Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2131-2144
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5354

On first excess level analysis of hysteretic bilevel control queue with multiple vacations

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this article, we consider queueing model with bilevel hysteretic control and multiple vacations. This system's mechanism depends on the queue size and the facility where the server goes to the series of vacation trips when the line is empty and returns but waiting in the system when the number of units is more than level M. In this case, the server doesn't start a new busy period unless this size is more than another level $N > M$. Furthermore, we employ N-policy and first excess level analysis to derive the probability generating function of queue size. Additionally, we assume that the vacation times are exponentially distributed random variables, and arrival batches are type 1 geometrically distributed random variables.

Keywords: hysteretic control, multiple vacations, N-policy, first excess level theory, input batches, bulk input, marked delayed renewal process, delayed renewal process, point process, marked point process, random walk.
2010 Msc MSC: 28AXX; 60G57; 60G55; 60HXX; 60H05.

1. Introduction

Control queue with vacations participates influential responsibility in several real-life situations and arises in industrial fields such operating systems, communications networks, call centers, computer systems, and transportation systems. In these systems, the server becomes idle and goes to multiple vacations when the queue drops to zero. After that, the sever returns to the system when the line up to a specific level of system size. In this study, the server stays in the design and waits

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Received: March 2021 *Accepted:* July 2021

until the line increases to another level. From the cost-economic perspective, it is advantageous to encourage the reneged customers not to quit the system and remain for their maintenances [4, 28, 31].

The initial study of batch arrival line with N-policy was rendered by Lee and Srinivasan [35]. They worked a system to get the optimal stationary operating policy under a suitable linear cost structure and other analyses. Subsequent Lee et al. [34] have deemed this category widely through several styles. In specific, some perspectives of this system have also been evaluated by Chae and Lee [8], Teghem [45], Medhi [38], Kalita and Choudhury [29], and Choudhury and Baruah [13].

Many researchers are offered batch arrival schemes under distinct vacation strategies because of their interdisciplinary environment. Lots of academics, including Baba [5], Choudhury [11, 12], Lee et al. [32, 33], Rosenberg and Yechiali [40], Madan and Abu-Dayyeh [37] and Teghem [46], and others have eXamined batch arrival line under several vacation rules.

The early notification for the control queue is given as a removable server by Romani since 1957 [4]. In [21], Yadin and Naor consider removable server and N-policy since 1963 that means the server is shut down after the line becomes empty and returns when the number of units rises to N. Later in 1970, they update their model to what call hysteretic control, where the server becomes idle when the size system is less than N and more than 1, but he will be busy if this size is large than N. There are different kinds of controllable queues: control of the number of servers, control of the service rate, control of the admission of customers, and control of the queueing discipline [29]. We consider the latter control in our work. There are numerous related studies which are significant in [4, 7, 9, 10, 14, 26, 27, 30, 36]. Some scholars alter the first level access technique to find the size system; see [16, 17, 19, 20, 21, 39, 42, 43].

In the neXt section, this article includes the fundamental concepts required to build our goals of this work, such as the first-level access analysis for the N-policy queue. Moreover, in the third section, we obtain the probability generating function (briefly, pgf) of queueing system size when this queue is controlled by two threshold levels M and N. The sever enters in idle period if the row drops to zero or returns to the system but waiting if the line is more than first level M. However, the server starts a new busy period whenever the line up to the second level N. We consider the arrival batches are type 1 geometrically distributed, and the vacation time is eXponentially distributed. The pgf of the number of units is found by using the level access analysis approach.

2. Model description

The most crucial purpose of this section is the evaluation of the stochastic process $\{Q(t); t > 0\}$ giving the accumulative number of units in the queue at time t. Let $(\Omega,\ \mathcal{F}(\Omega), P)$ be a probability space is with σ-algebra $\mathcal{F}(\Omega)$ where all processes in this work are defined in this space. Moreover, we assume these processes are right continuous. Let denote the nonnegative real numbers by $\mathbb{R}_0^+$.

2.1. Preliminaries

To begin with all notions and ideas, we should indicate notable works to Dr. Dshalalow in [2, 3, 6, 15, 18, 22, 23, 24, 25] that we select to give here as essential concepts.

A *filtration* $\{\mathcal{F}_t; t \geq 0\}$ is a monotone non-decreasing family of sub-σ-algebras in $\mathcal{F}(\Omega)$. It represents the expanded record of a specific stochastic process. The $\mathcal{F}_t$-adapted process X_t is a stochastic process such that for every Borel set $A \subseteq \mathbb{R}_0^+$, the set $\{w : X(t, w) \in A\}$ is an element of $\mathcal{F}_t$. A probability space $(\Omega,\ \mathcal{F}(\Omega), P)$ with filtration is called a filtered probability space and denoted by $(\Omega,\ \mathcal{F}(\Omega), \mathcal{F}_t, P)$. A stopping time T defined on the latter filtered probability space is a random variable (in short, r.v.) such that for any $t \geq 0$, the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$. The a.s. monotone increasing sequence of stopping times $\{T_n; n = 1, 2, \ldots, \}$ on $\mathbb{R}_0^+$ are called arrival times (point process). We provide another process associating with the point process (we called it, counting point process), where this process is the total number of arrivals in the time interval $[0, T]$ and defined as the following

$$C_t = C([0,t]) = \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(T_n) = \sum_{n=1}^{\infty} \epsilon_{T_n}([0,t]),$$

where $\mathbf{1}_B$ is the indicator function for set B and ϵ_X is a unit (or Dirac) mass, shown for any Borel set B and positive real number X as

$$\epsilon_X(B) = \begin{cases} 1, & X \in B \\ 0, & X \notin B \end{cases}$$

The (ordinary) Poisson point process $T = \{T_n; n = 1, 2, \ldots, \}$ is a point process on $\mathbb{R}_0^+$with the counting process C_t (so-called, (ordinary) Poisson counting point process) if the latter process C_t is Poisson distributed with parameter αt (termed, rate or intensity of C_t)and its increments is an independent and stationary or for $r < s$, $C_s - C_r$ is independent of $\mathcal{F}_r$ and $C_s - C_r$ has the same distribution of C_{s-r}. In fact, λ represents the mean number of arrivals in a unit time interval.
There is precious joint transformation connecting the point process with its counting process where it is provided as

$$E\left[z^{C_T} e^{-\theta T}\right]$$

where, when $z = 1$ we have

$$\beta(\theta) = E\left[e^{-\theta T}\right]$$

and it called the Laplace-Stieltjes transform of nonnegative r.v. T with the moment generating function $\beta(\theta) = m(-\tau)$. However, if $\theta = 0$, then $E\left[z^{C_T}\right]$ is just pgf of C_T. It is imperative to see if T is the independent of the Poisson counting process C_T with parameter, then

$$E\left[z^{C_T} e^{-\theta T}\right] = \beta(\theta + \lambda(1 - z))$$

Let C_T be Poisson counting point process associated with point process $T = \{T_n : n = 1, 2, \ldots\}$ and the sequence of independent and identically distributed (in short, iid) real-valued r.v.'s $X = \{X_n : n = 1, 2, \ldots\}$ with a common pgf $a(z)$ where X is the independence of T. We introduce $(X, T) = \{(X_n, T_n) : n = 1, 2, \ldots\}$ as the marked Poisson process and its counting as marked counting Poisson process with this following form

$$M_t = M([0,t]) = \sum_{n=1}^{\infty} X_n \epsilon_{T_n}([0,t]).$$

Consequently, the latter process is with position independent marking, and it has independent and stationary increments. Moreover, its pgf is given as compound Poisson r.v. by

$$E\left[z^{M_t}\right] = e^{\lambda(a(z)-1)t}$$

and the joint transform

$$E\left[z^{M_t} e^{-\theta t}\right] = \beta(\theta + \lambda(1 - a(z))).$$

2.2. Bulk input queue

The queue has bulk input (arrival batches) when the marked process represents these batches. Let $(X,T)=\{(X_n,T_n):n=1,2,\ldots\}$ be marked Poisson process corresponding with point process $T=\{T_n:n=1,2,\ldots\}$ of arriving rate λ. Moreover, the mark X_n is the random size of batches customers at arriving points T_n. Recall that, $X=\{X_n:n=1,2,\ldots\}$ is a sequence of iid nonnegative integer-valued r.v.'s with the pgf $a(z)$ and expectation a. Assume that this process (X,T) is with position independent marking. That means the marks X_n is independent of its position T_n.
Let Q_n be the number of units in the line upon exit of the nth unit at the time t_n, $n=0,1,\ldots$ Then $\{Q_n\}$ is a sequence with the following transitions:

$$Q_{n+1}=\begin{cases}Q_n-1+V_{n+1}\ , & Q_n>0\\ X_{R_n}-1+V_{n+1}, & Q_n=0\end{cases}$$

where X_{R_n} is the first group entering after the time t_n. In Fact, $\{Q_n\}$ is a time-homogeneous Markov chain embedded in $\{Q(t)\}$ upon leaving periods. The transition probability matrix (in brief, TPM) of $\{Q_n\}$ it is a $\triangle_2$-matrix. Hence, the chain is irreducible and aperiodic. Besides, we want to find the resulting

$$P_i(z)=E\left[z^{Q_1}\middle|Q_0=i\right]=\begin{cases}E\left[z^{i-1+V_1}\right], & i>0\\ E\left[z^{X_{R_n}-1+V_1}\right], & i=0\end{cases}=\begin{cases}z^{i-1}\beta(\lambda-\lambda a(z)), & i>0\\ z^{-1}a(z)\beta(\lambda-\lambda a(z)), & i=0\end{cases}$$

According to Abolnikov and Dukhovny [1], we should get $P_1^{'}(1-)$.

$$P_1^{'}(1-)=\beta^{'}(0)(-\lambda)a^{'}(1)=a\lambda b=\rho.$$

such that $\beta^{'}(0)=-b$, $a^{'}(1)=a$ and ρ is the offered load with the condition $\rho<1$. Consequently, $\{Q_n\}$ is recurrent positive, and then this chain is ergodic. To find pgf of distribution of Q_n, we have

$$P(z)=\sum_{i=0}^{\infty}p_iP_i(z)=p_0z^{-1}a(z)\beta(\lambda-\lambda a(z))+z^{-1}\beta(\lambda-\lambda a(z))\sum_{i=1}^{\infty}p_iz^i$$

By resolving the directly above, we get the comprehensive Pollaczed-Khinchine formula

$$P(z)=p_0\beta(\lambda-\lambda a(z))\frac{a(z)-1}{z-\beta(\lambda-\lambda a(z))}$$

To catch p_0, we can utilize the fact $P(1-)=1$, then $L'Hospital$ rule to get $p_0=1-\lambda ab$.

2.3. First access analysis theory

Let $(X,T)=\{(X_n,T_n):n=0,\ 1,\ 2,\ldots\}$ marked delayed renewal process with position dependent marking. This process represents a random walk. That means the mark X_n happens at the distinct time T_n. The delay when inter-renewal times $\{\Delta_n=T_n-T_{n-1}:\ n=0,\ 1,\ 2,...\}$ are independent, and all except for $T_0=\triangle_0$ are identically distributed. The mark X_n are position-dependent if it may depend on the inter-renewal time Δ_n. However, the mark X_n on Δ_n is conditionally independent of X_i such that $i<n$.

Let $\{T_n:\ n=0,1,2,...\}$ be a nondecreasing monotone. Thus, there is no clustering, and this is leading us to the associated counting process C_t is continuous in probability. Moreover, we presume that the marks X_n's are nonnegative integer-valued r.v.'s, and they have some joint transforms

$$\gamma(z,\theta)=Ez^{X_1}e^{-\Delta_1\theta},\qquad |z|\le 1,\ \mathrm{Re}\theta\ge 0,$$
$$\gamma_0(z,\theta)=Ez^{\mathcal{X}_0}e^{\Delta_0\theta},\qquad |z|\le 1,\ \mathrm{Re}\theta\ge 0,$$

We will see the emphasis on the performance of this random walk when the marking component

$$A_k = X_0 + \ldots + X_k$$

rises over fiXed level N. To do that, we will familiarize the following random index.

$$\upsilon = \inf \ \{n : A_n = X_0 + \ldots + X_n \geq N\} \tag{2.1}$$

at which the collective marks A_υ crosses threshold N. Similarly, the r.v. T_υ will be the first access time of this random walk.
We want to study the subsequent joint transform:

$$\Phi_\nu = \Phi_\nu \left(u \ , \upsilon, \ \vartheta, \ \theta\right) = E\xi^\nu u^{A_{\nu-1}} \upsilon^{A_\nu} e^{-\vartheta T_{\nu-1} - \theta T_{\nu-1}}, \tag{2.2}$$

To verify a directed value of the functional Φ_ν, first, we propose the secondary set of random indices

$$\{\nu\left(k\right) = \inf \ \{n : X_0 + \ldots + X_n > k\}, \ k = 0, 1, \ldots\}$$

as well as the set of the functionals

$$\left\{\Phi_{\nu(k)} = E\xi^{\nu(k)} u^{A_{\nu(k)-1}} \upsilon^{A_{\nu(k)}} e^{-\vartheta_{T_{\nu(k)-1}} - \theta_{T_{\nu(k)}}}, \ k = 0, 1, \ldots\right\} \tag{2.3}$$

Seeing from (2.1) and (2.2) that

$$\nu = \nu\left(N-1\right). \tag{2.4}$$

Next, we state the operator

$$D_k \left\{f\left(k\right)\right\}\left(X\right) := \sum_{k=0}^{\infty} X^k f\left(k\right)\left(1 - X\right), \ \|X\| < 1 \tag{2.5}$$

and the inverse operator can return f, if we employ it for every k:

$$\mathcal{D}_X^k \left(D_p \left\{f(p)\right\}\left(X\right)\right) = f(k), \ k = 0, 1, \ldots \tag{2.6}$$

where the inverse $\mathcal{D}^k$ is shown as

$$k \longmapsto \mathcal{D}_X^k \varphi\left(X, \ y\right) = \begin{cases} \lim_{X \to 0} \frac{1}{k!} \frac{\partial^k}{\partial X^k} \left[\frac{1}{1-X} \varphi\left(X, \ y\right)\right] & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{2.7}$$

Hence, we can restore $\Phi_{\nu(N-1)} = \Phi_\nu$ by employing $\mathcal{D}^{N-1}$ to $D_p \Phi_{\nu(p)}$.
In the succeeding theorem, we will deliberate the significant features of the inverse operator $\mathcal{D}^k$ which are directed in the consequent work.

Theorem 2.1 (Properties of operator $\mathcal{D}^k$). *Let $\mathcal{D}^k$ be the inverse operator of D_k as above equation* (2.7)*, then the following properties are true*

(i) *$\mathcal{D}^k$ is a linear functional.*
(ii) *$\mathcal{D}_X^k \left(\mathbf{1}\left(X\right)\right) = \mathbf{1}$, where $\mathbf{1}\left(X\right) = \mathbf{1}$ for all $X \in \mathbb{R}$*
(iii) *Let g be an analytic function at zero. Then, it holds true that*

$$\mathcal{D}_X^k \left(X^j g\left(X\right)\right) = \mathcal{D}_X^{k-j} g\left(X\right). \tag{2.8}$$

(iv) In particular of (iii), if $j=k$, we have

$$\mathcal{D}_X^k\left(X^k g(X)\right)=g(0). \tag{2.9}$$

(v) Let $a(X)=\sum_{i=0}^{\infty} a_i X^i$. Then,

$$\mathcal{D}_X^k(a(X))=\sum_{i=0}^{k} a_i \text{ and } \mathcal{D}_X^k(a(xy))=\sum_{i=0}^{k} a_i y^i \tag{2.10}$$

(vi) For any real number b it holds true that

$$\mathcal{D}_X^k\left\{\frac{1}{1-bX}\right\}=\begin{cases}\frac{1-b^{k+1}}{1-b}, & b\neq 1\\ k+1, & b=1\end{cases} \tag{2.11}$$

(vii) For any real number a and for a positive integer n, except for $a=n=1$, it holds true that

$$\mathcal{D}_X^k\left\{\frac{1}{(1-aX)^n}\right\}=\begin{cases}\sum_{j=0}^{k}\binom{n+j-1}{j}a^j & \text{except for } a=n=1\\ k+1, & a=n=1\end{cases} \tag{2.12}$$

(viii) For two real numbers a and b it holds

$$\mathcal{D}_X^k\left\{\frac{1}{1-bX}\frac{1}{(1-aX)^n}\right\}=\begin{cases}\frac{1}{1-bX}\sum_{j=0}^{k}\binom{n+j-1}{j}\left(a^j-b^{k+1}\left(\frac{a}{b}\right)^j\right), & b\neq 1\\ \sum_{j=0}^{k}\binom{n+j-1}{j}a^j(k-j+1), & b=1\end{cases} \tag{2.13}$$

The following theorem is critical to our further works.

Theorem 2.2 (The Key First Access Theorem). *Let the following functionals be given as*

$$\gamma:=\gamma(Xu\upsilon,\ \vartheta+\theta),\quad \gamma_0:=\gamma_0(Xu\upsilon,\ \vartheta+\theta) \tag{2.14}$$
$$\Gamma:=\gamma(X\upsilon,\theta),\quad \Gamma^1:=\gamma(\upsilon,\theta), \tag{2.15}$$
$$\Gamma_0:=\gamma_0(X\upsilon,\theta),\quad \Gamma_0^1:=\gamma_0(\upsilon,\ \theta). \tag{2.16}$$

Then, it holds true that

$$\Phi^*(X)=D_p(\Phi_{\nu(p)}(X))=\Gamma_0^1-\Gamma_0+\frac{\gamma_0\xi}{1-\gamma\xi}\left(\Gamma^1-\Gamma\right) \tag{2.17}$$

and the functional Φ_ν meets the following

$$\Phi_\nu=\Phi_\nu(u,\upsilon,\ \vartheta,\ \theta)=E\xi^\nu u^{A_{\nu-1}}\upsilon^{A_\nu}e^{-\vartheta\tau_{\nu-1}-\theta\tau_{\nu-1}}=\mathcal{D}_X^{N-1}\left(\Gamma_0^1-\Gamma_0+\frac{\gamma_0\xi}{1-\gamma\xi}\left(\Gamma^1-\Gamma\right)\right) \tag{2.18}$$

Corollary 2.3. *Let $\xi=u=1,\ \vartheta=0$, then*

$$\Phi_\nu=E\upsilon^{A_\nu}e^{-\theta_{\tau_\nu}}=\gamma_0(\upsilon,\theta)-(1-\gamma(\upsilon,\ \theta))\mathcal{D}_X^{M-1}\left(\frac{\gamma_0(X\upsilon,\ \theta)}{1-\gamma(X\upsilon,\ \theta)}\right) \tag{2.19}$$

and if $\tau_0=\Delta_0=0$ and $X_0=A_0=i\geq 0$, then $\gamma_0(\upsilon,\theta)=\upsilon^i$ and

$$\Phi_\nu=E\upsilon^{A_\nu}e^{-\theta_{\tau_\nu}}=\upsilon^i-\upsilon^i(1-\gamma(\upsilon,\ \theta))\mathcal{D}_X^{M-1}\left(X^i\frac{1}{1-\gamma(X\upsilon,\ \theta)}\right) \tag{2.20}$$

2.4. Queues with an N-Policy

The server shutdown to be idle in the system if the queue drops to zero. Nevertheless, the next busy period does not begin with the early reaching batch unless it meets a certain positive number N. The server returns maintenance, and the queue size A_v and T_v can be established by the formula

$$\Phi_v = Ez^{A_v}e^{-\theta_{T_v}} = z^i - z^i\left(1 - \Gamma^1\right)\mathcal{D}_X^{N-1-i}\left(\frac{1}{1-\Gamma}\right)$$

Since the number of units in the queue is zero, then $i = 0$.

$$\Phi_v = Ez^{A_v}e^{-\theta_{T_v}} = 1 - \left(1 - \gamma\left(z{,}\theta\right)\right)\mathcal{D}_X^{N-1}\left(\frac{1}{1-\gamma\left(zX,\theta\right)}\right)$$

where

$$\gamma\left(z{,}\theta\right) = a\left(z\right)\frac{\lambda}{\lambda+\theta}$$

and the marginal transform

$$\alpha\left(z\right) = Ez^{A_v} = 1 - \left[1 - a\left(z\right)\right]\ \mathcal{D}_X^{N-1}\left(\frac{1}{1-a\left(Xz\right)}\right)$$

and to create Kendall's formula, we start with

$$Q_{n+1} = \begin{cases} A_v - 1 + V_1, & Q_n = 0 \\ Q_{n-1} - 1 + V_1, & Q_n > 0 \end{cases}$$

to get the subsequent

$$P_i\left(z\right) = E\left[z^{Q_1}\middle|Q_0 = i\right] = \begin{cases} z^{-1}\alpha\left(z\right)\beta\left(\lambda - \lambda a\left(z\right)\right), & i = 0 \\ z^{i-1}\beta\left(\lambda - \lambda a\left(z\right)\right), & i > 0 \end{cases}$$

Thus, the pgf of this system

$$P\left(z\right) = \sum_{i=0}^{\infty} p_i P_i(z) = p_0\alpha\left(z\right)z^{-1}\beta\left(\lambda - \lambda a\left(z\right)\right) + z^{-1}\beta\left(\lambda - \lambda a\left(z\right)\right)\sum_{i=1}^{\infty} p_i z^i$$

By making only some steps, we find

$$P\left(z\right) = p_0\beta\left(\lambda - \lambda a\left(z\right)\right)\frac{\alpha\left(z\right) - 1}{z - \beta\left(\lambda - \lambda a\left(z\right)\right)}$$

and

$$p_0 = \frac{1 - \lambda ab}{\alpha} = \frac{1-\rho}{\alpha}$$

where

$$\alpha := EA_v = \alpha'\ \left(z\right)\big|_{z=1}$$

2.5. An n-policy queue with multiple vacations

This model deals with multiple vacations, which occurs when the server goes on a series of vacation segments. In this case, the server terminates his break and comes back to the system if the line contains to N or more customers.

Let $\mathcal{T}_1, \mathcal{T}_2, \cdots$ be a renewal point process as vacation times associating with a sequence of inter-renewal times $\Delta_k = \mathcal{T}_k - \mathcal{T}_{k-1}, k = 1, 2, \cdots,\ \mathcal{T}_0 = 0$, where these times are iid r.v.'s and they have LST $\gamma(\vartheta) = Ee^{-\Delta_1 \vartheta}$. We can represent this queue by using a marked renewal process $(A, \mathcal{T})$ with position dependent marking where its counting process is given as

$$M_t = \sum_{k=1}^{\infty} X_k \epsilon_{\mathcal{T}_k}([0, t))$$

and it is related to the following delayed renewal counting point process for process $\mathcal{T}_1, \mathcal{T}_2, \cdots$

$$\mathcal{T} = \sum_{k=1}^{\infty} \epsilon_{\mathcal{T}_k}$$

Such that X_k is the number of arrivals in the interval $[\mathcal{T}_{k-1},\ \mathcal{T}_k)$ and with $X_0 = 0$. The model is eXhaustive, then $\gamma_0(z, \theta) = 1$. We also have the following assumption

$$\gamma(z, \theta) = Ez^{X_1} e^{-\Delta_1 \theta} = \gamma(\theta + \lambda - \lambda a(z)) = \frac{\lambda}{\lambda + \theta} a(z), \gamma(\vartheta) = Ee^{-\Delta_1 \vartheta}$$

Let $\mathcal{T}_v$ be the first access level time by corollary 2.10, and we have with $i = 0$

$$\phi_v(z, \theta) = Ez^{A_v} e^{-\theta \mathcal{T}_v} = 1 - [1 - \gamma(\theta + \lambda - \lambda a(z))] \mathcal{D}_X^{N-1} \left(\frac{1}{1 - \gamma(\theta + \lambda - \lambda a(\mathrm{Xz}))} \right)$$

and the marginal transform

$$\alpha(z) = Ez^{A_v} = 1 - [1 - \gamma(\lambda - \lambda a(z))] \mathcal{D}_X^{N-1} \left(\frac{1}{1 - \gamma(\lambda - \lambda a(\mathrm{Xz}))} \right).$$

Therefore, we see that Kendall's formula for $P(z)$ has the following form

$$P(z) = p_0 \beta(\lambda - \lambda a(z)) \frac{1 - \alpha(z)}{\beta(\lambda - \lambda a(z)) - z} 0$$

where the latter formula is subjected to

$$P_0(z) = E\left[z^{Q_{n+1}} | Q_{\ n} = 0\right] = \beta(\lambda - \lambda a(z)) \alpha(z) z^{-1},$$
$$p_0 = \frac{1 - \lambda ab}{\alpha} = \frac{1 - \rho}{\alpha}, \qquad \alpha := EA_v = \alpha'(z)|_{z=1}.$$

2.6. A bilevel hysteretic control queue with multiple vacations

This system includes two thresholds M and N such that $M \leq N$. If this system is empty, the server leaves it for multiple vacations according to the threshold $M \geq 0$. However, the server returns to the system when upon the end of one of his vacation trips and the queue crosses M. In this case, if the queue turns out to be N, then the server initiates a new busy period. Otherwise, the server waits in the system. The input is assumed to be bulk. We called this model the queue with bilevel hysteretic control with multiple vacations.

Our model is the single-server queueing system with the Poisson bulk input, general service time, and bilevel hysteretic control with multiple vacations. Notice that if $M = N$, the system shrinks to the common N-policy with multiple vacations, and if $M = 0$, the system is N-policy without vacations. The system behaves within three distinct parts; the first one appears when the server is absent from the system, the second one is when the server is in the system but waiting, and the third one is when the server is busy with his primary work.

Let $\{Q(t); t \geq 0\}$ be queuing process and $\{\mathcal{T}_1, \mathcal{T}_2, \cdots\}$ be the end of vacation time. Let X_i number of customers during $(\mathcal{T}_{i-1}, \ \mathcal{T}_i]$ $, i = 1, 2, \ldots \ (\mathcal{T}_0 = 0)$. Assume $(A, \mathcal{T}) = \{(A_k, \mathcal{T}_k), \ k = 1, 2, \cdots\}$ be a random walk process where $A_k = \sum_{i=1}^{k} X_i$ is the total number of customers at $\mathcal{T}_k$. If $A_\mu \geq M$, then the server returns to the system and

$$\mu = \inf \ \{m = 1, 2, \cdots : X_1 + \cdots + X_m = A_m \geq M\}$$

is the first access level index.
If $A_\mu < N$, then the server is waiting in the buffer to fill up to N or more customers arriving in batches. Let (B_n, t_n^*), $n = 0, 1, \cdots$ be random walk process where t_n^* are arrival times of batches with $t_0^* = \mathcal{T}_\mu$, and $B_n = \sum_{i=0}^{n} U_i$, with $U_0 = A_\mu$. We called U_n is the nth batch of customers that arrives at t_n^*. If $B_v \geq N$, then the severer is busy where

$$v = \inf \ \{n = 0, 1, \cdots : B_n \geq N\}$$

is called the second access level index.
We are fascinating to find k s.t.

$$A_k = \sum_{i=0}^{k} X_i \geq M$$

and to find n s.t.

$$B_n = \sum_{i=0}^{n} U_i \geq N \ , \ U_0 = A_\mu$$

By assuming the input bulks $(A, \mathcal{T})$ be marked Poisson (compound Poisson), with position independent marking and identically A, where its counting process

$$M_t = \sum_{i=1}^{\infty} X_i \epsilon_{\mathcal{T}_i}([0, t))$$

where the marks $X_i's$ are iid r.v.'s with common pdf and the mean

$$a(z) = E z^{X_i} \ , \quad a = EX_i, \quad i = 1, 2, \ldots$$

Let the vacation times $\Delta_1, \Delta_2, \ \ldots, \ (\Delta_i = \mathcal{T}_i - \mathcal{T}_{i-1})$ be iid r.v.'s with the common LST

$$\gamma(\theta) = E e^{-\theta \Delta_1}$$

The marginal counting process of $(A, \mathcal{T})$ is given as

$$C_t = \sum_{i=1}^{\infty} \epsilon_{\mathcal{T}_i}([0, t))$$

where $\mathcal{T}_n = \sum_{i=1}^{n} \Delta_i$ and the functional

$$\gamma(z,\theta) = Ez^{X_1}e^{-\Delta_1\theta} = \gamma(\theta+\lambda-\lambda a(z))$$

gives the number of units entering the system during one vacation trip, and we have

$$\alpha_0(z) = \phi_\mu(z,0) = 1 - [1-\gamma(\theta+\lambda-\lambda a(z))]\,\mathcal{D}_X^{M-1}\left(\frac{1}{1-\gamma(\theta+\lambda-\lambda a(\mathrm{xz}))}\right)$$

If $M \leq A_\mu < N$, then the server is waiting in the system until the buffer is up to N or more. In this case, the input batches B modeled as a marked delayed Poisson process $(B, \mathcal{T}^*)$ with counting process

$$\widehat{M}_t = \sum_{k=0}^{\infty} U_k \epsilon_{t_k^*}([0,t))$$

where the marks $U_1, U_2, \ \ldots$ are iid r.v.'s with pdf and mean

$$a(z) = Ez^{U_i}\ ,\ a = EU_i,\ U_0 = A_\mu \text{ and } \quad t_0^* = \mathcal{T}_\mu, \qquad i = 1,2,\ldots$$

Let the waiting times $\Delta_1^*, \Delta_2^*, \ \ldots,\ \left(\Delta_i^* = t_i^* - t_{i-1}^*\right)$ be iid r.v.'s with the common LST

$$\Gamma(\theta) := Ee^{-\theta\Delta_1}\ , \quad \|z\| \leq 1, \quad Re(\theta) \geq 0$$
$$\Gamma_0(\theta) := Ee^{-\theta\Delta_0} = Ee^{-\theta t_0^*} = Ee^{-\theta\mathcal{T}_v}$$

where

$$\widehat{C}_t = \sum_{i=1}^{\infty} \epsilon_{t_k^*}([0,t))$$

is the Poisson point process.
Define the functional

$$\Gamma(z,\theta) = E\left[z^{U_i}e^{-\theta\Delta_i^*}\right] = a(z)\frac{\lambda}{\lambda+\theta}$$
$$\Gamma_0(z,\theta) = E\left[z^{U_0}e^{-\theta\Delta_0^*}\right] = Ez^{A_\mu}e^{-\theta\mathcal{T}_\mu} = \phi_\mu(z,\theta)$$

where we are curious to find

$$\psi_v(z,\theta) = Ee^{B_v}e^{-\theta\mathcal{T}_v^*}$$

where

$$v = \inf\left\{n : B_n = \sum_{i=0}^{n} U_i \geq N\right\}$$

So that, we have

$$\psi_v(z,\theta) = \phi_\mu(z,\theta) - \left[1 - a(z)\frac{\lambda}{\lambda+\theta}\right]\mathcal{D}_u^{N-1}\left(\frac{\phi_\mu(uz,\theta)}{1-a(uz)\frac{\lambda}{\lambda+\theta}}\right), \qquad \text{and}$$
$$(z) := Ez^{B_v} = \psi_v(z,0) = \alpha_0(z) - [1-a(z)]\,\mathcal{D}_u^{N-1}\left\{\frac{\alpha_0(uz)}{1-a(uz)}\right\}$$

with

$$\alpha = EB_v = ac\lambda\mathcal{D}_X^{M-1}\left\{\frac{1}{1-\gamma(\lambda-\lambda a(X))}\right\} + a\mathcal{D}_u^{N-1}\left\{\frac{1-\left([1-\gamma(\lambda-\lambda a(u))]\,\mathcal{D}_X^{M-1}\left\{\frac{1}{1-\gamma(\lambda-\lambda a(\mathrm{Xu}))}\right\}\right)}{1-a(u)}\right\}.$$

3. Applications on a bilevel hysteretic control queue with multiple vacations

In this sequel, we will consider the crucial targets of our effort. We intend to receive pgf of the number of elements for this system when vacations times are exponentially distributed with parameter v, and arriving batches are type 1 geometrically distributed with parameter $p = 1/a$. So, we have

$$\gamma(\theta) = \frac{v}{v+\theta} \quad and \quad a(z) = \frac{pz}{1-qz}$$

To find $P(z)$, we know that

$$P(z) = p_0 \beta(\lambda - \lambda a(z)) \frac{(\alpha(z) - 1)}{Z - \beta(\lambda - \lambda a(z))}$$

where

$$\alpha(z) = \alpha_0(z) - [1 - a(z)] \mathcal{D}_y^{N-1}\left[\frac{\alpha_0(\mathrm{yz})}{1 - a(\mathrm{yz})}\right]$$

and

$$\alpha_0(z) = 1 - [1 - \gamma(\lambda - \lambda a(z))] \mathcal{D}_X^{M-1}\left[\frac{1}{1 - \gamma(\lambda - \lambda a(\mathrm{Xz}))}\right]$$

First of all, we will find the following

$$1 - a(z) = \frac{1-z}{1-qz}$$

Therefore, we get

$$\gamma(\lambda - \lambda a(z)) = \gamma(\lambda(1 - a(z))) = \frac{v}{v + \lambda(1 - a(z))}$$

The latter is provided the following

$$1 - \gamma(\lambda - \lambda a(z)) = \frac{\lambda(1-z)}{v(1-qz) + \lambda(1-z)}$$

and so we obtain

$$\frac{1}{1 - \gamma(\lambda - \lambda a(z))} = 1 + \frac{v}{\lambda}\,\frac{1}{1-z} - \frac{\mathrm{vq}}{\lambda}\,\frac{z}{1-z}$$

Now, we are ready to find this operator by depending on properties (i), (iii), and (vi) as below

$$\begin{aligned}\mathcal{D}_X^{M-1}\left[\frac{1}{1 - \gamma(\lambda - \lambda a(\mathrm{Xz}))}\right] &= \mathcal{D}_X^{M-1}\left[1 + \frac{v}{\lambda}\,\frac{1}{1-zX} - \frac{\mathrm{vq}}{\lambda}\,\frac{\mathrm{zX}}{1-zX}\right] \\ &= \mathcal{D}_X^{M-1}(1) + \frac{v}{\lambda}\mathcal{D}_X^{M-1}\left(\frac{1}{1-zX}\right) - \frac{\mathrm{vqz}}{\lambda}\mathcal{D}_X^{M-1}\left(\frac{X}{1-zX}\right) \\ &= 1 + \frac{v}{\lambda}\,\frac{1-z^M}{1-z} - \frac{\mathrm{vqz}}{\lambda}\,\frac{1-z^{M-1}}{1-z}\end{aligned}$$

Therefore, we can see that

$$\alpha_0(z) = 1 - \frac{\lambda(1-z)}{v(1-qz) + \lambda(1-z)}\left[1 + \frac{v}{\lambda}\,\frac{1-z^M}{1-z} - \frac{\mathrm{vqz}}{\lambda}\,\frac{1-z^{M-1}}{1-z}\right] = \frac{\mathrm{pv}}{v+\lambda}\,\frac{z^M}{1 - \left(\frac{vq+\lambda}{v+\lambda}\right)z}$$

Second, we need to achieve

$$\frac{\alpha_0(z)}{1-a(z)} = \frac{\text{pv}}{v+\lambda}\ \frac{z^M}{1-\left(\frac{vq+\lambda}{v+\lambda}\right)z}\ \frac{1-qz}{1-z} = \frac{\text{pv}}{v+\lambda}\ \left[\frac{z^M}{\left(1-\left(\frac{vq+\lambda}{v+\lambda}\right)z\right)(1-z)} - \frac{qz^{M+1}}{\left(1-\left(\frac{vq+\lambda}{v+\lambda}\right)z\right)(1-z)}\right]$$

we will use this fact

$$\frac{1}{(1-az)(1-bz)} = \frac{A}{(1-az)} + \frac{B}{(1-bz)}$$

where $A = \frac{a}{a-b}$ and $B = \frac{b}{b-a}$. Let $a = \frac{vq+\lambda}{v+\lambda}$ and $b = 1$, then

$$A = \frac{\frac{vq+\lambda}{v+\lambda}}{\frac{vq+\lambda}{v+\lambda}-1} = -\frac{vq+\lambda}{\text{vp}}\ , B = \frac{1}{1-\frac{vq+\lambda}{v+\lambda}} = \frac{v+\lambda}{\text{vp}}$$

and we get as a result

$$\frac{\alpha_0(z)}{1-a(z)} = \frac{vq+\lambda}{v+\lambda}\ \frac{-z^M}{\left(1-\left(\frac{vq+\lambda}{v+\lambda}\right)z\right)} + q\frac{z^{M+1}}{(1-z)}$$

Thus, we will apply the operator on the latter and corresponding properties (i), (iii), and (vi). We get

$$\mathcal{D}_y^{N-1}\left[\frac{\alpha_0(z)}{1-a(z)}\right] = \frac{-(vq+\lambda)}{v+\lambda}\ \frac{1-\left[\frac{(vq+\lambda)}{(v+\lambda)}z\right]^{N-M}}{1-\left[\left(\frac{vq+\lambda}{v+\lambda}\right)z\right]} + q\frac{1-z^{N-M-1}}{1-z}$$

Currently, we can find

$$\alpha(z) = \frac{\text{pv}}{v+\lambda}\ \frac{z^M}{1-\left(\frac{vq+\lambda}{v+\lambda}\right)z} - \frac{1-z}{1-qz}\left[\frac{-(vq+\lambda)}{v+\lambda}\ \frac{1-\left[\frac{(vq+\lambda)}{(v+\lambda)}z\right]^{N-M}}{1-\left[\left(\frac{vq+\lambda}{v+\lambda}\right)z\right]} + q\frac{1-z^{N-M-1}}{1-z}\right]$$

By taking derivate for above pgf and let $z = 1$, the eXpectation is given as

$$= \frac{\text{pv}}{v+\lambda}\left[\frac{M}{\left(1-\frac{vq+\lambda}{v+\lambda}\right)} - \frac{1}{\left(1-\frac{v+\lambda}{vq+\lambda}\right)}\right] + \frac{vq+\lambda}{v+\lambda}\frac{1}{p}\left(1-\frac{vq+\lambda}{v+\lambda}\right)^{N-M-1}$$

To find p_0, we consider the formula

$$p_0 = \frac{1-\rho}{\alpha}\text{where } \rho = \lambda ab$$

To get a and b, we know that

$$a = \ a(z)|_{z=1} = 2 - \frac{1}{a}$$

and

$$b = \ -\beta(\theta)|_{\theta=0} = \frac{1}{v}$$

then

$$p_0 = \frac{1-\frac{\lambda}{v}\left(2-\frac{1}{a}\right)}{\alpha}$$

Again, we need to find

$$\beta(\lambda-\lambda a(z))=\frac{v}{v+\lambda(1-a(z))}$$

and

$$z-\beta(\lambda-\lambda a(z))=\frac{vz+\lambda(1-a(z))z-v}{v+\lambda(1-a(z))}$$

So, we have

$$\frac{\beta(\lambda-\lambda a(z))}{z-\beta(\lambda-\lambda a(z))}=\frac{v\left[1-\left(1-\frac{1}{a}\right)z\right]}{(1-z)\left[(\lambda-v)+\left(1-\frac{1}{a}\right)\mathrm{vz}\right]}$$

Therefore, we get

$$P(z)=\frac{1-\frac{\lambda}{v}\left(2-\frac{1}{a}\right)}{\alpha}\;\frac{v\left[1-\left(1-\frac{1}{a}\right)z\right]}{(\lambda-v)+\left(1-\frac{1}{a}\right)\mathrm{vz}}\;\frac{\alpha(z)-1}{1-z}$$

4. Conclusion

Our results are concerned with deriving the directed formula for pgf of system size of controllable queue with two threshold levels using the first level access theory. These levels affect the work of the server. The server becomes exhausted and rests when the line is empty. Moreover, he goes back to this system and waiting when the queue has M units or more. Finally, the busy period of the server will begin when the number of arrivals is N and more where $N > M$. We assume that the arrival batches are type 1 geometrically distributed, and the vacation trips are exponentially distributed.

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