Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2085-2107
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Epidemiological model Involving Two Diseases in Predator Population with Holling Type-II Functional Response

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, two types of diseases in the predator population in an ecological model are proposed and analyzed. The first (SIS infectious disease) transmitted horizontally, spread by contact between susceptible individuals and infected individuals. And the second (SI disease) is transmitted vertically from mothers to offspring with the effect of an external source (environmental effect). No transmission of the diseases can happen from predator to prey by predation or contact. Linear functional response and Holing type-II for describing the predation of the susceptible and the infected predators respectively also linear incidence for describing the transition of diseases are used. All possible equilibrium points were analyzed for this model. Locally and globally dynamics of the model have been discussed, numerical simulation is used to investigate the effect of the diseases on the system's dynamics.

Keywords: Eco-epidemiological model, SI disease, SIS epidemic disease, Prey-predator model, Lyapunov function.

1. Introduction

The previous belief was that only humans face epidemics. It turns out that animals, especially wild ones, face a range of diseases, from Ebola to cancer and even the plague. In recent decades, the extent of the impact of epidemics and their spread on both humans and animals has become

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Received: March 2021 *Revised:* July 2021

clear and the extent of the danger of these epidemics on life in general. Scientific studies in their medical or other aspects have become obligated to find a clear map with clear details of the course of these epidemics and the relationship of these epidemics in humans or animals at the same level, because of these epidemics that there is a mutual risk in some of them as a result of some epidemics related to infection and development between races of the same sex or mixing direct and indirect between different ethnicities. The study of disease prevalence between humans and animals is known as the epidemiological model, and there are several researchers who have independently studied the dynamics of mathematical environmental and epidemiological models [1, 8, 9, 10, 11, 12, 15, 18, 13]. Anderson and May [2] studied the link between the Lotka-Volterra prey-predator model with the infectious disease as well as the prevalence by contact among the population of prey without reproduction in infected prey.

Many types of epidemics are discussed on ecological models contain SI, SIS and SIR epidemic disease in one species for example [17, 20, 14], while researchers study spread of two epidemic disease in the same species for example [21, 6]. Further diseases can be spread in different ways among the individuals of the population, one of the most common way to prevalence infectious disease among infected and susceptible is by contact in the population, but there are no doubt that the environment play a vital role of spread these diseases which called an external sources for example of these researches are [16, 5].

The most recent models take into account random environments [7], and sometimes periodic structures [3]. The novelty in the study of two epidemics in the predator population is the destination of Fabio et al. [19].

In this paper, an eco- epidemiological model containing two types of diseases in the predator population have been presented, the first SIS with horizontal transmission and the second SI with vertical transmission, without intersection with each other in the same individuals of a predator population. Moreover, there is no spread of disease between predators and prey.

2. The mathematical model formulation

The ecosystem study proposed in this section included a prey, $P(T)$ a total population density of prey at time T, which interacting with susceptible predators $S(T)$ a total population at time T, $H(T)$ infected predators at time T with disease (SIS) and $V(T)$ infected predators at time T with disease (SI). A cross between diseases cannot occur in the same individual from a predator population. There is no spread of diseases between prey and predators. Moreover, the first disease (SIS) is transmitted horizontally between individuals of the predator population by contact. The second disease (SI) spreads vertically and also the influence of the external source (environmental sources) on the occurrence of this disease, the following assumptions are now adopted in formulating the basic environmental epidemiology model.

(1) The prey species, reproduction logistically with carrying capacity k_p, $(k_p > 0)$ and intrinsic growth rate denoted by r, $(r > 0)$.

(2) There are (SIS and SI diseases) in the predator population which divided the population in to the following:

  (i) According to the Lotka–Voltera type of functional response susceptible predators consumed the prey via predation rate $a < 0$ and participate part of this food with conversion rate $0 < e_1 < 1$ with natural rate of death due to absence of prey, $d_1 > 0$.

  (ii) The first disease (SIS disease) is passing within the same species by contact (horizontally) with an infected individual at infection rate $\theta > 0$ and recovery rate $\gamma > 0$, (means that the infected individual becomes susceptible again). Furthermore, the infected predators consumed the prey individuals according to Hollying type-II of functional response with maximum attack rate $c_1 > 0$ and half saturation $b_1 > 0$ that participate part of this food with conversion rate $0 < e_2 < 1$. with natural rate of death in absence of prey , $d_2 > 0$.

(iii) The second disease (SI disease) has the ability to pass vertically from mothers to new individuals (vertically) at an infection rate $\beta > 0$. As well as to the effect of an external source (environmental influence) that causes disease among predators with an external source rate $\alpha > 0$. The infected predators consumed the prey individuals according to Hollying type-II of functional response with maximum attack rate $c_2 > 0$ and half saturation $b_2 > 0$ that participate part of this food with conversion rate $0 < e_3 < 1$, with rate of death due to infected disease $d_3 > 0$.

In consonance to the previous assumptions, the following set of equations can represented the proposed model.

$$\begin{aligned}
\frac{\mathrm{dP}}{\mathrm{dT}} &= rP\left(1-\frac{P}{k_p}\right) - aPS - \frac{C_1\mathrm{PH}}{b_1+P} - \frac{C_2\mathrm{PV}}{b_2+P},\\
\frac{\mathrm{dS}}{\mathrm{dT}} &= e_1aPS + \gamma H - \theta SH - \beta SV - \alpha S - d_1S,\\
\frac{\mathrm{dH}}{\mathrm{dT}} &= \theta SH - \gamma H - d_2H + \frac{e_2C_1\mathrm{PH}}{b_1+P}\ ,\\
\frac{\mathrm{dV}}{\mathrm{dT}} &= \beta SV + \alpha S - d_3V + \frac{e_3C_1\mathrm{PV}}{b_{21}+P}.
\end{aligned} \tag{1.1}$$

Accompanied by initial conditions $P(0) \geq 0,\quad S(0) \geq 0,\quad H(0) \geq 0,\quad V(0) \geq 0$, that there are seventeen parameters which can be reduced to make the model easy to deal with it by dimensionless parameters and variables to simplify the system.

$$t = rT,\qquad p = \frac{P}{K_p},\qquad s = \frac{S}{K_p},\qquad h = \frac{H}{K_p},\qquad v = \frac{V}{K_p},$$

$$u_1 = \frac{a\ k_p}{r},\quad u_2 = \frac{c_1}{r},\quad u_3 = \frac{c_2}{r},\quad u_4 = \frac{b_1}{k_p},\quad u_5 = \frac{b_2}{k_p},\quad u_6 = \frac{e_1ak_p}{r},\quad u_7 = \frac{\gamma}{r},\quad u_8 = \frac{\theta k_p}{r},$$

$$u_9 = \frac{\beta k_p}{r},\quad u_{10} = \frac{\alpha}{r},\quad u_{11} = \frac{d_1}{r},\quad u_{12} = \frac{d_2}{r},\quad u_{13} = \frac{e_2c_1}{r},\quad u_{14} = \frac{d_3}{r},\quad u_{15} = \frac{e_3c_2}{r}$$

By accordance with the following dimensionless system:

$$\begin{aligned}
\frac{\mathrm{dp}}{\mathrm{dt}} &= p(1-p) - u_1ps - \frac{u_2\mathrm{ph}}{u_4+p} - \frac{u_3\mathrm{pv}}{u_5+p} = f_1(p,s,h,v),\\
\frac{\mathrm{ds}}{\mathrm{dt}} &= u_6ps + u_7h - u_8sh - u_9sv - (u_{10}+u_{11})s = f_2(p,s,h,v),\\
\frac{\mathrm{dh}}{\mathrm{dt}} &= u_8sh - u_7h - u_{12}h + \frac{u_{13}\mathrm{ph}}{u_4+p} = f_3(p,s,h,v),\\
\frac{\mathrm{dv}}{\mathrm{dt}} &= u_9sv + u_{10}s - u_{14}v + \frac{u_{15}\mathrm{pv}}{u_5+p} = f_4(p,s,h,v).
\end{aligned} \tag{1.2}$$

With $p(0) \geq 0,\ s(0) \geq 0,\ h(0) \geq 0,\ v(0) \geq 0$. Note that there is reduced in number of the parameters from seventeen in the system (2.1) to fifteen in the system (1.2). It is easy to exam about all the functions of the system (1.2) are continuous and have continuous partial derivatives on the following positive four dimensional space $R_+^4 = \left\{(p, s, h, v) \in R_+^4 : \ p(0) \geq 0, \quad s(0) \geq 0, \quad h(0) \geq 0, \quad v(0) \geq 0\right\}$. So the solution of the system (1.2) exists and unique. Moreover with the non-negative initial conditions all the solutions of the system (1.2)) are uniformly bounded as illustrated in the following theorem:

Theorem 1.1. *All the solutions of the system* (1.2) *which initiate in* R_+^4 *are uniformly bounded.*
Proof . *let* $(p(t), s(t), h(t), v(t))$ *be any solution of the system* (1.2) *with non-negative initial condition* $(p_0, s_0, \ h_0, \ v_0) \in R_+^4$.

From 1^{st} *equation of system* (1.2) *we have:*

$$\frac{dp}{dt} \leq p\,(1-p)\,.$$

Through the theory of differential inequality [4], we get:

$$\lim_{t\to\infty} \sup p(t) \leq 1$$

Define the function

$$M\,(t) = p\,(t) + s\,(t) + h\,(t) + v(t).$$

Therefore,

$$\frac{dM}{dt} = p\,(1-p) - (u_1 - u_6)ps - (u_2 - u_{13})\frac{ph}{u_4 + p} - (u_3 - u_{15})\frac{pv}{u_5 + p} - u_{12}h - \ u_{14}v - u_{11}s.$$

So, according to the biological facts always $u_1 > u_6$, $u_2 > u_{13}$, $u_3 > u_{15}$ *we get:*

$$\frac{dM}{dt} \leq 2p - (p + u_{11}s + u_{12}h + u_{14}v),$$

therefore $\frac{dM}{dt} \leq 2 - sM$, *where* $D = \min\{1, \quad u_{11}, \quad u_{12}, \quad u_{14}\}$, *then* $M\,(t) \leq \frac{2}{D} + (M_0 - \frac{2}{D})e^{-Dt}$.
Then

$$\lim_{t\to\infty} M(t) \leq \frac{2}{D},$$

so $0 \leq M\,(t) \leq \frac{2}{D}, \quad \forall\ t > 0.$

Hence the solutions of the system (1.2) *are uniformly bounded.* □

2. The existence of equilibrium points

In this section, it appears there are at most in system (1.2) six equilibrium points which will be studied of the stability at each of these points, explicit computation appears as follow:

(i) $E_0 = (0, 0, 0, 0)$ exists always.
(ii) The equilibrium point $E_1 = (1, 0, 0, 0)$ exists always.
(iii) The equilibrium point $E_2 = (\widehat{p}, \widehat{s}, 0, 0)$ where, $\widehat{p} = \frac{u_{11}}{u_6}$, and $\widehat{s} = \frac{u_6 - u_{11}}{u_1 u_6}$, exists provided that:

$$u_6 > u_{11}, \tag{2.1}$$

(iv) The Equilibrium Point $E_3 = \left(\bar{p}, \bar{s}, \bar{h}, 0\right)$
$\bar{p}$ is unrivaled and positive solution of the following equation:

$$A_1 p^3 + A_2 p^2 + A_3 p + A_4 = 0, \tag{2.2a}$$

where,

$$\begin{aligned}
A_1 &= u_8\left(u_{12} - u_{13}\right),\\
A_2 &= \left(u_7 + u_{12} - u_{13}\right)\left[u_2 u_6 - u_8\left(1 - u_4\right) + u_1\left(u_{12} - u_{13}\right)\right] + u_8[u_7\left(1 - u_4\right) + u_4 u_{12}],\\
A_3 &= \left(u_7 + u_{12} - u_{13}\right)\left[u_1 u_4\left(u_7 + 2u_{12}\right) - u_4 u_8 - u_2\left(u_{10} + u_{11}\right)\right]\\
&\quad + u_4 u_7 u_8\left(2 - u_4\right) + u_4\left(u_7 + u_{12}\right)\left[u_2 u_6 - u_1 u_7 - u_8\left(1 - u_4\right)\right],\\
A_4 &= u_4^2 u_7 u_8 - u_4\left(u_7 + u_{12}\right)\left[u_4\left(u_8 - u_1 u_{12}\right) + u_2\left(u_{10} + u_{11}\right)\right].\\
\bar{s} &= \frac{\left(u_7 + u_{12} - u_{13}\right)\bar{p} + u_4\left(u_7 + u_{12}\right)}{u_8\left(u_4 + \bar{p}\right)}, \qquad \bar{h} = \frac{\left(u_4 + \bar{p}\right)\left[1 - \bar{p} - u_1 \bar{s}\right]}{u_2}
\end{aligned}$$

Exist provided the following conditions:

$$u_{12} > u_{13} \tag{2.2b}$$
$$\bar{p} < 1, \tag{2.2c}$$
$$\bar{s} < \frac{1 - \bar{p}}{u_1}, \tag{2.2d}$$
$$u_4 < 1, \tag{2.2e}$$
$$u_2 u_6 + u_1\left(u_{12} - u_{13}\right) < u_8\left(1 - u_4\right), \tag{2.2f}$$
$$\left(u_7 + u_{12} - u_{13}\right)\left[u_2 u_6 + u_1\left(u_{12} - u_{13}\right) - u_8\left(1 - u_4\right)\right] < -u_8[u_7\left(1 - u_4\right) + u_4 u_{12}], \tag{2.2g}$$
$$u_2 u_6 < u_1 u_7 + u_8\left(1 - u_4\right), \tag{2.2h}$$
$$u_1 u_4\left(u_7 + 2u_{12}\right) < u_4 u_8 + u_2\left(u_{10} + u_{11}\right), \tag{2.2i}$$
$$\begin{aligned}&\left(u_7 + u_{12} - u_{13}\right)\left[u_1 u_4\left(u_7 + 2u_{12}\right) - u_4 u_8 - u_2\left(u_{10} + u_{11}\right)\right] + u_4\left(u_7 + u_{12}\right) \star\\ &\left[u_2 u_6 - u_1 u_7 - u_8\left(1 - u_4\right)\right] > -\, u_4 u_7 u_8\left(2 - u_4\right),\end{aligned} \tag{2.2j}$$
$$u_8 > u_1 u_{12}, \tag{2.2k}$$
$$u_4^2 u_7 u_8 < u_4\left(u_7 + u_{12}\right)\left[u_4\left(u_8 - u_1 u_{12}\right) + u_2\left(u_{10} + u_{11}\right)\right], \tag{2.2l}$$

(v) The equilibrium point $E_4 = \left(\bar{\bar{p}}, \bar{\bar{s}}, 0, \bar{\bar{v}}\right)$,
$\bar{\bar{p}}$ is unrivaled and positive solution of the following equation:

$$B_1 p^3 + B_2 p^2 + B_3 p + B_4 = 0, \tag{2.3a}$$

Where,

$$\begin{aligned}
B_1 &= -u_6 u_9,\\
B_2 &= u_6[u_9\left(1 - u_5\right) - u_3 u_6 - u_1\left(u_{14} - u_{15}\right)] + u_9 u_{11},\\
B_3 &= u_5 u_6\left(u_9 - u_1 u_{14}\right) + u_{10}[u_9\left(1 - u_5\right) - u_3 u_6] - \left(u_{10} + u_{11}\right)\left[u_9\left(1 - u_5\right) - 2u_3 u_6 - u_1\left(u_{14} + u_{15}\right)\right],\\
B_4 &= u_5 u_9 u_{10} - \left(u_{10} + u_{11}\right)\left[u_5\left(u_9 - u_1 u_{14}\right) + u_3 u_{11}\right],\\
\bar{\bar{v}} &= \frac{u_6 \bar{\bar{p}} - \left(u_{10} + u_{11}\right)}{u_9}, \qquad \bar{\bar{s}} = \frac{\left(1 - \bar{\bar{p}}\right)\left(u_5 + \bar{\bar{p}}\right) - u_3 \bar{\bar{v}}}{u_1\ \left(u_5 + \bar{\bar{p}}\right)}
\end{aligned}$$

Exists provided the following conditions:

$$\frac{u_{10}+u_{11}}{u_6} < \bar{\bar{p}} < 1, \tag{2.3b}$$

$$\left(1-\bar{\bar{p}}\right)\left(u_5+\bar{\bar{p}}\right) > u_3\bar{\bar{v}}, \tag{2.3c}$$

$$u_5 < 1, \tag{2.3d}$$

$$u_9\left(1-u_5\right) > 2u_3u_6 + u_1\left(u_{14}-u_{15}\right), \tag{2.3e}$$

$$u_{14} > u_{15}, \tag{2.3f}$$

$$u_9 > u_1u_{14}, \tag{2.3g}$$

$$u_5u_6\left(u_9-u_1u_{14}\right) + u_{10}[u_9\left(1-u_5\right) - u_3u_6] > \left(u_{10}+u_{11}\right)[u_9\left(1-u_5\right) - 2u_3u_6 - u_1\left(u_{14}+u_{15}\right)], \tag{2.3h}$$

$$u_5u_9u_{10} > \left(u_{10}+u_{11}\right)[u_5\left(u_9-u_1u_{14}\right)+u_3u_{11}]. \tag{2.3i}$$

(vi) The positive equilibrium point $E_5 = \left(\widetilde{p}, \widetilde{s}, \widetilde{h}, \widetilde{v}\right)$,
$\widetilde{p}$ is unrivaled and positive solution of the following equation:

$$F_1p^7 + F_2p^6 + F_3p^5 + F_4p^4 + F_5p^3 + F_6p^2 + F_7p + F_8 = 0, \tag{2.4a}$$

Where,

$F_1 = R_{11}\left(u_7R_3 - u_8R_1\right),$
$F_2 = R_{18} + \left(R_3R_{12} + R_4R_{11}\right) - u_8\left(R_1R_{12} + R_2R_{11}\right),$
$F_3 = R_{19} + \left(R_3R_{13} + R_4R_{12}\right) - u_8\left(R_1R_{13} + R_2R_{12}\right) + R_{24},$
$F_4 = R_{20} + \left(R_3R_{14} + R_4R_{13}\right) - u_8\left(R_1R_{14} + R_2R_{13}\right) + R_{25},$
$F_5 = R_{21} + \left(R_3R_{15} + R_4R_{14}\right) - u_8\left(R_1R_{15} + R_2R_{14}\right) + R_{26},$
$F_6 = R_{22} + \left(R_3R_{16} + R_4R_{15}\right) - u_8\left(R_1R_{16} + R_2R_{15}\right) + R_{27},$
$F_7 = R_{23} + \left(R_3R_{17} + R_4R_{16}\right) - u_8\left(R_1R_{17} + R_2R_{16}\right) + R_{28},$
$F_8 = R_{17}\left(R_4 - u_8R_2\right) + R_{29},$
$R_1 = u_7 + u_{12} - u_{13},$
$R_2 = u_4\left(u_7 + u_{12}\right),$
$R_3 = u_8,$
$R_4 = u_4u_8,$
$R_5 = u_{10}R_1,$
$R_6 = u_{10}\left(u_5R_1 + R_2\right),$
$R_7 = u_5u_{10},$
$R_8 = R_3\left(u_{14} - u_{15}\right) - u_9R_1,$
$R_9 = u_{14}\left(u_5R_3 + R_4\right) - u_9\left(u_5R_1 + R_3\right) - u_{15}R_4,$
$R_{10} = u_5\left(u_{14}R_4 - u_9R_2\right),$
$R_{11} = -u_7R_3R_8,$
$R_{12} = -u_7[R_8[\left(u_4R_3 + R_4\right) + u_1R_1 - R_3\left(1 - u_5\right)] + R_3R_9],$
$R_{13} = u_7[R_8[u_5R_3 + \left(u_4R_3 + R_4\right)\left(1 - u_5\right) - u_1u_5R_1 - u_4R_4 - u_1\left(u_4R_1 + R_2\right)] - R_9[R_3\left(1 - u_5\right)$
$+ \left(u_4R_3 + R_4\right) + u_1R_1] - R_3\left(R_{10} + u_3R_5\right)],$

$$R_{14}=u_7[R_8[u_5(u_4R_3+R_4)+u_4R_4(1-u_5)-u_1u_5(u_4R_1+R_2)-u_1u_4R_2]$$
$$+R_9[u_5R_3+(u_4R_3+R_4)(1-u_5)-u_1u_5R_1-u_4R_4-u_1(u_4R_1+R_2)]$$
$$+R_{10}[(u_4R_3+R_4)+u_1R_1-R_3(1-u_5)]-u_3[R_3R_6+R_5(u_4R_3+R_4)]],$$
$$R_{15}=u_7[u_4u_5R_8(R_4-u_1R_2)+R_9[u_5(u_4R_3+R_4)+u_4R_4(1-u_5)-u_1u_5(u_4R_1+R_2)-u_1u_4R_4]$$
$$+R_{10}[u_5R_3+(u_4R_3+R_4)(1-u_5)-u_1u_5R_1+u_4R_4-u_1(u_4R_1+R_2)]-u_3[R_3(R_7$$
$$+u_4R_6)+R_4(R_6+u_4R_5)]],$$
$$R_{16}=u_7[u_4u_5R_9(R_4-u_1R_2)+R_{10}[u_5(u_4R_3+R_4)-u_4R_4(1-u_5)-u_1[u_5(u_4R_1+R_2)+u_4R_2]]$$
$$-u_3[R_7(u_4R_3+R_4)+u_4R_6]],$$
$$R_{17}=u_4u_7[u_5R_{10}(R_4-u_1R_2)-u_3R_4R_7],$$
$$R_{18}=u_2u_6R_1R_3R_8,$$
$$R_{19}=u_2u_6[R_8[(R_1R_4+R_2R_3)+u_5R_1R_3]+R_1R_3R_9],$$
$$R_{20}=u_2u_6[R_8[R_2R_4+u_5(R_1R_4+R_2R_3)]+R_9[(R_1R_4+R_2R_3)+u_5R_1R_3]+R_1R_3R_{10}],$$
$$R_{21}=u_2u_6[u_5R_2R_4R_8+R_9[R_2R_4+u_5(R_2R_3+R_1R_4)]+R_{10}[(R_1R_4+R_2R_3)+u_5R_1R_3]],$$
$$R_{22}=u_2u_6[u_5R_2R_4R_9+R_{10}[R_2R_4+u_5(R_1R_4+R_2R_3)]],$$
$$R_{23}=u_2u_5u_6R_2R_4R_{10},$$
$$R_{24}=-u_2R_1R_3[(u_{10}+u_{11})R_8+u_9R_5],$$
$$R_{25}=-u_2[(u_{10}+u_{11})[R_8[R_2R_3+R_1(u_5R_3+R_4)+R_1R_3R_9]]+u_9[R_3(R_1R_6+R_2R_5)$$
$$+R_1R_5(u_5R_3+R_4)]],$$
$$R_{26}=-u_2[(u_{10}+u_{11})[R_8[R_2(u_5R_3+R_4)+u_5R_1R_4]+R_9[R_2R_3+R_1(u_5R_3+R_4)]+R_1R_3R_{10}]$$
$$+u_9[R_3(R_1R_7+R_2R_6)+(u_5R_3+R_4)(R_1R_6+R_2R_5)+u_5R_1R_4R_5]],$$
$$R_{27}=-u_2[(u_{10}+u_{11})[u_5R_2R_4R_8+R_9[u_5(R_1R_4+R_2R_3)+R_2R_4]+R_{10}[R_2R_3+R_1(u_5R_3+R_4)]]$$
$$+u_9[(R_1R_7+R_2R_6)(u_5R_3+R_4)+u_5R_4(R_1R_6+R_2R_5)]],$$
$$R_{28}=-u_2[(u_{10}+u_{11})[u_5R_2R_4R_9+R_{10}[R_2(u_5R_3+R_4)+u_5R_1R_4]]+u_9[R_2R_7(u_5R_3+R_4)$$
$$+u_5R_4(R_1R_7+R_2R_6)]],$$
$$R_{29}=-u_2R_2R_4[(u_{10}+u_{11})R_{10}+u_9R_7],$$
$$\check{s}=\frac{u_4(u_7+u_{12})+(u_7+u_{12}-u_{13})\widetilde{p}}{u_8(u_4+\check{p})}\quad,\quad \widetilde{v}=\frac{u_{10}\widetilde{s}}{u_{14}-u_9\widetilde{s}-\frac{u_{15}\widetilde{p}}{u_5+\widetilde{p}}}\ \text{ and}$$
$$\widetilde{h}=\frac{(u_4+\widetilde{p})[(u_5+\widetilde{p})(1-\widetilde{p}-u_1\widetilde{s})-u_3\widetilde{v}]}{u_2(u_5+\widetilde{p})}$$

Exist if in addition to the conditions (2.2b), (2.2b) and (2.2f), the following conditions hold:

$$u_{14} > u_9\widetilde{s} + \frac{u_{15}\widetilde{p}}{u_5+\widetilde{p}}, \tag{2.5a}$$

$$1 > \widetilde{p} + u_1\widetilde{s}, \tag{2.5b}$$

$$(u_5+\widetilde{p})(1-\widetilde{p}-u_1\widetilde{s}) > u_3\widetilde{v}, \tag{2.5c}$$

$$R_3(u_{14}-u_{15}) > u_9R_1, \tag{2.5d}$$

$$u_{14}(u_5R_3+R_4) > u_9(u_5R_1+R_2)+u_{15}R_4, \tag{2.5e}$$

$$u_{14}R_4 > u_9R_2, \tag{2.5f}$$

$$(u_4R_3+R_4)+u_1R_1 > R_3(1-u_5), \tag{2.5g}$$

$$u_5R_3+(u_4R_3+R_4)(1-u_5) > u_1u_5R_1+u_4R_4+u_1(u_4R_1+R_2), \tag{2.5h}$$

$$u_1R_1+(u_4R_3+R_4) > R_3(1-u_5), \tag{2.5i}$$

$$\begin{aligned}&R_8[u_5R_3+(u_4R_3+R_4)(1-u_5)-u_1u_5R_1-u_4R_4-u_1(u_4R_1+R_2)]\\&>R_9[u_1R_1+(u_4R_3+R_4)-R_3(1-u_5)]+R_3(R_{10}+u_3R_5),\end{aligned} \tag{2.5j}$$

$$u_5(u_4R_3+R_4)+u_4R_4(1-u_5) > u_1u_5(u_4R_1+R_2)+u_1u_4R_2, \tag{2.5k}$$

$$\begin{aligned}&R_8[u_5(u_4R_3+R_4)+u_4R_4(1-u_5)-u_1u_5(u_4R_1+R_2)-u_1u_4R_2]\\&+R_9[u_5R_3+(u_4R_3+R_4)(1-u_5)-u_1u_5R_1-u_4R_4-u_1(u_4R_1+R_2)]\\&>u_3[R_3R_6+R_5(u_4R_3+R_4)]-R_{10}[(u_4R_3+R_4)+u_1R_1-R_3(1-u_5)],\end{aligned} \tag{2.5l}$$

$$R_4 > max\{u_1R_2,\quad u_8R_2\}, \tag{2.5m}$$

$$\begin{aligned}&u_4u_5R_8(R_4-u_1R_2)+R_9[u_5(u_4R_3+R_4)+u_4R_4(1-u_5)-u_1u_5(u_4R_1+R_2)-u_1u_4R_4]\\&+R_{10}[u_5R_3+(u_4R_3+R_4)(1-u_5)-u_1u_5R_1+u_4R_4-u_1(u_4R_1+R_2)]\\&>u_3[R_3(R_7+u_4R_6)+R_4(R_6+u_4R_5)],\end{aligned} \tag{2.5n}$$

$$u_5(u_4R_3+R_4)+u_4R_2 > u_4R_4(1-u_5)+u_1[u_5(u_4R_1+R_2)], \tag{2.5o}$$

$$\begin{aligned}&u_4u_5R_9(R_4-u_1R_2)+R_{10}[u_5(u_4R_3+R_4)-u_4R_4(1-u_5)-u_1[u_5(u_4R_1+R_2)+u_4R_2]]\\&>u_3[R_7(u_4R_3+R_4)+u_4R_6],\end{aligned} \tag{2.5p}$$

$$u_5R_{10}(R_4-u_1R_2) > u_3R_4R_7, \tag{2.5q}$$

$$R_{18}-u_8(R_1R_{12}+R_2R_{11}) < -(R_3R_{12}+R_4R_{11}), \tag{2.5r}$$

$$R_4R_{12} > -R_3R_{13}, \tag{2.5s}$$

$$R_2R_{12} > -R_1R_{13}, \tag{2.5t}$$

$$R_{19}-u_8(R_1R_{13}+R_2R_{12}) < -[(R_3R_{13}+R_4R_{12})+R_{24}], \tag{2.5u}$$

$$R_{20}-u_8(R_2R_{13}+R_1R_{14}) < -[(R_4R_{13}+R_3R_{14})+R_{25}], \tag{2.5v}$$

$$R_{21}-u_8(R_2R_{14}+R_1R_{15}) < -[(R_4R_{14}+R_3R_{15})+R_{26}], \tag{2.5w}$$

$$R_{22}-u_8(R_2R_{15}+R_1R_{16}) < -[(R_4R_{15}+R_3R_{16})+R_{27}], \tag{2.5x}$$

$$R_{23}-u_8(R_2R_{16}+R_1R_{17}) < -[(R_4R_{16}+R_3R_{17})+R_{28}], \tag{2.5y}$$

$$R_{17}(R_4-u_8R_{12}) > -R_{29}, \tag{2.5z}$$

3. The local stability analysis

In this section, the local stability analysis of system 1.2 has been discussed by computing the Jacobian matrix $J(p,s,h,v)$ of system 1.2 about each of the previous equilibrium points.

$$J=\begin{pmatrix} 1-2p-u_1s-\frac{u_2u_4h}{(u_4+p)^2}-\frac{u_3u_5v}{(u_5+p)^2} & -u_1p & \frac{-u_2p}{u_4+p} & \frac{-u_3p}{u_5+p} \\ u_6s & u_6p-u_8h-u_9v-(u_{10}+u_{11}) & u_7-u_8s & -u_9s \\ \frac{u_{13}u_4h}{(u_4+p)^2} & u_8h & u_8s-(u_7+u_{12})+\frac{u_{13}p}{u_4+p} & 0 \\ \frac{u_5u_{15}v}{(u_5+p)^2} & u_9v+u_{10} & 0 & u_9s-u_{14}+\frac{u_{15}p}{u_5+p} \end{pmatrix}$$

- **Analysis of the local stability to system 1.2 at** $E_0=(0,0,0,0)$

At $E_0=(0,0,0,0)$, the Jacobian matrix of system 1.2 is

$$J_0=J(E_0)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(u_{10}+u_{11}) & u_7 & 0 \\ 0 & 0 & -(u_7+u_{12}) & 0 \\ 0 & u_{10} & 0 & -u_{14} \end{pmatrix}$$

Then the characteristic equation of J_0 is given by:
$(1-\lambda)(-(u_{10}+u_{11})-\lambda)(-(u_7+u_{12})-\lambda)(-u_{14}-\lambda)=0$,
$\lambda_{0p}=1>0$, $\lambda_{0s}=-(u_{10}+u_{11})<0$, $\lambda_{0h}=-(u_7+u_{12})<0$ and $\lambda_{0v}=-u_{14}<0$.
Therefore, E_0 is unstable.

- **Analysis of the local stability to system 1.2 at** $E_1=(1,0,0,0)$

At $E_1=(1,0,0,0)$, the Jacobian matrix of system 1.2 as follow:

$$J_1=J(E_1)=\begin{pmatrix} -1 & -u_1 & -\frac{u_2}{u_4+1} & -\frac{u_3}{u_5+1} \\ 0 & u_6-(u_{10}+u_{11}) & u_7 & 0 \\ 0 & 0 & -(u_7+u_{12})+\frac{u_{13}}{u_4+1} & 0 \\ 0 & u_{10} & 0 & -u_{14}+\frac{u_{15}}{u_5+1} \end{pmatrix},$$

The characteristic equation of J_1 take the form as following:
$(-1-\lambda)(u_6-(u_{10}+u_{11})-\lambda)\left(-(u_7+u_{12})+\frac{u_{13}}{u_4+1}-\lambda\right)\left(-u_{14}+\frac{u_{15}}{u_5+1}-\lambda\right)=0,\ \ so,\ \ \lambda_{1p}=-1<0,\ \ \lambda_{1s}=u_6-(u_{10}+u_{11}),\ \ \lambda_{1h}=-(u_7+u_{12})+\frac{u_{13}}{u_4+1}\ \ and\ \ \lambda_{1v}=-u_{14}+\frac{u_{15}}{u_5+1}$.
Therefore, $E_1=(1,0,0,0)$ locally asymptotically stable provided the following conditions hold

$$u_6<(u_{10}+u_{11}), \tag{3.1}$$
$$(u_7+u_{12})>\frac{u_{13}}{u_4+1}, \tag{3.2}$$
$$u_{14}>\frac{u_{15}}{u_5+1}. \tag{3.3}$$

It is unstable otherwise.

- **Analysis of the local stability to system 1.2 at** $E_2=(\widehat{p},\widehat{s},0,0)$

At $E_2 = (\widehat{p}, \widehat{s}, 0, 0)$, the Jacobian matrix of system 1.2 is

$$J_2 = J(E_2) = \begin{pmatrix} -\frac{u_{11}}{u_6} & -u_1\widehat{p} & \frac{-u_2\widehat{p}}{u_4+\widehat{p}} & \frac{-u_3\widehat{p}}{u_5+\widehat{p}} \\ u_6\widehat{s} & -u_{10} & u_7 - u_8\widehat{s} & -u_9\widehat{s} \\ 0 & 0 & u_8\widehat{s} - (u_7 + u_{12}) + \frac{u_{13}\widehat{p}}{u_4+\widehat{p}} & 0 \\ 0 & u_{10} & 0 & u_9\widehat{s} - u_{14} + \frac{u_{15}\widehat{p}}{u_5+\widehat{p}} \end{pmatrix}$$

The characteristic equation of J_2 take the form as following:

$(b_{33} - \lambda)\left[\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3\right] = 0.$

So, either

$(b_{33} - \lambda) = 0$, which gives $\lambda_{2h} = u_8\widehat{s} - (u_7 + u_{12}) + \frac{u_{13}\widehat{p}}{u_4+\widehat{p}} < 0$, provided that

$$(u_7 + u_{12}) > u_8\widehat{s} + \frac{u_{13}\widehat{p}}{u_4 + \widehat{p}} \tag{3.4}$$

Or

$$\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 = 0, \tag{3.5}$$

where:

$$\begin{aligned} L_1 &= -\left(b_{11} + b_{22} + b_{44}\right), \\ L_2 &= b_{11}b_{22} + b_{44}\left(b_{11} + b_{22}\right) - b_{24}b_{42} - b_{12}b_{21}, \\ L_3 &= b_{24}\left(b_{11}b_{42} - b_{14}b_{21}\right) + b_{44}\left(b_{12}b_{21} - b_{11}b_{22}\right). \end{aligned}$$

Using Routh Hurwitz criterion implies that equation (3.5) has roots where real part is negative if and only if: $L_1 > 0, \quad L_3 > 0 \qquad and \quad \Delta = (L_1L_2 - L_3)\,L_3 > 0.$

Now, $L_i > 0, \quad i = 1, 3$ the conditions satisfied below:

$$u_{14} > u_9\widehat{s} + \frac{u_{15}\widehat{p}}{u_5 + \widehat{p}}, \tag{3.6a}$$

$$b_{11}b_{42} > b_{14}b_{21}, \tag{3.6b}$$

Straightforward computation shows that: $\widehat{\Delta} = L_1L_2 - L_3 = \widehat{Q}_1 - \widehat{Q}_2,$ where,

$$\begin{aligned} \widehat{Q}_1 &= \left(b_{11} \ + b_{22}\right)\left[-b_{11}b_{22} - b_{44}\left(b_{11} + b_{22} + b_{44}\right) + b_{12}b_{21}\right] + b_{24}b_{42}\left(b_{11+}b_{22}\right), \\ \widehat{Q}_2 &= b_{14}b_{21}b_{42}, \end{aligned}$$

So, $\Delta > 0$ on the authority of both conditions (3.6a) as long as the condition below:

$$\widehat{Q}_1 > \widehat{Q}_2. \tag{3.6c}$$

Therefore, E_2 is locally asymptotically stable, however, it is unstable otherwise.

- **Analysis of the local stability to system 1.2 at** $E_3 = \left(\bar{p}, \bar{s}, \bar{h}, 0\right)$

At $E_3 = \left(\bar{p}, \bar{s}, \bar{h}, 0\right)$, the Jacobian matrix of system (1.2) as follows:

$$J_3 = J(E_3) = \begin{pmatrix} 1 - 2\bar{p} - u_1\bar{s} - \frac{u_2u_4\bar{h}}{(u_4+\bar{p})^2} & -u_1\bar{p} & \frac{-u_2\bar{p}}{u_4+\bar{p}} & \frac{-u_3\bar{p}}{u_5+\bar{p}} \\ u_6\bar{s} & u_6\bar{p} - u_8\bar{h} - (u_{10}+u_{11}) & u_7 - u_8\bar{s} & -u_9\bar{s} \\ \frac{u_{13}u_4\bar{h}}{(u_4+\bar{p})^2} & u_8\bar{h} & 0 & 0 \\ 0 & u_{10} & 0 & u_9\bar{s} - u_{14} + \frac{u_{15}\bar{p}}{u_5+\bar{p}} \end{pmatrix}$$

The characteristic equation of J_3 take the form as following:

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0, \tag{3.7a}$$

where,

$$M_1 = -\left(c_{11} + c_{22} + c_{44}\right),$$
$$M_2 = c_{44}\left(c_{11} + c_{22}\right) - c_{13}c_{31} - c_{23}c_{32} - c_{12}c_{21} - c_{24}c_{42} + c_{11}c_{22},$$
$$M_3 = c_{44}\left(c_{13}c_{31} + c_{23}c_{32} + c_{12}c_{21} - c_{11}c_{22}\right) + c_{13}\left(c_{22}c_{31} - c_{21}c_{32}\right) - c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right)$$
$$- c_{42}\left(c_{14}c_{21} - c_{11}c_{24}\right),$$
$$M_4 = c_{44}[c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right) + c_{13}\left(c_{21}c_{32} - c_{22}c_{31}\right)] - c_{31}c_{42}\left(c_{14}c_{23} + c_{13}c_{24}\right),$$

Using Routh Hurwitz criterion implies equation (3.7a) has roots where real part is negative if and only if: $M_i > 0, \quad i = 1,3 \quad$ *and* $\Delta = \left(M_1M_2 - M_3\right)M_3 - M_1^2M_4 > 0$. Now, $M_i > 0, \quad i = 1,3$ provided that the conditions satisfied below:

$$1 < 2\bar{p} + u_1\bar{s} + \frac{u_3u_5\bar{v}}{\left(u_5 + \bar{p}\right)^2}, \tag{3.7b}$$

$$u_6\bar{p} < u_8\bar{h} + \left(u_{10} + u_{11}\right), \tag{3.7c}$$

$$u_9\bar{s} + \frac{u_{15}\bar{p}}{u_5 + \bar{p}} < u_{14}, \tag{3.7d}$$

$$\frac{u_7}{u_8} < \bar{s}, \tag{3.7e}$$

$$u_8\bar{h}\left(1 - 2\bar{p} - u_1\bar{s} - \frac{u_2u_4\bar{h}}{\left(u_4 + \bar{p}\right)^2}\right) > -u_1\bar{p}\left(\frac{u_{13}u_4\bar{h}}{\left(u_4 + \bar{p}\right)^2}\right), \tag{3.7f}$$

$$c_{44}[c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right) + c_{13}\left(c_{21}c_{32} - c_{22}c_{31}\right)] > c_{31}c_{42}\left(c_{14}c_{23} + c_{13}c_{24}\right), \tag{3.7g}$$

Straightforward computation shows that: $\Delta = \bar{Q}_1 - \bar{Q}_2$, where,

$$\bar{Q}_1 = \{c_{44}\left(c_{13}c_{31} + c_{23}c_{32} + c_{12}c_{21} - c_{11}c_{22}\right) + c_{13}\left(c_{22}c_{31} - c_{21}c_{32}\right) - c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right)$$
$$- c_{42}\left(c_{14}c_{21} - c_{11}c_{24}\right)\}\{c_{11}[c_{12}c_{21} - c_{22}\left(c_{11} + c_{44}\right) - c_{44}\left(c_{11} + c_{22}\right)] + c_{31}\left(c_{11}c_{13} + c_{12}c_{23}\right)$$
$$+ c_{22}[c_{12}c_{21} + c_{23}c_{32} + c_{24}c_{42} - c_{22}\left(c_{11} + c_{44}\right)] + c_{44}[c_{24}c_{42} - c_{44}\left(c_{11} + c_{22}\right)]\} + c_{31}c_{42}\left(c_{11} + c_{22} + c_{44}\right)^2$$
$$\left(c_{14}c_{23} + c_{13}c_{24}\right),$$
$$\bar{Q}_2 = -c_{44}\left(c_{11} + c_{22} + c_{44}\right)^2[c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right) + c_{13}\left(c_{21}c_{32} - c_{22}c_{31}\right)] + c_{21}\left(c_{13}c_{32} + c_{14}c_{42}\right)$$
$$\{c_{44}[c_{13}c_{31} + c_{23}c_{32} + c_{12}c_{21} - c_{11}c_{22}] + c_{13}\left(c_{22}c_{31} - c_{21}c_{32}\right) - c_{23}\left(c_{12}c_{31} - c_{11}c_{32}\right) - c_{42}\left(c_{14}c_{21} - c_{11}c_{24}\right)\},$$

So, $\Delta > 0$ on the authority of conditions (3.7b)-(3.7g) as long as the condition below:

$$\bar{Q}_1 > \bar{Q}_2, \tag{3.7h}$$

Therefore, E_3 is locally asymptotically stable, however, it is unstable otherwise.

- **Analysis of the local stability to system 1.2 at** $E_4 = (\bar{\bar{p}}, \bar{\bar{s}}, 0, \bar{\bar{v}})$

At $E_4 = (\bar{\bar{p}}, \bar{\bar{s}}, 0, \bar{\bar{v}})$, the Jacobian matrix of system (1.2) as follows

$$J_4 = J(E_4) = \begin{pmatrix} 1 - 2\bar{\bar{p}} - u_1\bar{\bar{s}} - \frac{u_3u_5\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^2} & -u_1\bar{\bar{p}} & \frac{-u_2\bar{\bar{p}}}{u_4+\bar{\bar{p}}} & \frac{-u_3\bar{\bar{p}}}{u_5+\bar{\bar{p}}} \\ u_6\bar{\bar{s}} & 0 & u_7 - u_8\bar{\bar{s}} & -u_9\bar{\bar{s}} \\ 0 & 0 & u_8\bar{\bar{s}} - (u_7 + u_{12}) + \frac{u_{13}\bar{\bar{p}}}{u_4+\bar{\bar{p}}} & 0 \\ \frac{u_5u_{15}\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^2} & u_9\bar{\bar{v}} + u_{10} & 0 & u_9\bar{\bar{s}} - u_{14} + \frac{u_{15}\bar{\bar{p}}}{u_5+\bar{\bar{p}}} \end{pmatrix}$$

The characteristic equation of J_4 take the form as following:

$$(d_{33} - \lambda) \left[\lambda^3 + N_1\lambda^2 + N_2\lambda + N_3\right] = 0. \tag{3.8}$$

So, either

$(d_{33} - \lambda)$, which gives $\lambda_{4h} = u_8\bar{\bar{s}} - (u_7 + u_{12}) + \frac{u_{13}\bar{\bar{p}}}{u_4+\bar{\bar{p}}} < 0$, provided that

$$(u_7 + u_{12}) > u_8\bar{\bar{s}} + \frac{u_{13}\bar{\bar{p}}}{u_4 + \bar{\bar{p}}}. \tag{3.9a}$$

Or

$$\lambda^3 + N_1\lambda^2 + N_2\lambda + N_3 = 0,$$

Where,

$$\begin{aligned} N_1 &= -\left(d_{11} + d_{44}\right), \\ N_2 &= d_{11}d_{44} - d_{14}d_{41} - d_{24}d_{42} - d_{12}d_{21}, \\ N_3 &= d_{42}\left(d_{11}d_{24} - d_{14}d_{21}\right) + d_{12}\left(d_{21}d_{44} - d_{24}d_{41}\right). \end{aligned}$$

Using Routh Hurwitz criterion implies equation (3.8) has roots where real part is negative if and only if $N_i > 0, \quad i = 1, 3 \quad$ *and* $\Delta = (N_1N_2 - N_3)\, N_3 > 0$. Now, $N_i > 0, \quad i = 1, 3$ provided that the conditions satisfied below:

$$1 < 2\bar{\bar{p}} + u_1\bar{\bar{s}} + \frac{u_3u_5\bar{\bar{v}}}{(u_5 + \bar{\bar{p}})^2}, \tag{3.9b}$$

$$u_9\bar{\bar{s}} + \frac{u_{15}\bar{\bar{p}}}{u_5 + \bar{\bar{p}}} < u_{14}, \tag{3.9c}$$

$$u_6\bar{\bar{s}}[u_9\bar{\bar{s}} - u_{14} + \frac{u_{15}\bar{\bar{p}}}{u_5 + \bar{\bar{p}}}] > -u_9\bar{\bar{s}}\left(\frac{u_5u_{15}\bar{\bar{v}}}{(u_5 + \bar{\bar{p}})^2}\right). \tag{3.9d}$$

Straightforward computation shows that:

$$\bar{\bar{\Delta}} = N_1N_2 - N_3 = \bar{\bar{Q}}_1 - \bar{\bar{Q}}_2,$$

where,

$$\begin{aligned} \bar{\bar{Q}}_1 &= (d_{11} + d_{44})\left[d_{14}d_{41} - d_{44}d_{11} + d_{24}d_{42} + d_{12}d_{21}\right] + d_{12}d_{24}d_{41}, \\ \bar{\bar{Q}}_2 &= d_{42}\left(d_{14}d_{21} - d_{11}d_{24}\right) - d_{12}d_{21}d_{44}. \end{aligned}$$

So, $\Delta > 0$ on the authority of conditions (3.9a)-(3.9d) as long as the condition below:

$$\bar{\bar{Q}}_1 > \bar{\bar{Q}}_2. \tag{3.9e}$$

Therefore, E_4 is locally asymptotically stable, however, it is unstable otherwise.

- **Analysis of the local stability to system 1.2 at** $E_5 = \left(\widetilde{p}, \widetilde{s}, \widetilde{h}, \widetilde{v}\right)$

At $E_5 = \left(\widetilde{p}, \widetilde{s}, \widetilde{h}, \widetilde{v}\right)$, the Jacobian matrix of system (1.2) as follows

$$J_5 = J(E_5) = \begin{pmatrix} 1-2\widetilde{p}-u_1\widetilde{s}-\frac{u_2u_4\widetilde{h}}{(u_4+\widetilde{p})^2}-\frac{u_3u_5\widetilde{v}}{(u_5+\widetilde{p})^2} & -u_1\widetilde{p} & \frac{-u_2\widetilde{p}}{u_4+\widetilde{p}} & \frac{-u_3\widetilde{p}}{u_5+\widetilde{p}} \\ u_6\widetilde{s} & u_6\widetilde{p}-u_8\widetilde{h}-u_9\widetilde{v}-(u_{10}+u_{11}) & u_7-u_8\widetilde{s} & -u_9\widetilde{s} \\ \frac{u_{13}u_4\widetilde{h}}{(u_4+\widetilde{p})^2} & u_8\widetilde{h} & 0 & 0 \\ \frac{u_5u_{15}\widetilde{v}}{(u_5+\widetilde{p})^2} & u_9\widetilde{v}+u_{10} & 0 & u_9\widetilde{s}-u_{14}+\frac{u_{15}\widetilde{p}}{u_5+\widetilde{p}} \end{pmatrix}$$

The characteristic equation of J_5 take the form as following:

$$\lambda^4 + K_1\lambda^3 + K_2\lambda^2 + K_3\lambda + K_4 = 0, \tag{3.10}$$

where,

$$\begin{aligned} K_1 &= -\left(e_{11} + e_{22} + e_{44}\right), \\ K_2 &= e_{22}e_{44} + e_{11}\left(e_{22} + e_{44}\right) - e_{13}e_{31} - e_{23}e_{32} - e_{14}e_{41} - e_{24}e_{42} - e_{12}e_{21}, \\ K_3 &= e_{31}[e_{13}\left(e_{22} + e_{44}\right) - e_{12}e_{32}] - e_{32}[e_{13}e_{21} - e_{23}\left(e_{11} + e_{44}\right)] - e_{44}\left(e_{11}e_{22} - e_{12}e_{21}\right) + e_{14}e_{22}e_{41} + \\ &\quad e_{11}e_{24}e_{42}, \\ K_4 &= e_{31}[e_{23}\left(e_{12}e_{44} - e_{14}e_{42}\right) - e_{13}\left(e_{22}e_{44} - e_{24}e_{42}\right)] - e_{32}[e_{13}\left(e_{24}e_{41} - e_{21}e_{44}\right) - e_{23}\left(e_{14}e_{41} - e_{11}e_{44}\right)] \end{aligned}$$

Using Routh Hurwitz criterion implies equation (3.10) has roots where real part is negative if and only if: $K_i > 0, \quad i = 1, 3, 4 \quad \textit{and} \ \Delta = \left(K_1K_2 - K_3\right)K_3 - K_1^2K_4 > 0$.

Now, $K_i > 0, \quad i = 1, 3, 4$ provided the conditions satisfied below:

$$1 < 2\widetilde{p} + u_1\widetilde{s} + \frac{u_2u_4\widetilde{h}}{\left(u_4 + \widetilde{p}\right)^2} + \frac{u_3u_5\widetilde{v}}{\left(u_5 + \widetilde{p}\right)^2}, \tag{3.11a}$$

$$\widetilde{p} < \frac{u_8\widetilde{h} + u_9\widetilde{v} + \left(u_{10} + u_{11}\right)}{u_6}, \tag{3.11b}$$

$$u_9\widetilde{s} + \frac{u_{15}\widetilde{p}}{u_5 + \widetilde{p}} < u_{14}, \tag{3.11c}$$

$$\frac{u_7}{u_8} < \widetilde{s}, \tag{3.11d}$$

$$-u_9\widetilde{s}\left(\frac{u_5u_{15}\widetilde{v}}{u_5 + \widetilde{p}}\right) > u_6\widetilde{s}\left(u_9\widetilde{s} + \frac{u_{15}\widetilde{p}}{u_5 + \widetilde{p}} - u_{14}\right), \tag{3.11e}$$

$$e_{23}\left(e_{12}e_{44} - e_{14}e_{42}\right) < e_{13}\left(e_{22}e_{44} - e_{24}e_{42}\right), \tag{3.11f}$$

$$e_{13}\left(e_{24}e_{41} - e_{21}e_{44}\right) < e_{23}\left(e_{14}e_{41} - e_{11}e_{44}\right), \tag{3.11g}$$

Straightforward computation shows that: $\Delta = \widetilde{Q}_1 - \widetilde{Q}_2$, where,

$\widetilde{Q}_1 = \{e_{11}[e_{13}e_{31} + e_{14}e_{41} + e_{12}e_{21}] - e_{11}(e_{22} + e_{44})(e_{11} + e_{22} + e_{44}) - e_{22}e_{44}(e_{22} + e_{44}) + e_{22}[e_{23}e_{32} + e_{24}e_{42} + e_{12}e_{21}] + e_{44}(2e_{12}e_{21} + e_{14}e_{41} + e_{24}e_{42})\}\{e_{31}[e_{13}(e_{22} + e_{44}) - e_{12}e_{32}] - e_{32}[e_{13}e_{21} - e_{23}(e_{11} + e_{44})] - e_{44}(e_{11}e_{22} + e_{12}e_{21}) + e_{14}e_{22}e_{41} + e_{11}e_{24}e_{42}\}$

$\widetilde{Q}_2 = e_{32}(e_{12}e_{31} + e_{13}e_{21})\{e_{31}[e_{13}(e_{22} + e_{44}) - e_{12}e_{32}] - e_{32}[e_{13}e_{21} - e_{23}(e_{11} + e_{44})] - e_{44}(e_{11}e_{22} + e_{12}e_{21}) + e_{14}e_{22}e_{41} + e_{11}e_{24}e_{42}\} + (e_{11} + e_{22} + e_{44})^2\{e_{31}[e_{23}(e_{12}e_{44} - e_{14}e_{42}) - e_{13}(e_{22}e_{44} - e_{24}e_{42})] - e_{32}[e_{13}(e_{24}e_{41} - e_{21}e_{44}) - e_{23}(e_{14}e_{41} - e_{11}e_{44})]\}$.

So, $\Delta > 0$ on the authority of conditions (3.11a)-(3.11g) as long as the condition below:

$$\widetilde{Q}_1 > \widetilde{Q}_2. \tag{3.11h}$$

Therefore, E_5 is locally asymptotically stable, however, it is unstable otherwise.

4. Global Stability Analysis

In this section, by using a suitable Lyapunov method about the previous equilibrium points of system (1.2) to study the global stability analysis, which were represented early locally stability as illustrated in the following theorems:

Theorem 4.1. *The equilibrium $E_1 = (1, 0, 0, 0)$, of system (1.2) is globally asymptotically stable in the basin of attraction of Int.R^4_+ that satisfies the condition:*

$$(p-1)^2 + u_{11}s + u_{12}h + u_{14}v > u_1 s + \frac{u_2 h}{u_4 + p} + \frac{u_3 v}{u_5 + p}, \tag{4.1}$$

Proof . *Consider the following function*

$$W_1(p, s, h, v) = [p - 1 - \ln p] + s + h + v.$$

Clearly the function $W_1 : R^4_+ \to R$ is C^1 is positive definite.
Differentiating W_1 with regard to time t with handle algebraic treatments we get:

$$\frac{dW_1}{dt} = -(p-1)^2 - (u_1 - u_6)ps - \frac{(u_2 - u_{13})ph}{u_4 + p} - \frac{(u_3 - u_{15})pv}{u_5 + p} - u_{11}s + u_1 s + \frac{u_2 h}{u_4 + p} + \frac{u_3 v}{u_5 + p} - u_{12}h - u_{14}v,$$

Now, by the biological facts $u_1 > u_6, u_2 > u_{13}$ and $u_3 > u_{15}$ we get:

$$\frac{dW_1}{dt} < -(p-1)^2 - u_{11}s - u_{12}h - u_{14}v + u_1 s + \frac{u_2 h}{u_4 + p} + \frac{u_3 v}{u_5 + p},$$

Thus, $\frac{dW_1}{dt} < 0$, under the condition (4.1) and hence $\frac{dW_1}{dt}$ is negative definite. Thus E_1 is globally asymptotically stable. □

Theorem 4.2. *The equilibrium $E_2 = (\widehat{p}, \widehat{s}, 0, 0)$, of system (1.2) is globally asymptotically stable in the basin of attraction of Int.R^4_+ that satisfies the following condition:*

$$\widehat{\theta}_1 > \widehat{\theta}_2, \tag{4.2}$$

where,

$$\widehat{\theta}_1 = -(p-\widehat{p})^2 - \left(\frac{u_7\widehat{s}}{s} + u_{12}\right)h - u_{14}v,$$
$$\widehat{\theta}_2 = (u_1-u_6)(p\widehat{s}+\widehat{p}s) + [\frac{u_2h}{u_4+p} + \frac{u_3v}{u_5+p}]\widehat{p} + [u_8h+u_9v]\widehat{s} + u_{10}s.$$

Proof . *Consider the following function*

$$W_2 = [p-\widehat{p}-ln\frac{p}{\widehat{p}}] + [s-\widehat{s}-\ln\frac{s}{\widehat{s}}] + h + v.$$

Clearly the function $W_2 : R^4_+ \to R$ *is* C^1 *is positive definite.*
Differentiating W_2 *with regard to time* t *and handle algebraic treatments we get:*

$$\frac{dW_2}{dt} = -(p-\widehat{p})^2 - (u_1-u_6)(p-\widehat{p})(s-\widehat{s}) - \frac{(u_2-u_{13})\,ph}{u_4+p} - \frac{(u_3-u_{15})\,pv}{u_5+p} - \left(\frac{u_7\widehat{s}}{s}+u_{12}\right)h$$
$$-u_{14}v + (u_1-u_6)(p\widehat{s}+\widehat{p}s) + [\frac{u_2h}{u_4+p} + \frac{u_3v}{u_5+p}]\widehat{p} + [u_8h+u_9v]\widehat{s}.$$

Now, by the biological facts $u_1 > u_6, u_2 > u_{13}$ *and* $u_3 > u_{15}$ *, we get:*

$$\frac{dW_2}{dt} < -(p-\widehat{p})^2 - \left(\frac{u_7\widehat{s}}{s}+u_{12}\right)h - u_{14}v + (u_1-u_6)(p\widehat{s}+\widehat{p}s) + [\frac{u_2h}{u_4+p} + \frac{u_3v}{u_5+p}]\widehat{p} + [u_8h+u_9v]\widehat{s}.$$

Thus, $\frac{dW_2}{dt} < 0$*, under condition (4.2) and hence* $\frac{dW_2}{dt}$ *is negative definite. Thus* E_2 *is globally asymptotically stable.* □

Theorem 4.3. *The equilibrium* $E_3 = (\bar{p},\bar{s},\bar{h},0)$*, of system (1.2) is globally asymptotically stable in the Basin of attraction of* $Int.R^4_+$ *that satisfies the next condition:*

$$\frac{u_7}{\bar{s}} \le 2\sqrt{\frac{u_7h}{s\bar{s}}}, \tag{4.3}$$
$$\bar{\theta}_1 > \bar{\theta}_2. \tag{4.4}$$

Where,

$$\bar{\theta}_1 = -(p-\bar{p})^2 - \left[\sqrt{\frac{u_7h}{s\bar{s}}}(s-\bar{s}) - (h-\bar{h})\right]^2 - u_{13}\left[\frac{\bar{p}h}{u_4+\bar{p}} + \frac{p\bar{h}}{u_4+p}\right] - u_{14}v,$$
$$\bar{\theta}_2 = (u_1-u_6)(p\bar{s}+\bar{p}s) + (h-\bar{h})^2 + u_2\left[\frac{\bar{p}h}{u_4+p} + \frac{p\bar{h}}{u_4+\bar{p}}\right] + \frac{u_3\bar{p}v}{u_5+p} + u_9\bar{s}v.$$

Proof . *Consider the following function*

$$W_3 = \left[p-\bar{p}-ln\frac{p}{\bar{p}}\right] + \left[s-\bar{s}-ln\frac{s}{\bar{s}}\right] + \left[h-\bar{h}-ln\frac{h}{\bar{h}}\right] + v.$$

Clearly the function $W_3 : R^4_+ \to R$ *is* C^1 *is positive definite.*
Differentiating W_3 *with regard to time t and handle algebraic treatments we get:*

$$\frac{dW_3}{dt} = -(p-\bar{p})^2 - (u_1-u_6)\,ps - (u_1-u_6)\,\bar{p}\bar{s} + (u_1-u_6)\,(p\bar{s}+\bar{p}s) - \frac{(u_2-u_{13})\,ph}{u_4+p} - \frac{(u_2-u_{13})\,\bar{p}\bar{h}}{u_4+\bar{p}}$$
$$- \frac{(u_3-u_{15})\,pv}{u_5+p} + \left(h-\bar{h}\right)^2 + u_2[\frac{\bar{p}h}{u_4+p} + \frac{p\bar{h}}{u_4+\bar{p}}] + \frac{u_3\bar{p}v}{u_5+p} - u_{13}[\frac{\bar{p}h}{u_4+\bar{p}} + \frac{p\bar{h}}{u_4+p}] - \left(h-\bar{h}\right)^2$$
$$- u_{14}v + u_9\bar{s}v - \frac{u_7h}{s\bar{s}}\,(s-\bar{s})^2 + \frac{u_7}{\bar{s}}\,(s-\bar{s})\left(h-\bar{h}\right).$$

Now, by the biological facts $u_1 > u_6, u_2 > u_{13}$ *and* $u_3 > u_{15}$ *with the condition* (4.3) *we get:*

$$\frac{dW_3}{dt} < -(p-\bar{p})^2 - [\sqrt{\frac{u_7h}{s\bar{s}}}\,(s-\bar{s}) - \left(h-\bar{h}\right)]^2 + (u_1-u_6)\,(p\bar{s}+\bar{p}s) + \left(h-\bar{h}\right)^2 + u_2[\frac{\bar{p}h}{u_4+p}$$
$$+ \frac{p\bar{h}}{u_4+\bar{p}}] + \frac{u_3\bar{p}v}{u_5+p} - u_{13}[\frac{\bar{p}h}{u_4+\bar{p}} + \frac{p\bar{h}}{u_4+p}] - u_{14}v + u_9\bar{s}v.$$

Thus, $\frac{dW_3}{dt} < 0$, *under condition* (4.4) *and hence* $\frac{dW_3}{dt}$ *is negative definite. Thus* E_3 *is globally asymptotically stable.* □

Theorem 4.4. *the equilibrium* $E_4 = (\bar{\bar{p}}, \bar{\bar{s}}, 0, \bar{\bar{v}})$, *of system* (1.2) *is globally asymptotically stable in the Basin of attraction of* $Int.R^4_+$ *that satisfies the next condition:*

$$\frac{u_{10}}{\bar{\bar{v}}} \le 2\sqrt{\frac{u_{10}s}{v\bar{\bar{v}}}}, \tag{4.5}$$
$$\bar{\bar{\theta}}_1 > \bar{\bar{\theta}}_2. \tag{4.6}$$

Where,

$$\bar{\bar{\theta}}_1 = -(p-\bar{\bar{p}})^2 - [(s-\bar{\bar{s}}) - \sqrt{\frac{u_{10}s}{v\bar{\bar{v}}}}\,(v-\bar{\bar{v}})]^2 - u_{15}[\frac{p\bar{\bar{v}}}{u_5+p} + \frac{\bar{\bar{p}}v}{u_5+\bar{\bar{p}}}] - [\frac{u_7\bar{\bar{s}}}{s} + u_{12}]h,$$
$$\bar{\bar{\theta}}_2 = (u_1-u_6)\,(p\bar{\bar{s}}+\bar{\bar{p}}s) + (s-\bar{\bar{s}})^2 + u_3[\frac{\bar{\bar{p}}v}{u_5+p} + \frac{p\bar{\bar{v}}}{u_5+\bar{\bar{p}}}] + [u_8\bar{\bar{s}} + \frac{u_2\bar{\bar{p}}}{u_4+p}]h + [\frac{u_2\bar{\bar{p}}}{u_4+p} + u_8\bar{\bar{s}}]h,$$

Proof . *Consider the following function*

$$W_4 = [p - \bar{\bar{p}} - ln\frac{p}{\bar{\bar{p}}}] + [s - \bar{\bar{s}} - ln\frac{s}{\bar{\bar{s}}}] + h + [v - \bar{\bar{v}} - ln\frac{v}{\bar{\bar{v}}}].$$

Clearly the function $W_4 : R^4_+ \to R$ *is* C^1 *is positive definite.*
Differentiating W_4 *with regard to time t and handle algebraic treatments we get:*

$$\frac{dw_4}{dt} = -(p-\bar{\bar{p}})^2 - (u_1-u_6)\,ps - (u_1-u_6)\,\bar{\bar{p}}\bar{\bar{s}} + (u_1-u_6)\,(p\bar{\bar{s}}+\bar{\bar{p}}s) - (s-\bar{\bar{s}})^2 - \frac{(u_2-u_{13})\,ph}{u_4+p} -$$
$$\frac{(u_3-u_{15})\,\bar{\bar{p}}\bar{\bar{v}}}{u_5+\bar{\bar{p}}} - \frac{(u_3-u_{15})\,pv}{u_5+p} + (s-\bar{\bar{s}})^2 + \frac{u_2\bar{\bar{p}}h}{u_4+p} + u_3[\frac{\bar{\bar{p}}v}{u_5+p} + \frac{p\bar{\bar{v}}}{u_5+\bar{\bar{p}}}] - u_{15}[\frac{p\bar{\bar{v}}}{u_5+p} + \frac{\bar{\bar{p}}v}{u_5+\bar{\bar{p}}}]$$
$$- [\frac{u_7\bar{\bar{s}}}{s} + u_{12}]h - \frac{u_{10}}{\bar{\bar{s}}}\,(s-\bar{\bar{s}})\,(v-\bar{\bar{v}}) + \frac{u_{10}s}{v\bar{\bar{v}}}\,(v-\bar{\bar{v}})^2 + u_8\bar{\bar{s}}h.$$

Now, by the biological facts $u_1 > u_6, u_2 > u_{13}$ and $u_3 > u_{15}$ with the condition (4.5) we get:

$$\frac{dW_4}{dt} < -(p-\bar{\bar{p}})^2 + (u_1-u_6)(p\bar{\bar{s}}+\bar{\bar{p}}s) - (s-\bar{\bar{s}})^2 - [(s-\bar{\bar{s}}) - \sqrt{\frac{u_{10}v}{\bar{\bar{s}}s}}(v-\bar{\bar{v}})]^2 + (s-\bar{\bar{s}})^2$$
$$+\frac{u_2ph}{u_4+p} + u_3[\frac{\bar{\bar{p}}v}{u_5+p} + \frac{p\bar{\bar{v}}}{u_5+\bar{\bar{p}}}] - u_{15}[\frac{p\bar{\bar{v}}}{u_5+p} + \frac{\bar{\bar{p}}v}{u_5+\bar{\bar{p}}}] - [\frac{u_7\bar{\bar{s}}}{s} + u_{12}]h + [u_8\bar{\bar{s}} + \frac{u_2\bar{\bar{p}}}{u_4+p}]h.$$

Thus $\frac{dW_4}{dt} < 0$, under condition (4.6) and hence $\frac{dW_4}{dt}$ is negative definite.
Therefore E_4 is globally asymptotically stable. □

Theorem 4.5. *the equilibrium $E_5 = \left(\widetilde{p}, \widetilde{s}, \widetilde{h}, \widetilde{v}\right)$, of system (1.2) is globally asymptotically stable in the Basin of attraction of $Int.R^4_+$ that satisfies the next condition*

$$\frac{u_7}{\widetilde{s}} \le 2\sqrt{\frac{u_7h}{s\widetilde{s}}}, \tag{4.7}$$
$$\frac{u_{10}}{\widetilde{v}} \le 2\sqrt{\frac{u_{10}s}{v\widetilde{v}}}, \tag{4.8}$$
$$\widetilde{\theta}_1 > \widetilde{\theta}_2. \tag{4.9}$$

Where,

$$\widetilde{\theta}_1 = -(p-\widetilde{p})^2 - \left[\sqrt{\frac{u_7h}{\bar{\bar{s}}s}}(s-\bar{\bar{s}}) - \left(h-\widetilde{h}\right)\right]^2 - \left[(s-\widetilde{s}) - \sqrt{\frac{u_{10}s}{v\bar{\bar{v}}}}(v-\bar{\bar{v}})\right]^2$$
$$-u_{15}\left[\frac{p\widetilde{v}}{u_5+p} + \frac{\widetilde{p}v}{u_5+\widetilde{p}}\right] - u_{13}\left[\frac{p\widetilde{h}}{u_5+p} + \frac{\widetilde{p}h}{u_5+\widetilde{p}}\right]$$
$$\widetilde{\theta}_2 = (u_1-u_6)(p\widetilde{s}+\widetilde{p}s) + u_2\left[\frac{\widetilde{p}h}{u_4+p} + \frac{p\widetilde{h}}{u_4+\widetilde{p}}\right] + u_3\left[\frac{\widetilde{p}v}{u_5+p} + \frac{p\widetilde{v}}{u_5+\widetilde{p}}\right] + \left(h-\widetilde{h}\right)^2 + (s-\widetilde{s})^2.$$

Proof . *Consider the following function*

$$W_5 = \left[p - \widetilde{p} - ln\frac{p}{\widetilde{p}}\right] + \left[s - \widetilde{s} - ln\frac{s}{\widetilde{s}}\right] + \left[h - \widetilde{h} - ln\frac{h}{\widetilde{h}}\right] + \left[v - \widetilde{v} - ln\frac{v}{\widetilde{v}}\right].$$

Clearly the function $W_5 : R^4_+ \to R$ is C^1 is positive definite.
Differentiating W_5 with regard to time t and handle algebraic treatments we get:

$$\frac{dw_5}{dt} = -(p-\widetilde{p})^2 - (u_1-u_6)ps - (u_1-u_6)\widetilde{p}\widetilde{s} + (u_1-u_6)(p\widetilde{s}+\widetilde{p}s) - \left(h-\widetilde{h}\right)^2 - \frac{(u_2-u_{13})ph}{u_4+p}$$
$$-\frac{(u_2-u_{13})\widetilde{p}\widetilde{h}}{u_4+\widetilde{p}} - \frac{(u_3-u_{15})\widetilde{p}\widetilde{v}}{u_5+\widetilde{p}} - \frac{(u_3-u_{15})pv}{u_5+p} + \left(h-\widetilde{h}\right)^2 + u_2\left[\frac{\widetilde{p}h}{u_4+p} + \frac{p\widetilde{h}}{u_4+\widetilde{p}}\right]$$
$$+u_3\left[\frac{\widetilde{p}v}{u_5+p} + \frac{p\widetilde{v}}{u_5+\widetilde{p}}\right] - u_{15}\left[\frac{p\widetilde{v}}{u_5+p} + \frac{\widetilde{p}v}{u_5+\widetilde{p}}\right] - u_{13}\left[\frac{p\widetilde{h}}{u_5+p} + \frac{\widetilde{p}h}{u_5+\widetilde{p}}\right] - \frac{u_7}{\widetilde{s}}(s-\widetilde{s})\left(h-\widetilde{h}\right)$$
$$+\frac{u_7h}{s\bar{\bar{s}}}(s-\widetilde{s})^2 - \frac{u_{10}}{\widetilde{v}}(s-\widetilde{s})(v-\widetilde{v}) + \frac{u_{10}s}{v\bar{\bar{v}}}(v-\widetilde{v})^2 + (s-\widetilde{s})^2 - (s-\widetilde{s})^2.$$

Now, by the biological facts $u_1 > u_6, u_2 > u_{13}$ and $u_3 > u_{15}$ with the condition (4.7) *and* (4.8) *we get:*

$$\begin{aligned}\frac{dW_5}{dt} < &-(p-\widetilde{p})^2 - \left[\sqrt{\frac{u_7 h}{s\bar{\bar{s}}}}\,(s-\widetilde{s}) - \left(h-\widetilde{h}\right)\right]^2 - \left[(s-\widetilde{s}) - \sqrt{\frac{u_{10}s}{v\bar{\bar{v}}}}\,(v-\bar{\bar{v}})\right]^2\\ &+(u_1-u_6)\left(p\widetilde{s}+\widetilde{p}s\right) + u_2\left[\frac{\widetilde{p}h}{u_4+p}+\frac{p\widetilde{h}}{u_4+\widetilde{p}}\right] + u_3\left[\frac{\widetilde{p}v}{u_5+p}+\frac{p\widetilde{v}}{u_5+\widetilde{p}}\right]\\ &+\left(h-\widetilde{h}\right)^2+(s-\widetilde{s})^2 - u_{15}\left[\frac{p\widetilde{v}}{u_5+p}+\frac{\widetilde{p}v}{u_5+\widetilde{p}}\right] - u_{13}\left[\frac{p\widetilde{h}}{u_5+p}+\frac{\widetilde{p}h}{u_5+\widetilde{p}}\right].\end{aligned}$$

Thus, $\frac{dW_5}{dt} < 0$ *under condition* (4.9) *and hence* $\frac{dW_5}{dt}$ *is negative definite.*
Thus, E_5 *is globally asymptotically stable.* □

5. Numerical Simulation

Right now, in any dynamical system, the appropriate numerical test to the entire analytical calculations is the most benefited methods to support the analytic results. Here, model (1.2) represented an epidemic model in prey-predator populations. Actually, the other benefit is to understand the influence of mutable values of parameters. Runge–Kutta with Predictor corrector strategy to get output with the parameters in the structure of system (1.2), by using Matlab the obtained numerical results to outline drawings for system arrangements. Instead of natural data the hypothetical theoretical arrangement is used here:

$$\begin{array}{lllll} u_1 = 0.4, & u_2 = 0.4, & u_3 = 0.033, & u_4 = 0.5, & u_5 = 0.015\\ u_6 = 0.001, & u_7 = 0.085, & u_8 = 0.85, & u_9 = 0.001, & u_{10} = 0.0001,\\ u_{11} = 0.00001, & u_{12} = 0.0001, & u_{13} = 0.0001, & u_{14} = 0.001, & u_{15} = 0.0008. \end{array} \tag{5.1}$$

From Eq.(5.1) which represent the set of data starting from various initial values, it is observed the solution of system (1.2) approaches asymptotically to a positive equilibrium point
$E_5 = (0.822,\ 0.1,\ 0.402,\ 0.396)$, which illustrated in Figure 1(a-d)

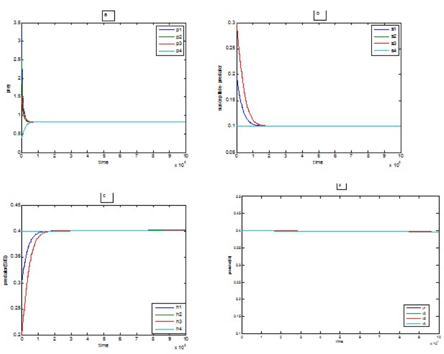

Figure 1: The time series of system (1.2) beginning with different initial points (3.5,0.2,0.3,0.4),(2.5,0.3,0.2,0.4) , (1.5,0.3,0.2,0.4) and (0.4,0.1,0.4,0.4), for the data given in eq. (24). The solution approaches asymptotically to the positive equilibrium point $E_5 = (0.822,\ 0.1,\ 0.402,\ 0.396)$, (a) trajectory of (p) as a function of time, (b) trajectory of (s) as a function of time, (c) trajectory of (h) as a function of time, (d) trajectory of (v) as a function of time.

To discuss importance of the parameters values of system (1.2) on the dynamical behavior of the proposed ecological system, the numerical solution for the data given in Eq.(5.1) with varying one or more than parameter at each time and the obtained results are given below.
Note that when $0 < u_1 \leq 2$, the solution of system (1.2) as yet approaches E_5, as illustrated in Figure 2, for typical value $u_1 = 0.5$.

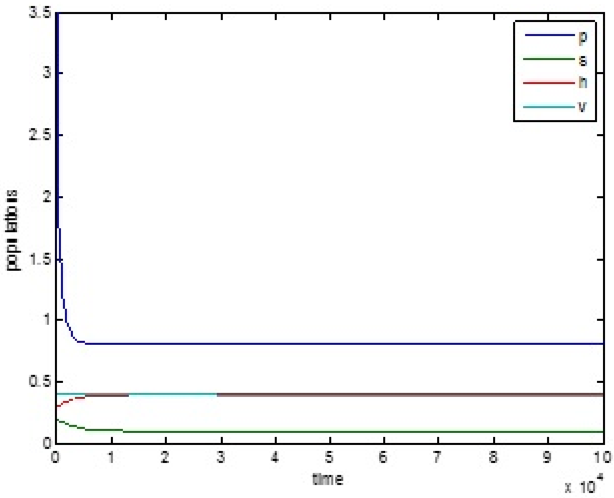

Figure 2: The Time series of the solution of system (1.2) which approach to $E_5 = (0.811,\ \ 0.1,\ \ 0.401,\ \ 0.396)$

Now, Table 1, illustrated the study of the residue of the parameters in the numerical results of and their effect on the ecological model.

Table 1: Numerical behavior of the system (1.2) at each time when changing one factor for the data it's provide Eq.(5.1).

Range of parameters	Behavior of solution
$0.4 \leq u_2 < 1.26$	Approach to E_5
$0.033 \leq u_3 < 0.4$	Approach to E_5
$u_4 \geq 0$	Approach to E_5
$u_5 \geq 0$	Approach to E_5
$0 \leq u_6 < 0.16$	Approach to E_5
$0.1 \leq u_8 \leq 2$	Approach to E_5
$0 \leq u_9 < 0.12$	Approach to E_5
$0 \leq u_{10} < 0.1$	Approach to E_5
$0 \leq u_{11} < 0.1$	Approach to E_5
$0 \leq u_{13} < 0.027$	Approach to E_5
$0 \leq u_{15} < 0.3$	Approach to E_5

When $0.01 \leq u_7 < 0.47$ the solution of system (1.2) remain approaches E_5, as illustrated in Figure(3a), for typical value $u_7 = 0.3$, while $0.47 \leq u_7 \leq 1$ the solution of system (1.2) approach to E_4, as illustrated in Figure (3b), for typical value $u_7 = 0.9$.

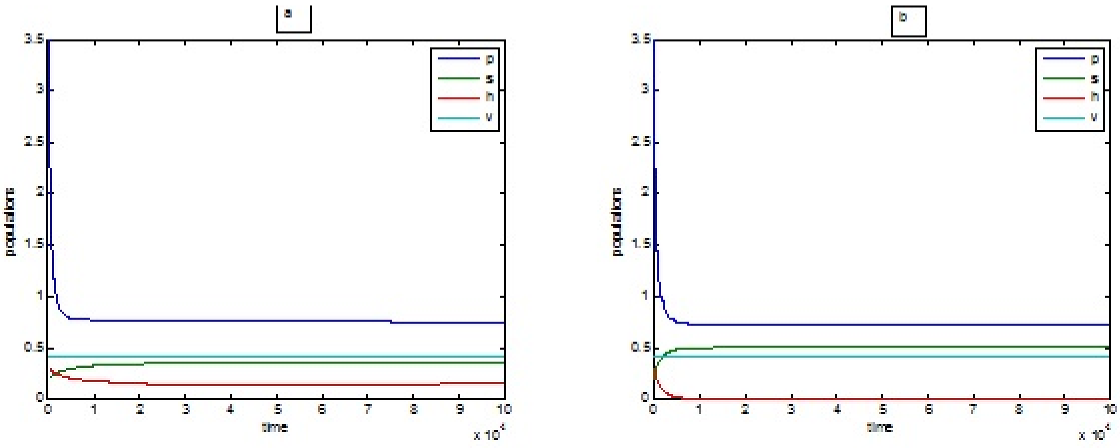

Figure 3: (a) Time series of the solution of system (1.2) for the data given in Eq. (5.1) with $u_7 = 0.3$ which approach to $E_5 = (0.756,\ \ 0.353,\ \ 0.154,\ \ 0.408)$. (b) Time series of the solution of system (1.2) for the data given in Eq. (4.9) with $u_7 = 0.9$ which approach to $E_4 = (0.725,\ \ 0.511,\ \ 0,\ \ 0.416)$

Note that, when $0 \leq u_{12} < 0.07$ the solution of system (1.2) remain approaches to E_5, as illustrated in Figure (4a), for typical value $u_{12} = 0.01$, while $0.07 \leq u_{12} < 1$ the solution of system (1.2) approach to E_4, as illustrated in Figure (4b), for typical value $u_{12} = 0.99$.

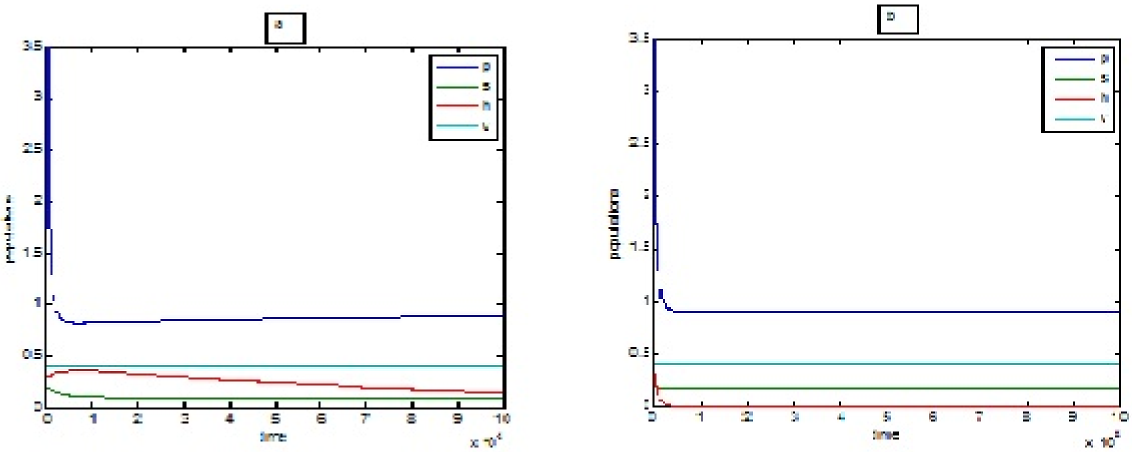

Figure 4: (a) Time series of the solution of system (1.2) for the data given in Eq.(4.9) with $u_{12} = 0.01$ which approach to $E_5 = (0.891,\ \ 0.1,\ \ 0.151,\ \ 0.396)$. (b) Time series of the solution of system (1.2) for the data given in Eq. (5.1) with $u_{12} = 0.99$ which approach to $E_4 = (0.894,\ \ 0.182,\ \ 0,\ 0.4)$.

In the range of $0 < u_{14} \leq 1$, note that when $0 \leq u_{14} < 0.088$ the solution of system (1.2) remain approach to E_5, as illustrated in Figure (4a), for typical value $u_{14} = 0.01$, furthermore $0.088 \leq u_{14} \leq 1$ the solution of system (1.2) approach to E_3, as illustrated in Figure(4b), for typical value $u_{14} = 0.9$.

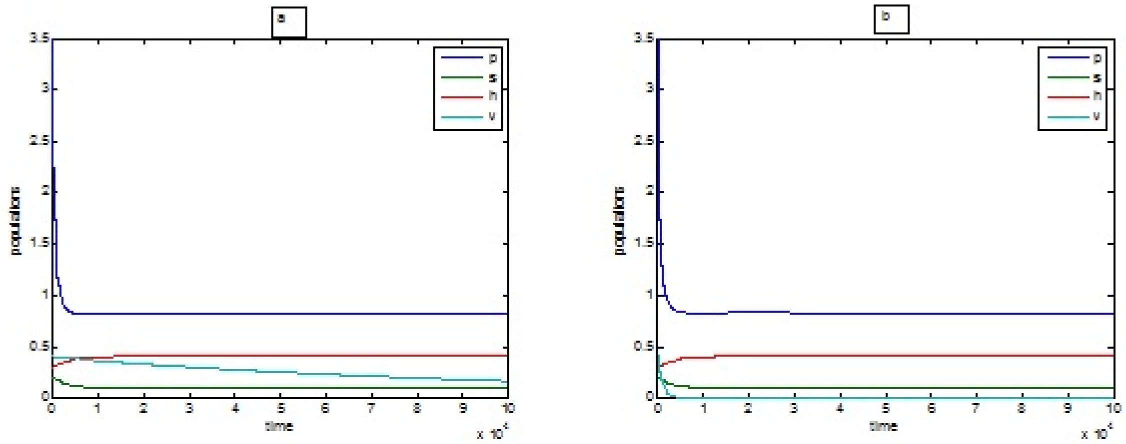

Figure 5: (a) Time series of the solution of system (1.2) for the data given in eq. (19) with $u_{14} = 0.01$ which approach to $E_5 = (0.821,\ \ 0.1,\ \ 0.403,\ \ 0.161)$. (b) Time series of the solution of system (1.2) for the data given in Eq. (5.1) with $u_{14} = 0.9$ which approach to $E_3 = (0.827,\ \ 0.1,\ \ 0.405,\ \ 0)$.

Now, when varying two parameters u_{12} and u_{14}) in the same time, in the range of $0.07 \leq u_{12} < 1$ and $0.088 \leq u_{14} \leq 1$ the solution of system (1.2) approach to E_2, as illustrated in Fig.(5), for typical values $u_{12} = 0.07$ and $u_{14} = 0.088$.

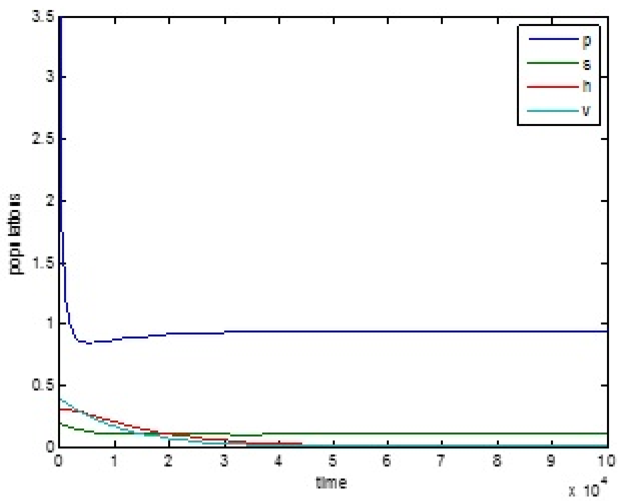

Figure 6: Time series of the solution of system (1.2) for the data given in Eq. (5.1) with $u_{12} = 0.07$ and $u_{14} = 0.088$ which approach to $E_2 = (0.946,\ \ 0.106,\ \ 0,\ \ 0)$.

Also varying three parameters u_{11}, u_{12} and u_{14}) in the same time , in the range of $0 \leq u_{11} \leq 1$, $0.07 \leq u_{12} < 1$ and $0.088 \leq u_{14} \leq 1$ the solution of system (1.2) approach to E_1, as illustrated in Figure 6, for typical values $u_{11} = 1$, $u_{12} = 0.9$ and $u_{14} = 0.99$.

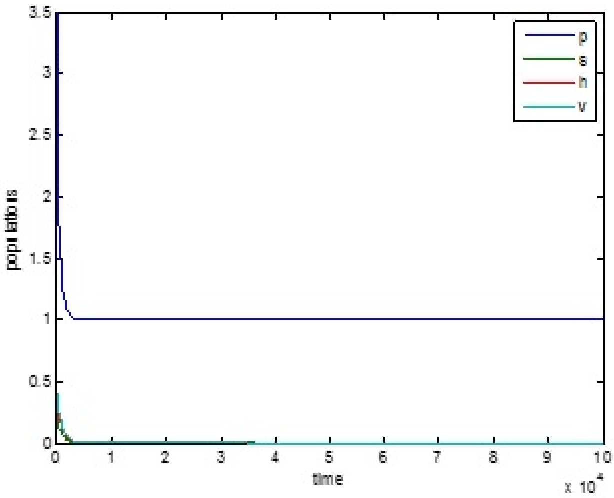

Figure 7: Time series of the solution of system (1.2) for the data given in Eq. (5.1) with $u_{11} = 1$, $u_{12} = 0.9$ and $u_{14} = 0.99$ which approach to $E_1 = (1, \ 0, \ 0, \ 0)$.

6. Discussion and Conclusions

In this paper, eco-epidemiological model has been proposed for study. Which includes SI disease in predator transmitted by an external source and vertically from mothers to offspring also SIS disease in predator species which is spread horizontally, by explicit contact between infected individuals and susceptible individuals. The two diseases cannot be transmitted from predator to prey by predation or by contact. Two types of functional response, linear and Holling type-II for depicting the predation as well as linear incidence for depicting the transition of diseases are used; the model is proposed and analyzed, and system (1.2) has been solved numerically for four initial points and the hypothetical set of parameters given by Eq. (5.1) and the following observation are obtained.

(i) No periodic solution is present through a set of hypothetical parameters in Eq. (5.1) in the system (1.2).
(ii) Varying of the parameters u_i , $i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15$, at each time and keeping the rest of parameters fixed as data given in Eq. (5.1) do not have any effect on the dynamical behavior of system (1.2) and the solution approach to E_5.
(iii) One of the most important results, the whole ecosystem cannot disappear altogether in the same species or with prey in the presence of the two diseases in the same time.
(iv) The parameters u_7, u_{11}, u_{12} and u_{14} play a vital role in this eco-epidemiological system.

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