\to t^+} \psi(s) > 0$ for all $t \geq 0$;*
3. *$\psi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$ for all $x, y \in X$ such that $Tx \neq Ty$.*

One of the most interesting generalizations of metric spaces is the concept of b-metric spaces (to be defined in Section 2) introduced by Czerwik in [19]. He proved the Banach contraction principle in this setting with the fact that d need not to be continuous. Thereafter, several results have been extended from metric spaces to b-metric spaces. In addition, a lot of results have been published on the fixed point theory of various classes of single-valued and multi-valued operators in the frame work of b-metric spaces (see [10, 17, 19, 40, 51] and the references therein).

Definition 1.10. *[19] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$;*

3. $d(x,y) \le s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space. The number $s \ge 1$ is called the coefficient of (X,d). It is clear that, the class of b-metric spaces is larger than that of metric spaces. If $s = 1$, a b-metric become a metric.

Example 1.11. *[10] Let $X = \mathbb{R}$ and $d(x,y) = |x-y|^2$ for all $x, y \in X$. It is easy to see that (x,d) is a b-metric space with coefficient $s = 2$, but (X,d) is not a metric space.*

Definition 1.12. *[17] Let (X,d) be a b-metric space. A sequence $\{x_n\}$ in X is said to be*

1. *b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.*
2. *b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Definition 1.13. *[17] Let (X,d) be a b-metric space. Then X is said to be complete if every b-Cauchy sequence in X is b-convergent.*

Let (X,d) be a b-metric space with $s \ge 1$ and $CB(X)$ $(N(X))$ denote family of all bounded and closed (nonempty) subset of X. For any $x \in X$ and $A, B \in CB(X)$, we define

$$D(x,A) = \inf_{a\in A} d(x,a) \quad \text{and} \quad D(A,B) = \sup_{a\in A} D(a,B).$$

Define a mapping $H : CB(X) \times CB(X) \to [0,\infty)$ by

$$H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\}$$

for any $A, B \in CB(X)$. Then the mapping H is a b-metric and it is called a Hausdorff b-metric induced by a b-metric.

Lemma 1.14. *[20] Let (X,d) be a b-metric space with $s \ge 1$. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following.*

1. *$D(x,B) \le d(x,b)$;*
2. *$D(x,B) \le H(A,B)$;*
3. *$D(x,A) \le s[d(x,y) + D(y,B)]$;*
4. *$D(x,A) = 0 \Leftrightarrow x \in A$;*
5. *$H(A,B) \le s[H(A,C) + H(C,B)]$.*

Let $S : X \to X$ and $T : X \to N(X)$. The pair (S,T) is said to satisfy range inclusion condition if $S(X) \subset T(X)$. A point $x \in X$ is a fixed point of T if $x \in Tx$. The set of all fixed point of T is denoted by $F(T)$. Also, a point $x \in X$ is called a coincidence point of S and T if $Sx \in Tx$. We denote the set of all coincidence point by $C(S,T)$. In addition, if for some $x \in X$, we have $x = Sx \in Tx$, then the point x is called a common fixed point of the pair (S,T). We denote the set of all common fixed point of S and T by $F(S,T)$.
Let x_0 be an arbitrary but fixed element in X. The set

$$D(S,T,x_0) = \{(x_n)_{n\in\mathbb{N}} \cup \{0\} : x_{n+1} = Sx_n \in Tx_{n-1} \ \ \forall \ \ n \in \mathbb{N}\}$$

is called a generalized dynamic process of S and T starting at x_0. It is worth mentioning that the set $D(S,T,x_0)$ reduces to the well-known dynamic process of T if $S=I$ (identity mapping), for details abour dynamic process see [25]. The generalized dynamic process $D(S,T,x_0)$ will simply be written as (Sx_n). The sequence $\{x_n\}$ for which (Sx_n) is a generalized dynamic process is called S iterative sequence of T starting with x_0. It is well-known that if the pair (S,T) satisfy the range inclusion condition, then for any $x_0 \in X$, construction of S iterative sequence of T starting with x_0 follows directly and consequently $D(S,T,x_0) \neq \emptyset$. More so, if $D(S,T,x_0) \neq \emptyset$, then such situation may arise that even the range inclusion condition does not hold.

Example 1.15. *Let $X=[0,\infty)$. Define $S: X \to X$ and $T: X \to N(X)$ by $S(x)=\frac{x-1}{3}$ and $T(x)=[0,\frac{x}{3}]$. The sequence $\{x_n\}$ defined by $x_n = x_{n-1}+1$ for all $n \in \mathbb{N}$. Suppose that $x_0=1$, we have that*

$$\begin{aligned}
S(x_1) &= \frac{1}{3} \in Tx_0 = [0, \frac{1}{3}],\\
S(x_2) &= \frac{2}{3} \in Tx_1 = [0, \frac{2}{3}],\\
S(x_3) &= 1 \in Tx_2 = [0, 1],\\
S(x_4) &= \frac{4}{3} \in Tx_3 = [0, \frac{4}{3}],\\
&\vdots
\end{aligned}$$

Thus $D(S,T,1)=\{\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\cdots\}$ is a generalized dynamic process of S and T with an initial point 1.

Example 1.16. *Let $X=[0,\infty)$. Define $S: X \to X$ and $T: X \to N(X)$ by $S(x)=x^2$ and $T(x)=[3+x,\frac{x}{3}]$. The sequence $\{x_n\}$ defined by*

$$x_n = \sqrt{x_{n-1}+3}$$

is an S iterative sequence of T with a starting point 0.

We denote by $\mathcal{F}$ the family of all functions $F: \mathbb{R}^+ \to \mathbb{R}$ and $\psi: [0,\infty) \to [0,\infty)$ which satisfy conditions
(F_1^*) F satisfies (F_1) and $(F_2^{'})$;
(F_2^*) F is continuous on $(0,\infty)$;
(F_3^*) $\liminf_{s\to t^+} \psi(s) > 0$ for all $t \geq 0$.
Motivated by the works of Kim [25], Wardowski [49, 50], and ongoing research interest in this direction, in this work we develop the notion of (ψ,F)-contraction in the framework of b-metric spaces. To do this, we introduce the notion of generalized multi-valued (ψ,S,F)-contraction type I mapping with respect to generalized dynamic process $D(S,T,x_0)$, generalized multi-valued (ψ,S,F)-contraction type II mapping with respect to generalized dynamic process $D(S,T,x_0)$, and establish common fixed point results for these classes of mappings in complete b-metric spaces. Finally, we apply our fixed point result in establishing the solutions of dynamic programming and integral equations.

2. Main Result

In this section, we introduce the notion of generalized multi-valued (ψ, S, F)-contraction type I mapping with respect to generalized dynamic process $D(S, T, x_0)$ with $x_0 \in X$, generalized multi-valued (ψ, S, F)-contraction type II mapping with respect to generalized dynamic process $D(S, T, x_0)$ with $x_0 \in X$, and establish common fixed point for these classes of mappings in complete b-metric spaces. In the sequel, we will consider only the dynamic process (Sx_n) satisfying the following condition:

$$(E) \quad \text{For any } n \in \mathbb{N}, d(Sx_n, Sx_{n+1}) > 0 \Rightarrow d(Sx_{n-1}, Sx_n) > 0.$$

If the dynamic process (Sx_n) does not satisfy property (E), then there exists $n_0 \in \mathbb{N}$ such that

$$d(Sx_{n_0}, Sx_{n_0+1}) > 0 \quad \text{and} \quad d(Sx_{n_0-1}, Sx_{n_0}) = 0$$

which implies that $Sx_{n_0-1} = Sx_{n_0} \in Tx_{n_0-1,}$ that is, the set of coincidence point of hybrid pair (S, T) is nonempty. It follows that under some suitable conditions on the pair (S, T), one can obtain the existence of common fixed point.

Lemma 2.1. *Suppose (X, d) is a b-metric space with $s \geq 1$. Let $\{Sx_n\}$ be a sequence in X such that $d(Sx_n, Sx_{n+1}) \to 0$ as $n \to \infty$. If $\{Sx_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{Sm_k\}$ and $\{Sn_k\}$ with $m_k > n_k > k$ satisfying $d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon$ and $d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon$ such that*

1. $\epsilon \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq s\epsilon$;
2. $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon$;
3. $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq s^2\epsilon$;
4. $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon$.

Proof . Suppose $\{Sx_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ satisfying

$$d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon. \tag{2.1}$$

We choose m_k, the least positive integer satisfying (2.1).

We now prove (1). Using (2.1)

$$\begin{aligned} \epsilon \leq d(Sx_{m_k}, Sx_{n_k}) &\leq sd(Sx_{m_k}, Sx_{n_{k-1}}) + sd(Sx_{n_{k-1}}, Sx_{n_k}) \\ &< s\epsilon + sd(Sx_{n_{k-1}}, Sx_{n_k}). \end{aligned} \tag{2.2}$$

Clearly, using our hypothesis, we have that

$$\epsilon \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq s\epsilon. \tag{2.3}$$

We now prove (2).
Now observe that

$$\begin{aligned} d(Sx_{m_k}, Sx_{n_k}) &\leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + sd(Sx_{m_{k+1}}, Sx_{n_k}) \\ &\leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + s^2d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + s^2d(Sx_{n_{k+1}}, Sx_{n_k})) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) &\leq sd(Sx_{m_{k+1}}, Sx_{m_k}) + sd(Sx_{m_k}, Sx_{n_{k+1}}) \\ &\leq sd(Sx_{m_{k+1}}, Sx_{m_k}) + s^2 d(Sx_{m_k}, Sx_{n_k}) + s^2 d(Sx_{n_k}, Sx_{n_{k+1}}). \end{aligned} \tag{2.5}$$

Using our hypothesis, (2.4) and (2.5), we have that

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon. \tag{2.6}$$

We now prove (3).
Note that,

$$d(Sx_{m_k}, Sx_{n_k}) \leq sd(Sx_{m_k}, Sx_{n_{k+1}}) + sd(Sx_{n_{k+1}}, Sx_{n_k}) \tag{2.7}$$

and

$$d(Sx_{m_k}, Sx_{n_{k+1}}) \leq sd(Sx_{m_k}, Sx_{n_k}) + sd(Sx_{n_k}, Sx_{n_{k+1}}). \tag{2.8}$$

Using our hypothesis, (2.8) and (2.7), we have that

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq s^2\epsilon. \tag{2.9}$$

We now prove (4).
Now observe that

$$\epsilon \leq d(Sx_{m_k}, Sx_{n_k}) \leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + s^2 d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + s^2 d(Sx_{n_{k+1}}, Sx_{n_k}) \tag{2.10}$$

and

$$d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq sd(Sx_{m_{k+1}}, Sx_{n_k}) + sd(Sx_{n_k}, Sx_{n_{k+1}}). \tag{2.11}$$

Thus, using our hypothesis, (2.10), (2.11) and (3), we have

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon. \tag{2.12}$$

□

We introduce the following class of functions: Let $\Psi = \Big\{ f : [0,\infty) \times [0,\infty) \to \mathbb{R} |\ f(t,t) = 0$ if and only if $t = 0$ and $f(t,s) \leq t - \frac{s}{2}\ \forall\ s,t \in [0,\infty) \Big\}$.

Example 2.2. 1. *Let $f_1 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as $f_1(t,s) = \phi(t) - \psi(s)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are functions such that $\phi(t) = t$ and $\psi(s) = \frac{s}{2}$, $\forall\ s,t \in [0,\infty)$. Clearly, $f_1 \in \Psi$.*

2. *Let $f_2 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as $f_2(t,s) = t - \frac{\phi(t,s)}{2\psi(t,s)} s$, where $\phi, \psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ are functions such that $\phi(t,s) \geq \psi(t,s)$, $\forall\ s,t \in [0,\infty)$. It is easy to see that $f_2 \in \Psi$.*
3. *Let $f_3 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as $f_3(t,s) = t - \phi(t) - \psi(s) - \frac{s}{2}$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are functions such that $\phi(t) > 0, \psi(s) > 0$, $\forall\ s,t \in [0,\infty)$ and $\phi(t) = 0 = \psi(s)$ if and only if $t,s = 0$. Clearly, $f_3 \in \Psi$.*

Definition 2.3. *Let (X,d) be a b-metric space with $s \geq 1$, $\psi : [0,\infty) \to [0,\infty)$ and S be a self map on X. A multi-valued mapping $T : X \to CB(X)$ is said to be a generalized multi-valued (ψ, S, F)-contraction type I mapping with respect to generalized dynamic process $D(S,T,x_0)$ with $x_0 \in X$ if $F \in \mathcal{F}$ and $L \geq 0$ such that*

$$f\left(\frac{1}{2s}D(Sx_{n-1},Tx_{n-1}), D(Sx_{n-1},T(Sx_{n-1}))\right) \leq d(Sx_{n-1},Sx_n)$$
$$\Rightarrow \psi(M(x_{n-1},x_n)) + F(s^5 d(Sx_n,S_{n+1})) \leq F(M(x_{n-1},x_n)) + LN(x_{n-1},x_n), \tag{2.13}$$

with $d(Sx_{n-1},Sx_n) > 0$, where $M(x_{n-1},x_n) = \max\{d(Sx_{n-1},Sx_n), D(Sx_{n-1},Tx_{n-1}), D(Sx_n,Tx_n), \frac{D(Sx_{n-1},Tx_{n-1})D(Sx_n,Tx_n)}{s+d(Sx_{n-1},Sx_n)}, \frac{D(Sx_n,Tx_{n-1})[1+D(Sx_{n-1},Tx_{n-1})]}{s+d(Sx_{n-1},Sx_n)}\}$ and
$N(x_{n-1},x_n) = \min\{D(Sx_{n-1},Tx_{n-1}), D(Sx_n,Tx_n), D(Sx_{n-1},Tx_n), D(Sx_n,Tx_{n-1})\}$.

Definition 2.4. *Let (X,d) be a b-metric space with $s \geq 1$, $\psi : [0,\infty) \to [0,\infty)$ and S be a self map on X. A multi-valued mapping $T : X \to CB(X)$ is said to be a generalized multi-valued (ψ, S, F)-contraction type II mapping with respect to generalized dynamic process $D(S,T,x_0)$ with $x_0 \in X$ if $F \in \mathcal{F}$ such that*

$$\psi(M(x_{n-1},x_n)) + F(s^3 d(Sx_n,S_{n+1})) \leq F(M(x_{n-1},x_n)), \tag{2.14}$$

with $d(Sx_{n-1},Sx_n) > 0$, where $M(x_{n-1},x_n) = \max\{d(Sx_{n-1},Sx_n), D(Sx_{n-1},Tx_{n-1}), D(Sx_n,Tx_n), \frac{D(Sx_{n-1},Tx_{n-1})D(Sx_n,Tx_n)}{s+d(Sx_{n-1},Sx_n)}, \frac{D(Sx_n,Tx_{n-1})[1+D(Sx_{n-1},Tx_{n-1})]}{s+d(Sx_{n-1},Sx_n)}\}$.

Theorem 2.5. *Let (X,d) be a complete b-metric space with $s \geq 1$, x_0 be an arbitrary point in X and $T : X \to CB(X)$ a generalized multi-valued (ψ, S, F)-contraction type I mapping with respect to generalized dynamic process $D(S,T,x_0)$ such that $S(X)$ is a complete subspace of X, then the pair (S,T) has a point of coincidence in X. More so, if S is T-weakly commuting, $Sx = S^2x$ for some $x \in C(S,T)$, then the pair (S,T) has a common fixed point.*

Proof . Let x_0 be any given point in X. In generalized multi-valued (ψ, S, F)-contraction type I mappings with respect to a generalized dynamic process, a sequence can be formulated as follows:

$$D(S,T,x_0) = \{(x_n)_{n\in\mathbb{N}\cup\{0\}} : x_{n+1} = Sx_n \in Tx_{n-1}\ \forall\ n \in \mathbb{N}\}.$$

Observe that if there exists $n_0 \in \mathbb{N}$ such that $Sx_{n_0} = Sx_{n_0+1}$, then we have nothing to show. As such, we suppose that $d(Sx_n, Sx_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since

$$\begin{aligned} f\left(\frac{1}{2s}D(Sx_{n-1},Tx_{n-1}), D(Sx_{n-1},T(Sx_{n-1}))\right) &\leq \frac{1}{2s}D(Sx_{n-1},Tx_{n-1}) - \frac{D(Sx_{n-1},T(Sx_{n-1}))}{2} \\ &\leq \frac{1}{2s}d(Sx_{n-1},Sx_n) \\ &< d(Sx_{n-1},Sx_n), \end{aligned}$$

so, we have

$$\psi(M(x_{n-1},x_n)) + F(s^5 d(Sx_n,S_{n+1})) \leq F(M(x_{n-1},x_n)) + LN(x_{n-1},x_n), \tag{2.15}$$

where

$$M(x_{n-1},x_n)=\max\{d(Sx_{n-1},Sx_n),D(Sx_{n-1},Tx_{n-1}),D(Sx_n,Tx_n),\frac{D(Sx_{n-1},Tx_{n-1})D(Sx_n,Tx_n)}{s+d(Sx_{n-1},Sx_n)},$$
$$\frac{D(Sx_n,Tx_{n-1})[1+D(Sx_{n-1},Tx_{n-1})]}{s+d(Sx_{n-1},Sx_n)}\}$$
$$=\max\{d(Sx_{n-1},Sx_n),d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),\frac{d(Sx_{n-1},Sx_n)d(Sx_n,Sx_{n+1})}{s+d(Sx_{n-1},Sx_n)},$$
$$\frac{d(Sx_n,Sx_n)[1+d(Sx_{n-1},Sx_n)]}{s+d(Sx_{n-1},Sx_n)}\}$$
$$=\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),\frac{d(Sx_{n-1},Sx_n)d(Sx_n,Sx_{n+1})}{s+d(Sx_{n-1},Sx_n)}\}.$$

Now observe that $\frac{d(Sx_{n-1},Sx_n)}{s+d(Sx_{n-1},Sx_n)}<1$, which implies that $\frac{d(Sx_{n-1},Sx_n)d(Sx_n,Sx_{n+1})}{s+d(Sx_{n-1},Sx_n)}<d(Sx_n,Sx_{n+1})$ as such, we have that

$$M(x_{n-1},x_n)=\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\}$$

and

$$N(x_{n-1},x_n)=\min\{D(Sx_{n-1},Tx_{n-1}),D(Sx_n,Tx_n),D(Sx_{n-1},Tx_n),D(Sx_n,Tx_{n-1})\}$$
$$=\min\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_{n+1}),d(Sx_n,Sx_n)\}=0.$$

If we suppose that $M(x_{n-1},x_n)=\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\}=d(Sx_n,Sx_{n+1})$, then (2.15) becomes

$$F(s^5d(Sx_n,S_{n+1}))\leq F(d(Sx_n,Sx_{n+1}))-\psi(d(Sx_n,Sx_{n+1}))<F(d(Sx_n,Sx_{n+1})),$$

which is a contradiction, as such we have that $M(x_{n-1},x_n)=\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\}=d(Sx_{n-1},Sx_n)$, and so (2.15) becomes

$$F(s^5d(Sx_n,S_{n+1}))\leq F(d(Sx_{n-1},Sx_n))-\psi(d(Sx_{n-1},Sx_n)).$$

Using a similar approach, we also have that

$$F(s^5d(Sx_{n-1},S_n))\leq F(d(Sx_{n-2},Sx_{n-1}))-\psi(d(Sx_{n-2},Sx_{n-1})).$$

From the properties of ψ, there exists $c>0$ and $n_0\in\mathbb{N}$ such that $\psi(d(Sx_n,Sx_{n+1}))>c$ for all $n>n_0$. We obtain the following inequalities

$$\begin{aligned}F(s^5d(Sx_n,S_{n+1}))&\leq F(d(Sx_0,Sx_1))-(\psi(d(Sx_0,Sx_1))+\cdots+\psi(d(Sx_{n_0-1},Sx_{n_0})))\\&\quad-(\psi(d(Sx_{n_0},Sx_{n_0+1}))+\cdots+\psi(d(Sx_{n-1},Sx_n)))\\&\leq F(d(Sx_0,Sx_1))-(n-n_0)c.\end{aligned}\tag{2.16}$$

Since $F\in\mathcal{F}$, taking limit as $n\to\infty$ in (2.16) and using Lemma 1.8, we have

$$\lim_{n\to\infty}F(s^5d(Sx_n,Sx_{n+1}))=-\infty\Leftrightarrow\lim_{n\to\infty}d(Sx_n,Sx_{n+1})=0.\tag{2.17}$$

In what follows, we show that $\{Sx_n\}$ is a b-Cauchy sequence. Suppose that $\{Sx_n\}$ is not a b-Cauchy sequence, then by Lemma 2.1, there exist an $\epsilon>0$ and sequences of positive integers $\{Sx_{n_k}\}$ and

$\{Sx_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon, d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon$ and $(1)-(4)$ of Lemma 2.1 hold. Since

$$\begin{aligned} f\left(\frac{1}{2s}D(Sx_{m_k}, Tx_{m_k}), D(Sx_{m_k}, T(Sx_{m_k}))\right) &\leq \frac{1}{2s}D(Sx_{m_k}, Tx_{m_k}) - \frac{D(Sx_{m_k}, T(Sx_{m_k})}{2} \\ &\leq \frac{1}{2s}d(Sx_{m_k}, Sx_{m+1}) \\ &< \frac{\epsilon}{2s} < \epsilon \leq d(Sx_{m_k}, Sx_{n_k}), \end{aligned}$$

we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} \psi(M(x_{m_{k-1}}, x_{n_{k-1}})) + F(d(Sx_{m_k}, Sx_{n_k})) &\leq \psi(M(x_{m_{k-1}}, Sx_{n_{k-1}})) + F(s^5 d(Sx_{m_k}, Sx_{n_k})) \\ &\leq F(M(x_{m_{k-1}}, x_{n_{k-1}})) + L \min N(x_{m_{k-1}}, x_{n_{k-1}}) \end{aligned} \tag{2.18}$$

Since $F \in \mathcal{F}$, using Lemma 2.1, and (2.17), we have that

$$\begin{aligned} \liminf_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_{k-1}})) + F(s\epsilon) &= \liminf_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_{k-1}})) + F(s\epsilon) \\ &= \liminf_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_{k-1}})) + F(s^3 \frac{\epsilon}{s^2}) \\ &\leq \liminf_{k\to\infty} [\psi(M(x_{m_{k-1}}, x_{n_{k-1}}) + F(s^5 d(Sx_{m_k}, Sx_{n_k}))] \\ &\leq F(\liminf_{k\to\infty} d(Sx_{m_{k-1}}, Sx_{n_{k-1}})) \\ &\leq F(s\epsilon), \end{aligned}$$

where $0 < \liminf_{d(Sx_n, Sx)\to 0^+} \psi(d(Sx_{m_k}, Sx)) = \mu$. That is

$$\mu + F(s\epsilon) \leq F(s\epsilon)$$

which is a contradiction. We therefore have that $\{Sx_n\}$ is b-Cauchy in $S(X)$. Since $S(X)$ is complete, there exists $x \in S(X)$ such that $\lim_{n\to\infty} Sx_n = x$. In addition, there exists $x^* \in X$ such that $Sx^* = x$. We claim that x^* is the coincidence point for pair the pair (S, T). To establish our claim, we first show that

$$f\left(\frac{1}{2s}D(Sx_{n-1}, Tx_{n-1}), D(Sx_n, T(Sx_n))\right) < d(Sx_{n-1}, x^*)$$

or

$$f\left(\frac{1}{2s}D(Sx_n, Tx_n), D(Sx_{n+1}, T(Sx_n))\right) < d(Sx_n, x^*).$$

Suppose on the contrary that there exists $m \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} &f\left(\frac{1}{2s}D(Sx_{m-1}, Tx_{m-1}), D(Sx_m, T(Sx_m))\right) \geq d(Sx_{m-1}, x^*) \\ &\text{or } \ f\left(\frac{1}{2s}d(Sx_m, Tx_m), d(Sx_{m+1}, T(Sx_m))\right) \geq d(Sx_m, x^*). \end{aligned} \tag{2.19}$$

Now observe that

$$\begin{aligned} d(Sx_{m-1},x^*) &\leq f\left(\frac{1}{2s}D(Sx_{m-1},Tx_{m-1}),D(Sx_m,T(Sx_m))\right) \\ &\leq \frac{1}{2s}D(Sx_{m-1},Tx_{m-1})-\frac{1}{2}D(Sx_m,T(Sx_m)) \\ &\leq \frac{1}{2s}d(Sx_{m-1},Sx_m) \\ &\leq \frac{1}{2}d(Sx_{m-1},x^*)+\frac{1}{2}d(x^*,Sx_m), \end{aligned} \tag{2.20}$$

which implies that $d(Sx_{m-1},x^*) \leq d(x^*,Sx_m)$. It follows from (2.19) and (2.20), that

$$d(Sx_{m-1},x^*) \leq d(x^*,Sx_m) \leq f\left(\frac{1}{2s}d(Sx_m,Tx_m),d(Sx_{m+1},T(Sx_m))\right) \leq \frac{1}{2s}d(Sx_m,Sx_{m+1}). \tag{2.21}$$

Since $f\left(\frac{1}{2s}D(Sx_{m-1},Tx_{m-1}),D(Sx_m,T(Sx_m))\right) \leq d(Sx_{m-1},Sx_m)$, we have that

$$\begin{aligned} \psi(M(x_{m-1}),x_m)+F(d(Sx_m,Sx_{m+1})) &\leq \psi(M(x_{m-1},x_m))+F(s^5d(Tx_m,Tx_{m+1})) \\ &\leq F(M(x_{m-1},x_m))+LN(x_{m-1},x_m) \\ &\leq F(d(Sx_{m-1},Sx_m)). \end{aligned} \tag{2.22}$$

It follows that

$$\psi(d(Sx_{m-1},Sx_m))+F(d(Sx_m,Sx_{m+1})) \leq F(d(Sx_{m-1},Sx_m)). \tag{2.23}$$

Using the fact that F is strictly increasing, we have that

$$d(Sx_m,Sx_{m+1}) < d(Sx_{m-1},Sx_m).$$

Using this fact and (2.21), we have

$$\begin{aligned} d(Sx_m,Sx_{m+1}) &< d(Sx_{m-1},Sx_m) \\ &\leq sd(Sx_{m-1},x^*)+sd(x^*,Sx_m) \\ &\leq \frac{1}{2}d(Sx_m,Sx_{m+1})+\frac{1}{2}d(Sx_m,x_{m+1}) \\ &= d(Sx_m,Sx_{m+1}), \end{aligned} \tag{2.24}$$

which is a contradiction. Thus we must have that

$$f\left(\frac{1}{2s}d(Sx_{n-1},Tx_{n-1}),d(Sx_n,T(Sx_n))\right) < d(Sx_{n-1},x^*)$$

or

$$f\left(\frac{1}{2s}d(Sx_n,Tx_n),d(Sx_{n+1},T(Sx_n))\right) < d(Sx_{n+1},x^*).$$

Thus, we have that

$$\begin{aligned} \psi(M(x_n,x^*))+F(D(Sx_{n+1},Tx^*)) &\leq \psi(M(x_n,x^*))+F(s^5d(Sx_{n+1},Sx^*)) \\ &\leq F(M(x_n,x^*))+LN(x_n,x^*) \\ &< F(M(x_n,x)). \end{aligned}$$

Using the fact that $F \in \mathcal{F}$ and taking limit, we have that

$$\lim_{n\to\infty} M(x_n, x^*) = D(Sx^*, Tx^*) \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x^*) = 0,$$

so, we obtain

$$\psi(D(Sx^*, Tx^*)) + F(D(Sx^*, Tx^*)) < F(D(Sx^*, Tx^*)),$$

which is a contradiction, as such $D(Sx^*, Tx^*) = 0$ and so we obtain

$$Sx^* \in Tx^*.$$

Hence, x^* is the coincidence point for the pair (S, T).
Suppose that S is T-weakly commuting, for $x^* \in C(S, T)$, we have that $Sx^* = S^2x^*$. Using this fact and the fact that $Sx^* \in Tx^*$, we have that

$$Sx^* = S^2x^* = S(Sx^*) \in TSx^*,$$

we obtain that $Sx^* \in TSx^*$. Thus Sx^* is the common fixed point for the pair (S, T). This complete the proof. □

Theorem 2.6. *Let (X, d) be a complete b-metric space with $s \geq 1$, x_0 be an arbitrary point in X and $T : X \to CB(X)$ a generalized multi-valued (ψ, S, F)-contraction type II mapping with respect to generalized dynamic process $D(S, T, x_0)$ such that $S(X)$ is a complete subspace of X, then the pair (S, T) has a point of coincidence in X. More so, if S is T-weakly commuting, $Sx = S^2x$ for some $x \in C(S, T)$, then the pair (S, T) has a common fixed point.*

Proof . The prove follow a similar approach as in Theorem 2.5, as such we omit it. □

3. Applications

In this section, we establish the existence of a solution for a system of functional equations and a class of volterra integral type equations.

3.1. System of functional equations in dynamic programming:

Dynamic programming problem is made up of two critical components, the decision space and the state space. The state space is a set of parameter representing different states. This space include initial states, action states and transitional states. A decision space is the set of possible actions that can be taken to solve the problem. The problem of dynamic programming related to multistage process reduces to the problem of solving functional equations of the form:

$$p(x) = \sup_{y\in G}\{g(x, y) + D_1(x, y, p(\eta(x, y)))\}, \tag{3.1}$$

$$q(x) = \sup_{y\in G}\{f(x, y) + D_2(x, y, p(\eta(x, y)))\} \tag{3.2}$$

for all $x \in W$, where U, V are Banach spaces, $W \subseteq U, G \subseteq V$, and

$$\eta : W \times G \to W,$$
$$f, g : W \times G \to \mathbb{R},$$
$$D_1, D_2 : W \times G \times \to \mathbb{R}.$$

For details about dynamic programming see ([13, 14] and the references therein). Suppose that W and G are the state and decision spaces respectively. Our purpose is to establish the existence of a common and bounded solution of function equations](3.2) and (3.1). Suppose $B(W)$ denote the set of all bounded real valued function on W. For any $h, k \in B(W)$, define

$$d(h, k) = \sup_{x \in W} |h(x) - k(x)|^2.$$

It is well-known that $(B(W), d)$ is a complete b-metric space with $s = 2$.

Suppose the following conditions hold:

1. D_1, D_2, f and g are bounded.
2. For any $x \in W, h \in B(W)$ and $b > 0$, define $A, C : B(W) \to B(W)$ by

$$A(h(x)) = \sup_{y \in G}\{g(x, y) + D_1(x, y, h(\eta(x, y)))\}$$
$$C(h(x)) = \sup_{y \in G}\{f(x, y) + D_2(x, y, h(\eta(x, y)))\}.$$

   More so, suppose that $\psi : [0, \infty) \to [0, \infty)$ such that for every $(x, y) \in W \times G, h, k \in B(W)$ and $t \in W$ implies that

$$|D_1(x, y, h(t)) - D_2(x, y, k(t))| \leq \sqrt{\frac{M(h(t), k(t))}{e^{\psi(M(h(t),k(t)))s^3}}}, \tag{3.3}$$

   where $M(h(t), k(t)) = \max\{d(Ch(t), Ck(t)), d(Ch(t), Ak(t)), d(Ch(t), Ak(t)),$ $\frac{d(Ch(t),Ak(t))d(Ch(t),Ak(t))}{s+d(Ch(t),Ck(t))}, \} \frac{d(Ch(t),Ah(t))[1+d(Ch(t),Ah(t))]}{s+d(Ch(t),Ck(t))}$.
3. For any $h \in B(W)$, there exists $k \in B(W)$ such that $x \in W$,

$$A(h(x)) = C(k(x)).$$

4. There exists $h \in B(W)$ such that

$$A(h(x)) = C(h(x)) \Rightarrow A(C(h(x))) = C(A(h(x))).$$

Theorem 3.1. *Suppose that the conditions* $(1) - (4)$ *are satisfied and* $C(B(W))$ *is a closed convex subspace of* $B(W)$, *then the functional equation* (3.1) *and* (3.2) *have a bounded solution.*

Proof . Let γ be an arbitrary positive number and $h_1, h_2 \in B(W), x \in W$ and $y_1, y_2 \in G$ such that

$$Ah_1 < g(x, y_1) + D_1(x, y_1, h_1(\eta(x, y_1))) + \gamma, \tag{3.4}$$

$$Ah_2 < g(x, y_2) + D_2(x, y_2, h_2(\eta(x, y_2))) + \gamma, \tag{3.5}$$

$$Ah_1 \geq g(x, y_2) + D_1(x, y_2, h_1(\eta(x, y_2))), \tag{3.6}$$

$$Ah_2 \geq g(x, y_1) + D_2(x, y_1, h_2(\eta(x, y_1))). \tag{3.7}$$

Using (3.4), (3.7) and (3.3), we have that

$$\begin{aligned} Ah_1(x) - Ah_2(x) &< D_1(x, y_1, h_1(\eta(x, y_1))) - D_2(x, y_1, h_2(\eta(x, y_1))) + \gamma \\ &\leq |D_1(x, y_1, h_1(\eta(x, y_1))) - D_2(x, y_1, h_2(\eta(x, y_1)))| + \gamma \\ &\leq \sqrt{\frac{M(h_1(x), h_2(x))}{e^{\psi(M(h_1(x), h_2(x)))} s^3}}. \end{aligned} \tag{3.8}$$

Also (3.5), (3.6) and (3.3), we have that

$$\begin{aligned} Ah_2(x) - Ah_1(x) &\leq D_2(x, y_2, h_2(\eta(x, y_2))) - D_1(x, y_2, h_1(\eta(x, y_2))) + \gamma \\ &\leq |D_1(x, y_2, h_1(\eta(x, y_2))) - D_2(x, y_2, h_2(\eta(x, y_2)))| + \gamma \\ &\leq \sqrt{\frac{M(h_1(x), h_2(x))}{e^{\psi(M(h_1(x), h_2(x)))} s^3}}. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have that

$$|Ah_1(x) - Ah_2(x)|^2 \leq \frac{M(h_1(x), h_2(x))}{e^{\psi(M(h_1(x), h_2(x)))} s^3}, \tag{3.10}$$

which implies that

$$\begin{aligned} e^{\psi(M(h_1(x), h_2(x)))} s^3 d(Ah_1(x), Ah_2(x)) &\leq M(h_1(x), h_2(x)), \\ \psi(M(h_1(x), h_2(x))) + \ln(s^3 d(Ah_1(x), Ah_2(x))) &\leq \ln(M(h_1(x), h_2(x))), \end{aligned}$$

taking $F(x) = \ln(x)$, we have that

$$\psi(M(h_1(x), h_2(x))) + F(s^3 d(Ah_1(x), Ah_2(x))) \leq F(M(h_1(x), h_2(x))).$$

It is easy to see that all the conditions in Theorem 2.6 are satisfied and thus the pair (A, C) has a common fixed point h^*, that is $h^*(x)$ is a bounded solution of (3.1) and (3.2). □

3.2. Existence of solution for a class of Volterra type integral inclusion:

In this section, we apply our fixed point result to the following Volterra type integral equations:

$$x(t) = \int_0^1 G_1(t, s, x(s))ds + g(t), \tag{3.11}$$

$$y(t) = \int_0^1 G_2(t, s, y(s))ds + f(t), \tag{3.12}$$

where $G_1, G_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0,1] \to \mathbb{R}$ are continuous functions. Let $C([0,1], \mathbb{R})$ be the space of all continuous function defined on $[0,1]$ endowed with the b-metric as defined by

$$d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|^2.$$

It is well-know that (X, d) is a complete b-metric space with $s = 2$. We define

$$Tx(t) = \int_0^1 G_1(t, s, x(s))ds + g(t),$$

$$Ty(t) = \int_0^1 G_2(t, s, y(s))ds + f(t).$$

Theorem 3.2. *Let $X = C([0,1], \mathbb{R})$ and suppose the following conditions hold:*

1. *suppose there exist $\psi : [0,\infty) \to [0,\infty)$ and $\Gamma : X \to [0,\infty)$ such that*

$$|G_1(t, s, x) - G_2(t, s, y)| \leq \Gamma(x(s))e^{-\psi(M(x,y))}\sqrt{\frac{M(x,y)}{s^3}}, \tag{3.13}$$

for all $t, s \in [0,1]$ and $x, y \in X$, where $\left(\int_0^1 \Gamma(x(s))e^{-\psi(M(x,y))}ds\right)^2 \leq e^{-\psi(M(x,y))}$ and $M(x,y) = \max\{|Sx - Sy|, |Sx - Tx|, |Sx - Ty|, \frac{|Sx-Tx||Sy-Ty|}{s+|Sx-Sy|}, \frac{|Sy-Tx|[1+|Sx-Tx|]}{s+|Sx-Sy|}\}$;

2. *there exists $x \in X$ such that $Tx(t) = Sx(t)$, which implies that $TSx(t) = STx(t)$.*

Then the system of integral equations (3.11) and (3.12) has a solution.

Proof .

$$\begin{aligned}
|Tx(t) - Ty(t)|^2 &\leq \left(\int_0^1 |G_1(t, s, x(s)) - G_2(t, s, y(s))|ds\right)^2 \\
&\leq \left(\int_0^1 \Gamma(x(s))e^{-\psi(M(x,y))}\sqrt{\frac{M(x,y)}{s^3}}ds\right)^2 \\
&= \frac{M(x,y)}{s^3}\left(\int_0^1 \Gamma(x(s))e^{-\psi(M(x,y))}ds\right)^2 \\
&\leq \frac{M(x,y)}{s^3}e^{-\psi(M(x,y))},
\end{aligned}$$

which implies that

$$\begin{aligned}
e^{\psi(M(x,y))}s^3 d(Tx, Ty) &\leq M(x,y) \\
\psi(M(x,y)) + \ln(s^3 d(Tx, Ty)) &\leq \ln(M(x,y)),
\end{aligned}$$

taking $F(x) = \ln(x)$, we have

$$\psi(M(x,y)) + F(s^3 d(Tx, Ty)) \leq F(M(x,y)).$$

It is easy to see that all the conditions in Theorem 2.6 are satisfied. Thus, the system of integral equation (3.11) and (3.12) has a common solution.

□

Declaration

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