Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2145-2152
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5355

q-homotopy analysis method for solving nonlinear Fredholm integral equation of the second kind

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(Communicated by Madjid Eshaghi Gordji)

Abstract

Several scientific and engineering applications are usually described as integral equations. We propose a numerical approach for solving the type of Fredholm nonlinear boundary value problems in a finite domain. The paper aims to use the q-homotopy analysis method to estimate the solution to test the efficiency of the proposed method. Comparison with updated work is exacted. The obtained results show that the proposed method is very effective and convenient for nonlinear Fredholm integral equations. The interval of convergence of homotopy analysis method, if exists, is increased when using q-homotopy analysis method is more to converge. The result reveals that the q- homotopy analysis method is considered a good method for solving NFIES.

Keywords: Nonlinear Integral equation, Fredholm Integral equation of the second kind, q-homotopy analysis method
2010 MSC: Primary 90C33; Secondary 26B25.

1. Introduction

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, mathematical physics, engineering potential theory, electrostatics and radiative heat transfer problems, a nonlinear problem can be represented in general by a set of governing equations with corresponding initial boundary conditions. The nonlinear Fredholm integral equation as defined below:

$$u(x) = f(x) + \int_a^b k(x,t)F(u(t))dt \tag{1.1}$$

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Received: April 2021 *Accepted:* July 2021

where $k(x,t)$ is an arbitrary function called the kernel of the integral equation (1.1) defined over square $G:[a,b]\times[a,b]$ and $f(x)$ is a given function of $x\in[a,b]$u(t) unknown function. Several scientific and engineering applications are usually described as integral equations. Also, there is a connection between integral and differential equations. This kind has been described analytically and numerically ([2], [5]).

Different mathematical methods have been applied to solve integral problems in recent articles. Some of these approaches are based on different types of wavelets [1, 4, 7, 8, 9, 11, 14, 16, 17]. Also, it proved there are a lot of numerical methods to solve the kinds of integral equations, such as the Trapezoidal method, Nystrom method, the Galerkin method; Simpson's Rule and Romberg Integration ([4, 10]). polynomial approximation methods ([3, 15]) using different functions, Chebyshev polynomials [31], Bernstein polynomials ([26, 32]), numerical methods of integration [27]. As a basis, a different approach for obtaining approximate solutions was developed. Liao [18] proposed a general analytic approach for linear and nonlinear problems based on the core principles of homotopy in topology, called the homotopy analysis method (HAM). This approach has been successfully applied in recent years to handle a huge variety of nonlinear problems in science and engineering [6, 19, 20, 21, 22, 23, 24, 25, 28, 29]. The homotopy analysis method (HAM) is an analytic method. It is valid even if a given nonlinear problem does not contain any small/large parameters and can provide us with a convenient way to adjust and control the convergence region and rate of approximation series. It can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.A more general method of homotopy analysis method (HAM) was introduced to solve non-linear integral equations. it is called (q-HAM). The interval of convergence of HAM, if it exists, is increased when using q-HAM. The analysis shows that the series solution in the case of q-HAM is more likely to converge than that on HAM. q-HAM contains an auxiliary parameter as well as h such that the case of r =1 (q-HAM; r =1) the standard homotopy analysis method (HAM) can be reached ([12, 13]). The q-HAM has been successfully applied to numerous problems in science and engineering. All calculations are executed by MATLAB.

2. The q-homotopy analysis method

To describe the dynamic of homotopy analysis method (HAM) under homotopy theory in topology called q-HAM [12],by considering the following equation of the form:

$$N[u(x)]=0 \tag{2.1}$$

where N is a nonlinear operator, x is an independent variable and $u(x)$ is an unknown function, according to the zero-order deformation equation:

$$qH(x;q)=(1-rq)L\left[\theta(x;q)-u_0(x;q)\right]-qhH(x)N[\theta(x;q)]=0 \tag{2.2}$$

where $q\in[0,1/r]$ is an embedding parameter, $h\neq 0$ is a convergence-control parameter, $H\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x;\mathrm{q})$ is the initial guess of $u(x)$ and $\theta(\mathrm{x};\mathrm{q})$ is the auxiliary function that should be satisfied in the function initial conditions, it should be noted that h and $H(x)$ are necessary to the qHAM series solution rate of the convergence. The parameter q changes from 0 to $1/r$ to generate series solution such that when q = 0, one has

$$qH(x;0)=L\left[\theta(x;0)-u_0(x;0)\right]=0 \tag{2.3}$$

while if $q=1/r$, we get

$$qH(x;1)=-qhH(x)N[\theta(x;1)]=0 \tag{2.4}$$

$$\text{Thus, by imposing } qH(x;q)=0 \tag{2.5}$$

we can obtain

$$(1-rq)L\left[\theta(x;q)-u_0(x;q)\right]=qhH(x)N[\theta(x;q)] \tag{2.6}$$

if q = 0 and q = 1/r, the homotopy equations becomes

$$\begin{cases} \theta(x;0)=u_0(x) \\ \theta(x;1/r)=u(x) \end{cases} \tag{2.7}$$

As q changes from 0 to $1/r$ the solution $\theta(x;q)$ varies from the initial guess $u_0(x)$ to the qHAM solution $u(x)$. By expanding $\theta(x;\text{q})$ as a Taylor series in terms of q, we can yield the series solution in the following form:

$$\theta(x;q)=u_0(x)+\sum_{i=1}^{\infty}u_i(x)q^i \tag{2.8}$$

where

$$u_i(x)=\frac{1}{i!}\frac{\partial^i\theta(x;Q)}{\partial Q^i}\bigg|_{Q=0} \tag{2.9}$$

One has to know that, the auxiliary linear operator L, the initial guess $u_0(x)$, the convergence control h and the auxiliary function $H(x)$ are very important for the best q -homotropy series solution. Note that if $q=1/r$ then we have

$$\theta(x;1/r)=u_0(x)+\sum_{i=1}^{\infty}u_i(x)(1/r)^i \tag{2.10}$$

By defining the vectors

$$\overrightarrow{u_m}(x)=\{u_0(x),u_1(x),\ldots,u_i(x)\}$$

For i time derivatives of Eq. (2.6) in terms of q and then put $q=0$ and after that dividing them by $i!$, we obtain the i^{th} -order deformation equation [31]

$$L\left[u_i(x)-\chi_i u_{i-1}(x)\right]=h\mathcal{R}_i\left(\overline{u_{l-1}}(x)\right) \tag{2.11}$$

where

$$\mathcal{R}_i\left(\overline{u_{l-1}}(x)\right)=\frac{1}{(i-1)!}\frac{\partial^{i-1}N[\theta(x;q)]}{\partial q^{i-1}}\bigg|_{q=0} \tag{2.12}$$

$$\chi_i=\begin{cases} 0, i\leq 1 \\ r,\ \text{othewise} \end{cases}$$

3. The proposed method for solving NFIES using the q-HAM method

q-HAM in previous section to Eq. (1.1) as follows:
Construct the zeroth-order deformation for Eq. (1.1) as

$$(1-rq)L[u(x;q)-f(x)] = qh\left[u(x;q)-f(x)-\int_a^b k(x,t)F(u(x))dt\right] \tag{3.1}$$

Set the values of $q=0$ and $q=1/r$, implying

$$\begin{cases} u(x;0)=f(x) \\ u(x;1)=u(x) \end{cases} \tag{3.2}$$

From Eq. (3.2) we conclude that the initial guess $u_0(x;q)$ can be selected from $f(x)$ as there are no restrictions to select the proper q-HAM initial guess to obtain the suitable solution of any given equation According to previous section the Taylor series in terms of q for $u(x;q)$ such that:

$$U(x;q) = u(x;0) + \sum_{i=1}^{\infty} \frac{u_i(x;h)}{i!} q^i \tag{3.3}$$

where $u_i(x;h) = \frac{1}{i!}\frac{\partial^i u_i(x;q;h)}{\partial q^i}\Big|_{q=0}$.
Now for $q=1/r$ in Eq. (3.3) the i^{th} -order deformation equation is obtained:

$$U(x) = f(x) + \sum_{i=1}^{\infty} \frac{u_i(x;h)}{i!}\left(\frac{1}{r}\right)^i \tag{3.4}$$

The i^{th} -order deformation equation

$$u_i(x) = \chi_i u_{i-1}(x) + h\left[\mathcal{R}_i\left(u_{i-1}(x)\right)\right] \tag{3.5}$$

$\mathrm{i} = 0, 1, \mathrm{n}$

$$\chi_i = \begin{cases} 0 & , i \le 1 \\ r & , \text{otherwise} \end{cases}$$

$$\mathcal{R}_i\left(u_{i-1}(x)\right) = u_{i-1}(x) - \int_a^b \left(k(x,t)F\left(u_{i-1}(t)\right)dt - \left(1-\chi_i\right)f(x)\right. \tag{3.6}$$

from Eq. (3.5) and (3.6) we obtain

$$u_1(x;h) = hu_0(x) - hf(x) - \left[h\int_a^b \left(k(x,t)F(u_0(t))\right)dt\right] \tag{3.7}$$

where $u_0(x;q)$ is the initial guesses obtained from $f(x)$ and for $i \ge 2$, we have Then the q-HAM series solution on of the equation is given by the following form:

$$U(x;h) = u_0(x) + \sum_{i=1}^{\infty} \frac{u_i(x;h)}{i!}\left(\frac{1}{r}\right)^i \tag{3.8}$$

So the sequence in Eq. (3.8) will be convergent if we select a right value of h. The solution is therefore obtained in series form (q-homotropy solution series):

$$U(x) = \lim_{i\to\infty} \sum_{i=0}^{\infty} u_i(x) \tag{3.9}$$

To find the valid region of h, the h-curves given by the i^{th} order q-HAM approximation at different values. We choose the horizontal line parallel to as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution (3.8). the valid intersection region of for the values of in the curves becomes larger as increase.

4. Explanation and comparison of proposed method

4.1. Explanation proposed method

Example 4.1. *Consider the nonlinear Fredholm integral equation of the second kind*

$$u(x) = \cos(x) - \pi^2/48 + 1/12 \int_0^{\pi} tu^2(t)dt$$

with the exact solution $u(x) = \cos(x)$

Table 1: Numerical results of Example (4.1)

X	E x a c t 1 = $\cos(x)$	q-HAM	Error = $\lvert$E x a c t 1 $- u\rvert$
0	1	0.9988	0.0012
0.1	0.9950042	0.9938	0.001204
0.2	0.9800666	0.9789	0.001167
0.3	0.9553365	0.9542	0.001136
0.4	0.9210610	0.9199	0.001161
0.5	0.8775826	0.8764	0.001183
0.6	0.8253356	0.8242	0.001136
0.7	0.7648422	0.7637	0.001142
0.8	0.6967067	0.6955	0.001207
0.9	0.6216100	0.6204	0.00121
1.0	0.5403023	0.5391	0.001202

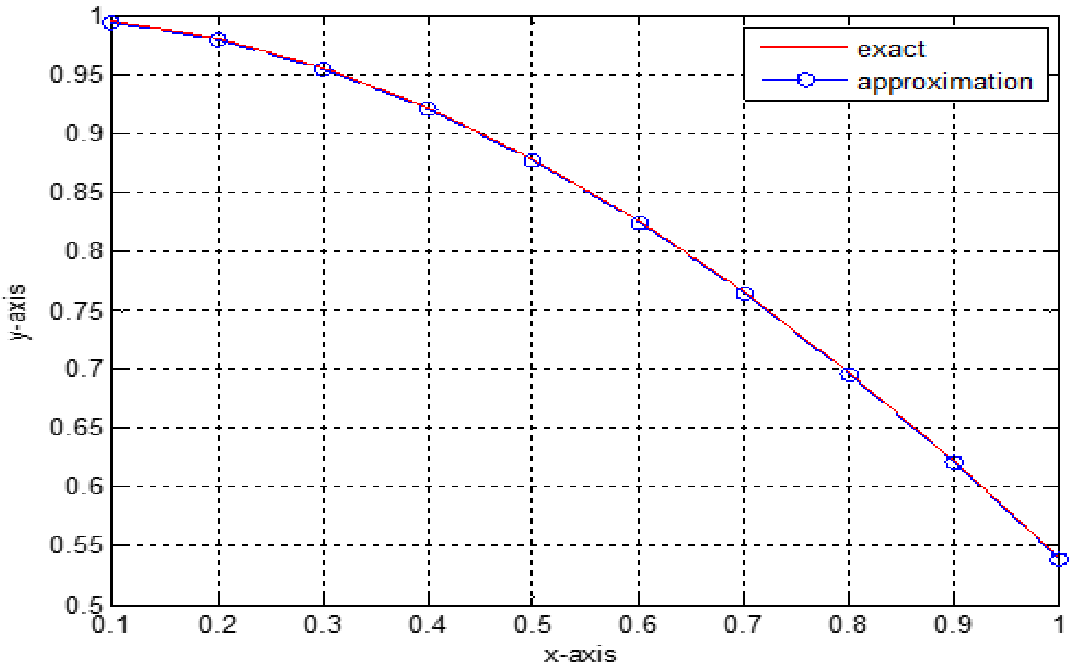

Figure 1: solution with q-ham method of example (4.1) at different values of x, (r=3) and (h= -1.75).

4.2. Comparison with update works

This section was conceded for present the result deduced from the comparison with the approximate solution method [32]

Example 4.2. *to solve the nonlinear Fredholm integral equation by using the successive approximations method and q-ham method, x is the exact solution.*

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xtu^2dt$$

Table 2: Numerical results of Example (4.2) and compassion with exact solution

X	Approximate method	Error=|exact − approximate|	q-ham method	Error=|exact − q-ham|
0	0	0	0	0
0.1	0.326758	0.226758	0.0997	0.0003
0.2	0.653516	0.453516	0.1994	0.0006
0.3	0.980273	0.680273	0.2992	0.0008
0.4	1.307031	0.907031	0.3989	0.0011
0.5	1.633789	1.133789	0.4986	0.0014
0.6	1.960547	1.360547	0.5983	0.0017
0.7	2.287305	1.587305	0.6980	0.002
0.8	2.614063	1.814063	0.7978	0.0022
0.9	2.940820	2.04082	0.8975	0.0025
1	3.267578	2.267578	0.9972	0.0028

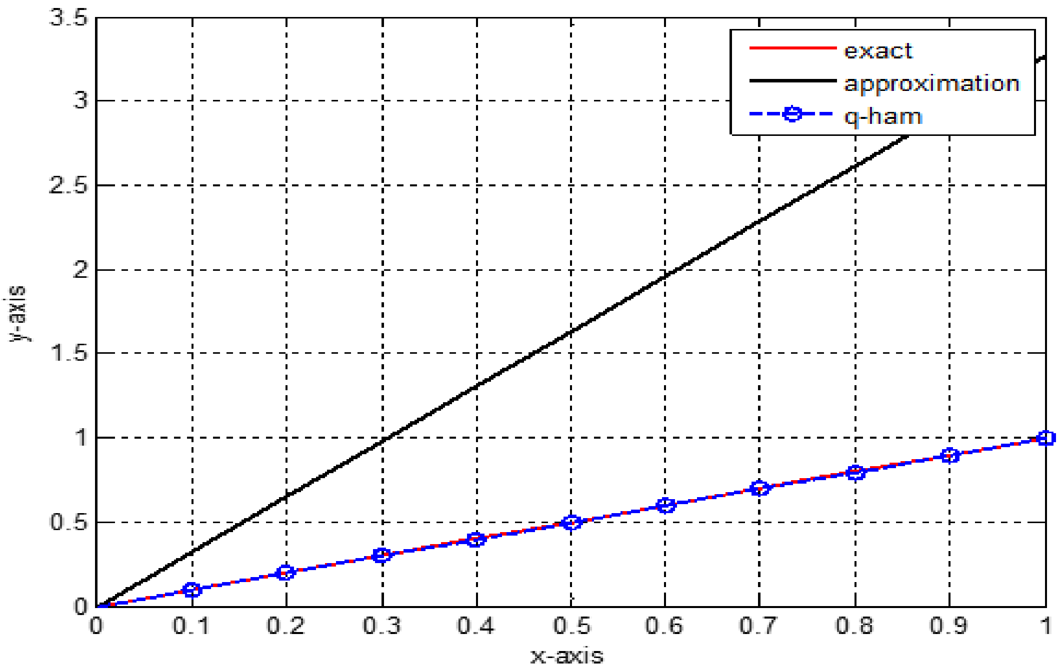

Figure 2: a compression between approximate solution and q-ham method of example (4.2) at different values of x, (r=5) and (h=-1.25).

5. Conclusion

In this work, the q-HAM technique for solving the nonlinear Fredholm integral equation of the second kind with Matlab7 is presented. The advantage of our technique is that it is still convergent for nonlinear problems, as shown by the tables showing the right accuracy. by comparison of exact solution and approximate solution We can see that the q-HAM is very applicable to the exact solution and its application is displayed through some examples. Numerical results show that the accuracy of the solutions obtained is good.

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