Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 677-687
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5404

# A second order fitted operator finite difference scheme for a singularly perturbed degenerate parabolic problem

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*(Communicated by Hossein Jafari)*

**Abstract**

A second order finite difference scheme is constructed to solve a singularly perturbed degenerate parabolic convection diffusion problem via Rothe's method. The solution of the problem exhibits a boundary layer on the left side of the spatial domain. By means of the Crank Nicolson finite difference scheme, the time derivative is discretised to obtain a set of semi-discrete boundary value problems. Using a fitted operator finite difference scheme based on the midpoint downwind scheme, the system of boundary value problems are discretized and analysed for convergence. Second order accuracy is established for each discretisation process. Numerical simulations are carried out to validate the theoretical error estimate.

*Keywords:* Singular perturbations; parabolic degenerate problem; fitted operator finite difference methods; convergence.
*2010 MSC:* 47A52; 35K05; 35R30; 65M70.

## 1. Introduction

Consider the singularly perturbed degenerate parabolic convection diffusion problem

$$\begin{aligned} -u_t(x,t) \quad + \quad & \varepsilon u_{xx}(x,t) + a(x)u_t(x,t) - b(x,t) = f(x,t), \;\; (x,t) \in Q, \\ & Q = \Omega \times (0,T], \;\; \Omega = (0,1), \end{aligned} \tag{1.1}$$

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*Received:* July 2020 *Accepted:* October 2020

subject to the boundary and the initial condition

$$u(0,t) = \eta_0(t), \;\; u(1,t) = \eta_1(t), \;\; t \in [0,T], \;\; u(x,t) = \varphi(x), \;\; x \in [0,1]. \tag{1.2}$$

We assume the source term $f(x,t)$ and the coefficient functions $a(x),\; b(x,t)$ are smooth and bounded. Moreover, the coefficient function satisfy

$$a(x) \geq a_0(x)x^p > \alpha > 0, \;\; p \geq 1, \;\; b(x,t) \geq \beta \geq 0.$$

The perturbation parameter $\varepsilon$ is such that $0 < \varepsilon << 1$. Problems of type (1.1) are known as singularly perturbed turning point problem since the coefficient of the convective term vanishes at $x = 0$. Subsequently, the point $x = 0$ is the turning point. Depending on the value of $p$ we can have either a simple turning point or a multiple point. Setting $\varepsilon = 0$, in Problem (1.1)–(1.2) yields the reduced problem

$$-(u_0)_t(x,t) + a(x)(u_0)_x(x,t) - b(x,t)(u_0)(x,t) = f(x,t), \tag{1.3}$$

which is a first order hyperbolic problem along with the initial and boundary conditions

$$u(0,t) = \eta_0(t), \;\; u(1,t) = \eta_1(t), \;\; t \in [0,T], \;\; u(x,0) = \varphi(x), \;\; (x,t) \in [0,1]. \tag{1.4}$$

Evidently, solving the reduced problem (1.3)–(1.4) will require only one boundary condition and thus, one of the boundary conditions will be underutilized leading to a boundary layer in that neighborhood. Notice that the convective term is positive throughout the domain $\Omega$. Therefore, the boundary layer will occur in the neighborhood of $x = 0$. The width of that layer takes the form $\mathcal{O}(\sqrt{\varepsilon})$, [20] similar to that of a reaction diffusion problem.

It is well known that classical numerical methods can not solve these problems satisfactorily. Thus some modifications of these classical numerical methods have been made to elude this difficulty. See for example the survey articles [4, 5, 6, 8] and the books [14, 18]. Below we cite some of the works done in line with Problem (1.1)–(1.2).

Clavero et. al [1] investigated a singularly perturbed convection diffusion problem with degenerate coefficient having a discontinuous source term and obtained a first order accuracy in time and almost second order accuracy in space. They discretised the space and the time variables at the same time via the simple upwind scheme on a non-uniform mesh and the backward Euler schemes respectively.

Dunne et. al [2] solved Problem (1.1)–(1.2) by the classical finite difference operator on a Shishkin mesh and the backward Euler finite difference scheme. These authors established an almost first order accuracy in space and first order in time.

Gupta et. al [3] designed a hybrid scheme to solve a stationary singularly perturbed problem with an interior turning point. They analysed the scheme for convergence and obtained a second order accuracy except for a logarithmic factor.

Majumdar and Natesan [10] proposed a numerical scheme which employed the backward Euler and the classical finite difference operator on Shishkin mesh to solve the time and the space variables respectively. They established that their scheme was first order accurate in both variables except for a logarithmic factor in space. Using Richardson extrapolation technique, they enhanced the accuracy to almost second order in space and second order in time.

In [11], Majumdar and Natesan studied a two-dimensional version of Problem (1.1)–(1.2) using an alternating direction scheme for the time derivative and an upwind finite difference operator on Shishkin mesh for the space derivative. These authors established an almost first order accuracy for their scheme.

In [12], Majumdar and Natesan proposed an $\varepsilon$-uniform scheme of first order accuracy in time and almost second order in space to solve Problem (1.1)–(1.2) via the Rothe's method. The scheme employed a combination of the classical central difference and the midpoint upwind scheme on a Shishkin mesh to solve the spatial derivatives and the backward Euler was used for the time derivative.

In [17], Rai and Yadav discretised the time and the space variables of a singularly perturbed delayed parabolic convection diffusion problem with degenerate coefficients via the backward Euler and a hybrid scheme on a non-uniform mesh respectively. Their analysis was almost second order in space and first order in time. Using Richardson extrapolation on the time variable, they improved the time accuracy to second order.

Viscor and Stynes [19] studied the reaction diffusion version of Problem (1.1)–(1.2) and obtained an almost second order accuracy in space and first order in time.

Inspired by the simplicity of the analysis, its accuracy and noticing that all the above papers focused on non-uniform mesh, we propose a scheme which uses a uniform mesh to solve Problem (1.1)–(1.2). The scheme uses the Crank Nicolson finite difference scheme to solve the time derivatives and a fitted operator finite difference scheme based on non-standard finite difference scheme [7] and midpoint downwind scheme for the spatial derivatives. The authors wish to state that, although the scheme which was proposed in [13] was also based on the non-standard finite difference scheme, Richardson extrapolation was employed to enhance the accuracy of the scheme to second order whilst the proposed scheme uses the midpoint rule to obtain the second order accuracy.

The paper unfolds as follows: In Section 2, some theoretical results are presented. The discretisation of the continuous problem in time by the Crank Nicolson finite difference scheme and its convergence are presented in Section 3. Section 4 focuses on the spatial discretisation which uses a fitted operator finite difference scheme based on the midpoint rule. In Section 5, the stability of the scheme is presented and then analysed for convergence in Section 6. In Section 7, numerical experiments are conducted to validate the theoretical findings whilst a summary of the main result is presented in Section 8.

## 2. Bounds on the Solution and Its Derivatives

In this section, we estimate the solution of (1.1)–(1.2) by providing the bounds of the solution and its partial derivatives. The proofs of the lemmas in this section follow the lines of the proofs in [9, 12, 15, 20].

**Lemma 2.1.** *Continuous minimum principle*
*Suppose $\Psi(x,t)$ is a smooth functions which satisfies $\Psi(x,0) \geq 0,\ \Psi(0,t) \geq 0,\ \Psi(1,t) \geq 0$ and $\mathcal{L}_\varepsilon \Psi(x,t) \leq 0, \forall\, (x,t) \in Q$. Then $\Psi(x,t) \geq 0, \forall\, (x,t) \in \bar{Q}$.*

**Proof .** Let $(x^*,t^*) \in \bar{Q}$, such that

$$\Psi(x^*,t^*) = \min_{(x,t)\,\in\,\bar{Q}} \Psi(x,t) \ \text{ and } \ \Psi(x^*,t^*) < 0.$$

It is clear that $(x^*,t^*) \in Q$. At the point $(x^*,t^*)$ the operator $\mathcal{L}_\varepsilon$ on $\Psi(x^*,t^*)$ becomes

$$\mathcal{L}_\varepsilon \Psi(x^*,t^*) = -\Psi_t(x^*,t^*) + \varepsilon \Psi_{xx}(x^*,t^*) + a(x^*)\Psi_x(x^*,t^*) - b(x^*,t^*)\Psi(x^*,t^*).$$

Since $\Psi_{xx}(x^*,t^*) > 0$ and $\Psi_x(x^*,t^*) = \Psi_t(x^*,t^*) = 0$, we obtain $\mathcal{L}_\varepsilon \Psi(x^*,t^*) \geq 0$, which is a contradiction. Therefore, $\Psi(x,t) \geq 0,\ \forall\ (x,t) \in \bar{Q}$, which completes the proof. $\square$

**Lemma 2.2.** *(Stability estimate)*
*Suppose $u(x,t)$ is the solution of Problem (1.1)–(1.2). Then the following estimate holds*

$$|u(x,t)| \leq \left[\beta^{-1}||f|| + \max(|\varphi(x)|, |\eta_0(t)|, |\eta_1(t)|)\right].$$

**Proof .** The proof of this Lemma is similar to the proof of Lemma 2.2 of [12]. □

**Theorem 2.1.** *The solution and its partial derivatives are such that,*

$$\left|\frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}\right| \leq C(1 + \varepsilon^{-i/2}\exp(-\mu x)), \qquad (x,t) \in Q,$$

*where $\mu = \sqrt{q_*/\varepsilon}$ and $i$ and $j$ are non-negative integers which satisfy $0 \leq i + j \leq 6$.*

**Proof .** The proof of this Theorem is similar to the proof of Lemma 2.3 of [12]. □

## 3. Time Discretisation

Here we transform the PDE (1.1)–(1.2) to a boundary value problem by discretising in time, whilst leaving the spatial domain continuous. The discretisation in time via the Crank Nicolson finite difference scheme on a uniform mesh yields the following semi-discrete boundary value problems

$$\begin{aligned} u(x,0) &= \varphi(x), \qquad x \in \bar{\Omega}, && (3.1)\\ \mathcal{L}^*_\varepsilon u(x,t_{k+1/2}) &\equiv -\frac{u^{k+1}-u^k}{\Delta t} + \varepsilon u_{xx}(x,t_{k+1/2}) + a(x)u_x(x,t_{k+1/2}) \\ &\quad -b(x,t_{k+1/2})u(x,t_{k+1/2}) = f(x,t_{k+1/2}), \quad k = 0,1,2,...,m, && (3.2)\end{aligned}$$

along with the boundary conditions

$$u(0,t_{k+1}) = \eta_0(t_{k+1}), \quad u(1,t_{k+1}) = \eta_1(t_{k+1}). \tag{3.3}$$

The notation $Z(x,t_{k+1/2})$ is given by

$$Z(x,t_{k+1/2}) = \frac{Z(x,t_{k+1}) + Z(x,t_k)}{2},$$

and $m$ is the number of sub-intervals in the time direction. The scheme (3.1)–(3.3) is rewritten as

$$\begin{aligned} u(x,0) &= \varphi(x), \qquad x \in \Omega, && (3.4)\\ \mathcal{L}^m_\varepsilon u^{k+1}(x) &\equiv \frac{\varepsilon}{2}u^{k+1}_{xx}(x) + \frac{a(x)}{2}u^{k+1}_x(x) - d(x)u^{k+1}(x) = g(x,t_{k+1}), && (3.5)\\ u^{k+1}(0) &= \eta_0(t_{k+1}), \quad u^{k+1}(1) = \eta_1(t_{k+1}), && (3.6)\end{aligned}$$

where

$$g(x,t_{k+1}) = f(x,t_{k+1/2}) - \frac{\varepsilon}{2}u^k_{xx}(x) - \frac{a(x)}{2}u^k_x(x) - e(x)u^k(x),$$
$$d(x) = (1/\Delta t + b(x,t_{k+1})/2)$$

and

$$e(x) = (1/\Delta t) - b(x,t_{k+1})/2,$$

or we write the scheme (3.1)–(3.3) as

$$u(x,0) = \varphi(x), \qquad x \in \Omega, \tag{3.7}$$

$$\mathcal{L}_\varepsilon^{**}u^{k+1}(x) \equiv \frac{\Delta t\varepsilon}{2}u_{xx}^{k+1}(x) + \frac{\Delta t a(x)}{2}u_x^{k+1}(x) - c(x)u^{k+1}(x) = g(x, t_{k+1}), \tag{3.8}$$

$$u^{k+1}(0) = \eta_0(t_{k+1}), \quad u^{k+1}(1) = \eta_1(t_{k+1}), \tag{3.9}$$

where

$$g(x, t_{k+1}) = f(x, t_{k+1/2}) - \frac{\Delta t\varepsilon}{2}u_{xx}^k(x) - \frac{a(x)\Delta t}{2}u_x^k(x) - p(x)u^k(x),$$
$$c(x) = 1 + \Delta t(b(x, t_{k+1})/2)$$

and

$$p(x) = 1 - \Delta t b(x, t_{k+1})/2.$$

**Lemma 3.1.** *Semi-discrete minimum principle*
*Let* $\Psi^{k+1}(x) \in \mathcal{C}^4(\bar\Omega)$. *If* $\Psi^{k+1}(x) \geq 0,\ \forall\ x \in \partial\Omega$ *and* $\mathcal{L}_\varepsilon^{**}\Psi^{k+1}(x) \leq 0$, *for* $x \in \Omega$ *then,* $\Psi^{k+1}(x) \geq 0,\ \forall\ x \in \bar\Omega$.

**Proof .** Suppose there exist $x^* \in \bar\Omega$ such that $\Psi^{k+1}(x^*) = \min_{x\in\bar\Omega}\Psi^{k+1}(x) < 0$. It is evident that $x^* \neq 0$, nor 1 but $x^* \in \Omega$. Now applying the differential operator $\mathcal{L}_\varepsilon^{**}$ on $\Psi^{k+1}$ gives $\mathcal{L}_\varepsilon^{**}\Psi^{k+1}(x^*) \geq 0$. which is a contradiction. Thus $\Psi^{k+1}(x) \geq 0,\ \ \forall\ x \in \bar\Omega$, as required. □

**Lemma 3.2.** *The local truncation error of the time semi-discretisation satisfies*

$$||e^{k+1}||_\infty \leq C^*(\Delta t)^3. \tag{3.10}$$

**Proof .** The local truncation error of the time semi-discetization method is given by

$$e^{k+1} = u(x, t_{k+1}) - \hat u(x, t_{k+1}), \tag{3.11}$$

where $\hat u^{k+1}$ is the computed solution of

$$\hat u(x,0) = \varphi(x), \qquad x \in \Omega, \tag{3.12}$$

$$\mathcal{L}_\varepsilon^{**}\hat u^{k+1}(x) \equiv \frac{\Delta t}{2}\varepsilon\hat u_{xx}^{k+1}(x) + \frac{\Delta t}{2}a(x)\hat u_x^{k+1}(x) - c(x)\hat u^{k+1}(x) = g(x, t_{k+1}), \tag{3.13}$$

$$\hat u^{k+1}(0) = \eta_0(t_{k+1}), \quad \hat u^{k+1}(1) = \eta_1(t_{k+1}), \tag{3.14}$$

where

$$g(x, t_{k+1}) = f(x, t_{k+1/2}) - \Delta t\frac{\varepsilon}{2}\hat u_{xx}^k(x) - \Delta t\frac{a(x)}{2}\hat u_x^k(x) - p(x)\hat u^k(x),$$
$$c(x) = (1 + \Delta t b(x, t_{k+1})/2)$$

and

$$p(x) = (1 - \Delta t b(x, t_{k+1})/2).$$

A truncated Taylor series expansion of $u(x, t_{k+1})$ and $u(x, t_k)$ takes the form

$$u(x, t_{k+1}) = u(x, t_{k+1/2}) + \frac{\Delta t}{2}u_t(x, t_{k+1/2}) + \frac{\Delta t^2}{8}u_{tt}(x, t_{k+1/2}) + \mathcal{O}(\Delta t)^3, \tag{3.15}$$

$$u(x, t_k) = u(x, t_{k+1/2}) - \frac{\Delta t}{2}u_t(x, t_{k+1/2}) + \frac{\Delta t^2}{8}u_{tt}(x, t_{k+1/2}) + \mathcal{O}(\Delta t)^3. \tag{3.16}$$

From Equations (3.15) and (3.16) we have

$$\begin{aligned}\frac{u(x,t_{k+1})-u(x,t_k)}{\Delta t} &= u_t(x,t_{k+1/2})+\mathcal{O}((\Delta t)^2)\\ &= -\varepsilon u_{xx}(x,t_{k+1/2})-a(x)u_x(x,t_{k+1/2})+c(x)u(x,t_{k+1/2})\\ &\quad +g(x,t_{k+1/2})+\mathcal{O}(\Delta t)^2. \end{aligned}\tag{3.17}$$

Further simplification yields

$$\mathcal{L}_\varepsilon^{**}u^{k+1}(x)\equiv\frac{\Delta t}{2}\varepsilon u_{xx}^{k+1}(x)+\frac{\Delta t}{2}a(x)u_x^{k+1}(x)-c(x)u^{k+1}(x)=g(x,t_{k+1})+\mathcal{O}(\Delta t)^3.\tag{3.18}$$

From Equation (3.13) and (3.18) the local truncation error satisfies

$$\begin{aligned}\mathcal{L}_\varepsilon^{**}e^{k+1}&=\mathcal{O}((\Delta t)^3),\\ e^{k+1}(0)&=e^{k+1}(1)=0.\end{aligned}$$

Since the operator $\mathcal{L}_\varepsilon^{**}$ satisfies a minimum principle, the result

$$|e^{k+1}(x)|\leq C(\Delta t)^3,$$

follows. □

The global error of the time discretisation satisfies the next result.

**Theorem 3.1.** *The global error $\mathcal{E}^m$ is such that*

$$||\mathcal{E}^m||_\infty\leq C(\Delta t)^2.\tag{3.19}$$

**Proof .**

$$||\mathcal{E}^m||_\infty \leq \left|\sum_{k=1}^{m}e^{k+1}(x)\right|\leq Cm(\Delta t)^3=C(\Delta t)^2.\tag{3.20}$$

□

Next we design a fitted operator finite difference scheme based on the midpoint downwind scheme to discretise Problem (3.4)–(3.6) in space.

## 4. Spatial Discretisation

In this section, the boundary value problem (3.4)–(3.6) is discretised via a fitted operator finite difference scheme. We employ the notation $U_i^{k+1}\equiv U^{k+1}(x_i)\equiv U(x_i,t_{k+1})$ as the numerical solution of $u^{k+1}(x)$ and perform the discretisation on a uniform mesh as follows:

$$u(x,0) = \varphi(x_i),\qquad x\in\bar{\Omega}^n\tag{4.1}$$

$$\begin{aligned}\mathcal{L}_\varepsilon^{m,n}U^{k+1}(x_i) &\equiv \frac{\varepsilon}{2}\delta^2(U^{k+1}(x_i))+\frac{a(x_{i+1/2})}{2}(D^+U^{k+1}(x_i))-d(x_{i+1/2})U^{k+1}(x_i)\\ &= g(x_{i+1/2},t_{k+1}),\end{aligned}\tag{4.2}$$

$$u^{k+1}(0) = \eta_0(t_{k+1}),\quad u^{k+1}(1)=\eta_1(t_{k+1}),\tag{4.3}$$

The source term $g(x_{i+1/2},t_{k+1})$ is given by

$$\begin{aligned}g(x_{i+1/2},t_{k+1}) &= \frac{f(x_{i+1/2},t_{k+1})+f(x_{i+1/2,t_k})}{2}-\frac{\varepsilon}{2}\delta^2U^k(x_i)\\ &\quad -\frac{a(x_{i+1/2})}{2}D^+U^k(x_i)-e(x_{i+1/2})U^k(x_i).\end{aligned}\tag{4.4}$$

The notations $D^+,\ \delta^2$ are the first and second order finite difference schemes on appropriate denominator functions and are given by

$$D^+ z_i = \frac{z_{i+1} - z_i}{h}, \quad \delta^2 z_i = \frac{z_{i+1} - 2z_i + z_{i-1}}{\phi_i^2(\varepsilon, h)}, \tag{4.5}$$

respectively. The denominator function $\phi_i^2(h, \varepsilon)$ is given by

$$\phi_i^2(h, \varepsilon) \;=\; \frac{\varepsilon h}{a(x_{i+1/2})}\left(1 - \exp\left(\frac{-a(x_{i+1/2})h}{\varepsilon}\right)\right). \tag{4.6}$$

The scheme (4.1)–(4.3) can be represented in a compact form

$$U + AU = F, \tag{4.7}$$

where $U$ and $F$ are vectors of size $(n-1)$ and $A$ is a tridiagonal matrix of size $(n-1) \times (n-1)$, where $n$ is the number of sub-intervals. The entries of F and A are as follows:

$$\begin{aligned} F_1 &= g_1(x_{i+1/2}, t_{k+1}) - r_0^- \eta_0(t_{k+1}), \\ F_i &= g_i, \qquad\qquad\qquad\qquad i = 2(1)n-2, \\ F_{n-1} &= g_{n-1}(x_{i+1/2}, t_{k+1}) - r_{n-1}^+ \eta_n(t_{k+1}), \end{aligned}$$

and

$$A_{ij} = \begin{cases} r_i^- & i = 2, 3, ...n-1,\ j = i-1, \\ r_i^c & i = 1, 2, 3, ...n-1,\ i = j, \\ r_i^+ & i = 1, 2, 3, ...n-2,\ j = i+1, \end{cases}$$

where the $r_i^-, r_i^c$ and $r_i^+$ are given by

$$r_i^- = \frac{\varepsilon}{2\phi_i^{-2}(h, \varepsilon)}, \ \ r_i^c = -\frac{\varepsilon}{\phi_i^2(h, \varepsilon)} - \frac{a(x_{i+1/2})}{2h} - d(x_{i+1/2}), \ \ r_i^+ = \frac{\varepsilon}{2\phi_i^{-2}(h, \varepsilon)} + \frac{a(x_{i+1/2})}{2h},$$

respectively. The operator $\mathcal{L}_\varepsilon^{m,n}$ satisfies a discrete minimum principle which is presented in the next Lemma. For simplicity, we have dropped the superscript indices.

## 5. Stability of the Scheme

Below we show that the spatial discretisation is stable and that the scheme (4.1)–(4.3) has a unique solution.

**Lemma 5.1.** *Discrete minimum principle*
*Let $\xi(x_i)$ be a mesh function defined on $\bar{\Omega}^n$, satisfies $\xi(x_i) \geq 0,\ \forall\ x_i \in\ \partial\Omega^n$ and $\mathcal{L}_\varepsilon^{m,n}\xi(x_i) \leq 0,\ \forall\ x_i \in \Omega^n$. Then $\xi(x_i) \geq 0,\ \forall\ x_i \in\ \bar{\Omega}^n$.*

**Proof .** The matrix $A$ is a square tridiagonal matrix which assumes the properties of an M-matrix thus the scheme is positive. □

From Lemma 5.1 we prove further that the scheme (4.1)–(4.3) satisfies a uniform stability estimate below.

**Lemma 5.2.** *The operator $\mathcal{L}_\varepsilon^{m,n}$ is uniformly stable. That is the solution $u_i$ of the discrete problem* (4.1)–(4.3) *satisfies*

$$|u_i| \leq \rho^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,n} u_i| + \max_{x_i \in \bar{\Omega}} (|\eta_0|, |\eta_1|).$$

**Proof .** Let $z = \rho^{-1}\max_{x_i\in\bar{\Omega}}|\mathcal{L}_\varepsilon^{m,n}u_i| + \max_{x_i\in\bar{\Omega}}(|\eta_0|, \eta_1|)$ and define the mesh function $\Psi_i^{\pm}$ by $\Psi_i^{\pm} = z \pm u_i$. At $i=0$ and $i=1$, we have $\Psi_0^{\pm} = z \pm u_i \geq 0$. On the domain $x_i \in \Omega^n$ we obtain

$$\begin{aligned}
\mathcal{L}_\varepsilon^{m,n}\Psi_i^{\pm} =& \frac{\varepsilon}{2}\left(\frac{z+u_{i+1}-2(z+u_i)+z\pm u_{i-1}}{\phi_i^2(h,\varepsilon)}\right) + \frac{a_{i+1/2}}{2}\left(\frac{z\pm u_{i+1}-(z\pm u_i)}{h}\right) - d_{i+1/2}(z\pm u_i)\\
=& \; -d_{i+1/2}z \pm \mathcal{L}_\varepsilon^{m,n}u_i\\
=& \; -d_{i+1/2}\left(\rho^{-1}\max_{(x_i)\in\bar{\Omega}}|\mathcal{L}_\varepsilon^{m,n}u_i| + \max_{x_i\in\bar{\Omega}}(|\eta_0|, \eta_n|)\right) \pm g_{i+1/2} \leq 0.
\end{aligned}$$

From Lemma 5.1, $\Psi_i^{\pm} \geq 0, \; \forall \; (x_i) \in \bar{\Omega}^n$. $\square$

## 6. Error Estimate

The error associated with the spatial discretisation is estimated as follows:

$$\begin{aligned}
\mathcal{L}_\varepsilon^{m,n}(u(x_i) - U(x_i)) &= \mathcal{L}_\varepsilon^{m,n}u(x_i) - \mathcal{L}_\varepsilon^{m,n}U(x_i) = \mathcal{L}_\varepsilon^{m,n}u(x_i) - g_{i+1/2}\\
&= \mathcal{L}_\varepsilon^{m,n}u(x_i) - \mathcal{L}_\varepsilon^{m}u(x_{i+1/2})\\
&= \frac{\varepsilon}{2}\left(\delta^2 u_i - u''(x_{i+1/2})\right) + \frac{a_{i+1/2}}{2}\left(D^+u_i - u'(x_{i+1/2})\right).
\end{aligned}$$

A truncated Taylor series expansion of the terms $u_{i+1}$ and $\phi_i^2$ and simplifying further yields

$$\begin{aligned}
\mathcal{L}_\varepsilon^{m,n}(u(x_i) - U(x_i)) =& \frac{\varepsilon}{2h^2}\left(h^2u'' + \frac{h^4}{4!}u^{iv} + \frac{h^6}{16.5!}u^{vi} + \ldots\right) - \frac{\varepsilon}{2}u'' + \frac{a_{i+1/2}h^2}{8.3!}u'''\\
&+ \frac{a_{i+1/2}h^4}{32.5!}u^{v} + \ldots\\
=& \left(\frac{\varepsilon}{2.4!}u^{iv} + \frac{a(x_{i+1/2})}{8.3!}u^{iii}\right)h^2 + \left(\frac{\varepsilon}{32.5!}u^{vi} + \frac{a(x_{i+1/2})}{32.5!}u^{v}\right)h^4.
\end{aligned}$$

From Theorem 2.1, and noticing that the exponential terms vanish as $\varepsilon \to 0$ (see [16] for proof) yields

$$|\mathcal{L}_\varepsilon^{m,n}(u_i - U_i)| \leq Ch^2.$$

Application of Lemma 5.1 leads to the result below.

**Lemma 6.1.** *The error associated with the fitted operator finite difference scheme based on the midpoint satisfies*

$$|u_i - U_i| \leq Ch^2.$$

**Theorem 6.1.** *(Uniform Convergence)*
*Let $u$ be the exact solution of the continuous Problem* (1.1)-(1.2) *and $U$ be the approximate solution. The error associated with the proposed numerical scheme satisfies*

$$\max_{0\leq i\leq n; 0\leq k\leq m}|u(x_i,t_k) - U(x_i,t_k)| \leq C(h^2 + \Delta t^2).$$

Next we apply the method on test problem to validate the theoretical estimate.

## 7. Numerical Results

In this section, we test the efficiency of the proposed scheme through numerical experiments. The exact solution of the problem is not known, thus to compute the maximum pointwise errors we use the double mesh principle given by

$$E_{n,\Delta t} = \max_{0\le i\le n; 0\le k\le m} |U_{i:n}^{k:m} - U_{i:2n}^{k:2m}|. \tag{7.1}$$

To compute the corresponding rate of convergence we use the formula

$$r = \log_2 \left( \frac{E_{n,\Delta t}}{E_{2n,\Delta t}} \right) \tag{7.2}$$

The $\varepsilon$ uniform maximum errors and the $\varepsilon$-uniform rates of convergence are computed with the formulae

$$E = \max_{0<\varepsilon\le 1} E_{n,\Delta t} \qquad \text{and} \qquad R = \max_{0<\varepsilon\le 1} r,$$

respectively.

**Example 7.1.** *[12] Consider the problem given by*

$$\begin{aligned} -u_t(x,t) &+ \varepsilon u_{xx}(x,t) + x^p u_x(x,t) - (x+p)u(x,t) = p(x^2-1)\exp(-t), \quad (x,t)\in Q, \\ u(x,0) &= (1-x)^2,\ x\in(0,1), \\ u(0,t) &= 1+t^2,\ u(1,t)=0,\ t\in[0,1]. \end{aligned}$$

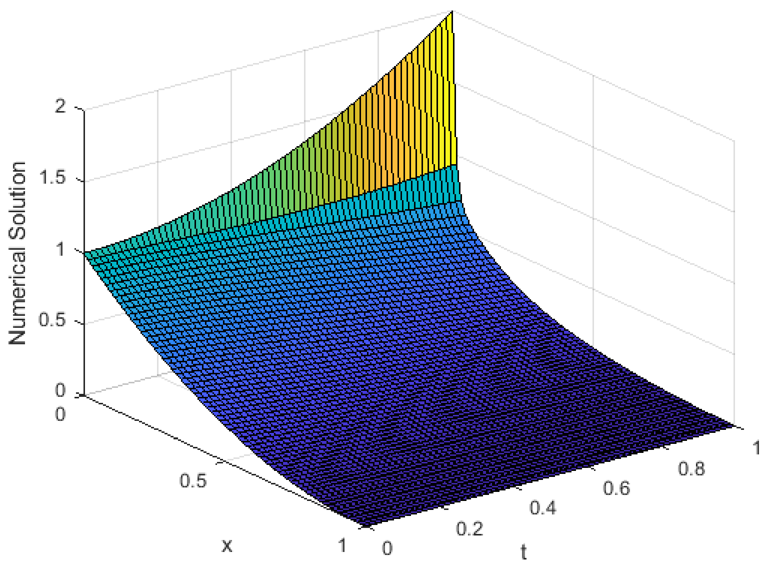


Figure 1: Surface plot of the numerical solution Example 7.1 for $p = 1$, $n = 64$ and $\varepsilon = 10^{-24}$.

Table 1: Maximum pointwise error and rate of convergence for Example 7.1.

| $\varepsilon$ | n=m=8 | 16 | 32 | 64 | 1028 |
|---|---|---|---|---|---|
| $10^{-4}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-5}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-6}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-7}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-8}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-9}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $10^{-10}$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |
| $E$ | 2.91E-02 | 1.55E-02 | 4.83E-03 | 1.28E-03 | 3.23E-04 |
| $R$ | 0.9103 | 1.6813 | 1.9209 | 1.9803 | |

## 8. Conclusion

A second order numerical method was proposed to solve a parabolic convection diffusion problem with degenerate coefficient. The scheme employed the Crank Nicolson finite difference scheme and a fitted operator finite difference scheme to solve the aforementioned problem via the Rothe's method. It is important to note that the fitted operator finite difference scheme which was designed was based on the midpoint rule. Some properties of the discrete problem which ensured the stability of the scheme were presented and used to analyse the scheme for convergence. This analysis resulted in a full second order in both the space and the time variables. To test the efficiency of the scheme, numerical experiments were conducted and the results were displayed in Table 1. From the table, the uniform second order convergence rate can be observed. Again surface plots of the numerical solution can be observed in Figure1. From the Figure 1, the left boundary layer in the solution can be observed. Currently the scheme is being explored on degenerate problems with delays.

## References


[1] C. Clavero, J. L. Gracia, G. I. Shishkin and L. P. Shishkina, *Convergent $\varepsilon$-uniformly for parabolic singularly perturbed problems with a degenerating convective term and a discontinuous source*, Math. Model. Anal. 20(5) (2015) 641–657.

[2] R. K. Dunne, E. O'Riordan and G. I Shishkin, *A fitted mesh method for a class of singularly perturbed parabolic problems with a boundary turning point*, Comput. Meth. Appl. Math. 3(3) (2003) 361–372.

[3] V. Gupta, S.K. Sahoo and R.K. Dubey, *Parameter-uniform fitted mesh higher order finite difference scheme for singularly perturbed problem with an interior turning point*, arXiv preprint arXiv:1909.07128.

[4] M.K. Kadalbajoo and K.C. Patidar, *A survey of numerical techniques for solving singularly perturbed ordinary differential equations*, Appl. Math. Comput. 130(2) (2002) 457–510.

[5] M.K. Kadalbajoo and K.C. Patidar, *Singularly perturbed problems in partial differential equations*, Appl. Math. Comput. 134(2) (2003) 371–429.

[6] M.K. Kadalbajoo and V. Gupta, *A brief survey on numerical methods for solving singularly perturbed problems*, Appl. Math. Comput. 217 (2010) 3641–3716.

[7] M.M. Khalsaraei, *Nonstandard explicit third-order Runge-Kutta method with positivity property*, Int. J. Nonlinear Anal. Appl. 8(2) (2017) 37–46.

[8] J.P. Kauthen and V. Gupta, *A survey of singularly perturbed Volterra equations*, Appl. Numerical Math. 24(23) (1997) 95–114.

[9] R.B. Kellog and A. Tsan, *Analysis of some difference approximations for a singular perturbation problem without turning points*, Math. Comput. 31(144) (1978) 1025–1039.

[10] A. Majumdar and S. Natesan, *Second-order uniformly convergent Richardson extrapolation method for singularly perturbed degenerate parabolic PDEs*, Int. J. Appl. Comput. Math. 3(1) (2017) 31–53.

[11] A. Majumdar and S. Natesan, *Alternating direction numerical scheme for singularly perturbed 2D degenerate parabolic convection-diffusion problems*, Appl. Math. Comput. 313 (2017) 453–473.

[12] A. Majumdar and S. Natesan, *An $\varepsilon$-uniform hybrid numerical scheme for a singularly perturbed degenerate parabolic convection-diffusion problem*, Int. J. Comput. Math. 96(7) (2019) 1313–1334.

[13] N.A. Mbroh, S.C. Oukouomi Noutchie and R.Y. Mpika Massoukou, *A uniformly convergent finite difference scheme for Robin type singularly perturbed parabolic convection diffusion problem*, Math. Comput. Simul. 174 (2020) 218–232.

[14] J.J.H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singularly Perturbed Problems: Error Estimates in The Maximum Norm for Linear Problems in One and Two Dimension*, World Scientific Publications, Singapore, 2012.

[15] M.J. Ng-Stynes, E. O'Riordan and M. Stynes, *Numerical methods for time-dependent convection-diffusion equations*, J. Comput. Appl. Math. 21(3) (1988) 289–310.

[16] K. C. Patidar, *High order fitted operator numerical method for self-adjoint singular perturbation problems*, Appl. Math. Comput., 171(1) (2005) 547–566.

[17] P. Rai and S. Yadav, *Robust numerical schemes for singularly perturbed delay parabolic convection-diffusion problems with degenerate coefficient*, Int. J. Comput. Math. 98(1) (2020) 195–221.

[18] H. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion-Reaction and Flow Problems*, Springer Science & Business Media, 2008.

[19] M. Viscor and M. Stynes, *A robust finite difference method for a singularly perturbed degenerate parabolic problem Part I,Int. J. Numerical Anal. Model.* 7(3) (2010).

[20] R. Vulanović and P. A. Farrell, *Continuous and numerical analysis of a multiple boundary turning point problem*, SIAM J. Numerical Anal. 30(5) (1993) 1400–1418.