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Semi local hollow modules

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Abstract

To consider R is a commutative ring with unitary, X be a non –Zero unity left $R-module$; X is known hollow module if each proper sub module of X is semi small local hollow module if X has a unique maximal submodule which contains each semi small submodule. The current study deals with this class of modules and give several fundamental properties related with this concept.

Keywords: Semi small submodules, Local hollow modules, Semi local hollow module.

1. Introduction

Throughout the following paper R represents a commutative ring with identity, and each $R-module$ are left unit. A proper submodule N of an $R-module$ X is called small if $N+K \neq X$ for each proper submodule K of X[2]. An on –Zero module X is called hollow module if each proper submodule of X is small [1]. A proper submodule A of an $R-module$ X is called maximal submodule in X, if B is a submodule of X with $A \subset B$, so $B = X$.[6]. A sub-module N of an $R-module$ X is called semi-small submodule of X, if and only if, $N+K \neq X$ for each primary submodule K of X.[1]. An $R-module$ X is called local if X has a unique maximal submodule which contains each proper submodule of X.[8].

A proper submodule A of an R-module X is said quasi essentially peudo prime submodule if [A:K] is primary ideal of R, for each a quasi essential submodule B of X such that $A \subset B$ [5]. In this paper, we give a strong from of local hollow module, we called semi local hollow module which is a module has unique maximal submodule which contains each semi small submodule of X. This work contains three sections. In section one, we definition of semi local hollow module a strong from of local hollow module, we investigate the properties of this class of modules. In section two, we investigate some conditions under which hollow modules and semi local hollow module are equivalent. In the third section investigate the relation between the semi local hollow modules and other modules such as amply supplemented, indecomposable modules and lifting modules.

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2. Semi local hollow module

In this part the study present the concept of semi local hollow module, and study the basic properties of this kind of modules.

Definition 2.1. *An $R-module$ X is called semi-local hollow modules if X has a unique maximal submodule which contains each semi small submodule of X.*

Remarks and examples

(i) Each semi-local hollow module is hollow module.
Proof . Suppose that X is semi-local hollow modules. Then there exist a unique maximal submodule contains every semi-small submodule say A in X. And since A is a submodule of X. Then each semi-small is contains in X. By definition hollow module so; A is semi-small submodule of X; Implies that X is hollow module. □
While the converse Remark (i) is not true (in general), for example:
Z_P^∞ is hollow module; but Z_P^∞ is not semi-local hollow module.

(ii) The $Z-module$ Z_4 is semi local hollow module, while the $Z-module$ Z_6 is not semi local hollow module.

(iii) Each local module is $semi-local-$ $hollow$ module, while the converse is not true (in general). For example:
$Z_2 \bigoplus Q$ is $semi-$ local$-$ module, while is not local module, since $\{o\} \bigoplus Q$ is a unique maximal submodule of $Z_2 \bigoplus Q$ and $\{o\} \bigoplus \{o\}$; is a small submodule of $Z_2 \bigoplus Q$ and contained in $\{o\} \bigoplus Q$ but $Z_2 \bigoplus \{o\}$is a proper submodule of $Z_2 \bigoplus Q$. But $Z_2 \bigoplus \{o\}$ is not contained in $\{o\} \bigoplus Q$.

(iv) Each simple module is not semi-local-hollow. For example: The $Z-module$ Z_5 is simple module, while not semi-local-hollow module, and each semi-local-hollow module is not simple module; for example the $Z-module$ Z_8 is semi-local- hollow, while is not simple module.

The following proposition the study present some of the basic properties of local-hollow module.

Proposition 2.2. *Epimorphic image of semi-local hollow module is semi local hollow module.*
Proof . *Suppose that X_1 is semi local hollow module. Let $f : X_1 \longrightarrow X_2$ be an epimorphism with X_2 is $R-module$. Assume that A is a unique maximal submodule of X_2 and $A + B = X_2$ where B is a proper submodule of X_2. Now $f^{-1}(A)$ unique maximal submodule of X_1, since other wise $f^{-1}(A) = X_1$, hence $f(f^{-1}(A)) = f(X_1) = X_2$ implies that $A = X_2$ which is contradiction, with A is unique maximal submodule of X_2. Thus $f^{-1}(A)$ is a unique maximal submodule of X_1. Since X_1 is semi-local hollow module, therefor $f^{-1}(A)$ contains each semi small submodule of X_1. Hence $f(f^{-1}(A))$ is a semi small submodule of $f(X_1)$. That is to say that A is semi small submodule of X_2 therefor X_2 is semi local hollow module.* □

Proposition 2.3. *To consider B semi-small submodule of module X, if X/B is semi local hollow module, then X is semi local hollow module.*
Proof . *Suppose that X/B is semi local hollow module, with B is semi small submodule of X then there exists a unique maximal submodule A/B of X/B with $N + M = X$ where M is submodule of X and N is proper submodule of X then $\frac{N+M}{B} = X/B$. Implies that $(\frac{N+B}{B}) + (\frac{M+B}{B}) = X/B$, since $(N + B)/B$ is proper submodule of A/B and X/B is semi local hollow module then $(N + B)/B$ is semi small submodule of X/B. Thus $\frac{M+B}{B} = X/B$, so $M + B = X$.*
Since B is semi-small submodule of X then $M = X$. Therefor X is semi local hollow module. □

Corollary 2.4. *To consider X an $R-$module, if X is semi local hollow module then X/A is semi local hollow module, for each proper submodule A of X.*
Proof . *It's clear by (prop.2.2).* □

Definition 2.5. *[6] A pair (P, F) is a projective cover of the module X in case P is a projective module and $f : P \longrightarrow X$ where f is an epimorphism and kerf is a semi small submodule of P (we call P itself a projective cover of X).*

Proposition 2.6. *Let $f : X_1 \longrightarrow X_2$ is projective cover of X_2; if X_2 is semi local hollow module then X_1 is semi local hollow module.*
Proof . *Suppose that X_2 is semi-local hollow module. Since $f : X_1 \longrightarrow X_2$ is an epimorphism, therefor $X_1/kerf$ is isomorphism to X_2. Hence it is semi local hollow module and kerf is semi small submodule of X_1. Thus by (prop.2.3) we get X_1 is semi-local hollow module.* □

Proposition 2.7. *Let X an $R-$module; so X is semi local hollow module and finitely generated module if and only if; X is a cyclic module, and has a unique maximal submodule.*
Proof .
$\Longrightarrow$ *To consider X is finitely generated semi local hollow module, therefore $X = R_{M_1} + R_{M_2} \ldots\ldots +$ R_{M_n} if $X \neq R_{M_1}$, then R_{M_1} is proper submodule of X. Implies that R_{M_1} is semi-small submodule of X. Hence $X = R_{M_2} + R_{M_3} \ldots + R_{M_n}$.*
Therefore we cancel the summand one by one unite, we have $X = R_{M_i}$, for some i. Thus X is cyclic module and since X is semi local hollow module, so X has unique maximal submodule by (def.2.1).
$\Longleftarrow$ *Suppose that X is cyclic module having unique maximal submodule say A, so X finitely generation. To consider M is proper submodule of X, with $M + B = X$ where B is a submodule of X. Now; when M is not semi small submodule of X, implies that $B \neq X$. So B is a proper. Submodule of X, B is submodule of A and since X is finitely generated. Then B is contained in a maximal submodule but by assumption X has a unique maximal submodule A. Thus M is submodule of A(M is contained in A). Therefore $M + A = A = X$ which is a contradiction. Hence $B = X$; M is submodule of A and M is semi small submodule of X. So X is semi local hollow module.* □

Proposition 2.8. *Let A a maximal submodule of a module X. When X is semi local hollow module and $X|A$ is finitely generated then X is finitely generated.*
Proof . *To consider A is a maximal submodule of semi local hollow module X with $X|A$ is finitely generated then $X|A = R\,(m_1 + A) + R\,(m_2 + A) \ldots + R(m_n + A)$ where $m_i \in X$ for all $i = 1, 2, \ldots n$. We claim that $X = R_{m_1} + R_{m_2} \ldots + R_{m_n}$. Let $x \in X$, so $x + A \in X|A$; implies that $x + A =$ $r_1\,(m_1 + A) + r_2\,(m_2 + A) + \ldots + r_n(m_n + A) = r_1m_1 + r_2m_2 + \ldots + r_nm_n + A$.*
This implies that $x = r_1m_1+r_2m_2+\ldots+r_nm_n$; for some $n \in A$. Thus $X = r_1m_1+r_2m_2+\ldots+r_nm_n+A$ and since X semi local hollow module, so A is a semi small submodule of X which implies that $X = r_1m_1 + r_2m_2 + \ldots + r_nm_n$. Thus M is finitely generated. □

3. Semi-local hollow module and hollow module

The second section suggests that each semi local hollow module is hollow module, and we give an example shows that the converse is not true. In this section we investigate conditions under which hollow module can be semi local hollow module.

Proposition 3.1. *Let X be an $R-$module, X is a semi local hollow module if and only if, X is hollow and cyclic module.*

Proof .

$\Longrightarrow$ *Suppose that X is semi local hollow module, So it has a unique maximal submodule A; such that A contains each small submodule of X. To consider $m \in X$ with $m \notin A$, so R_m is submodule of X. We claim that $R_m = X$. If $R_m \neq X$ then R_m is a proper semi-small submodule of X. Hence R_m is submodule of A which implies that $m \in A$, which is contradiction. Thus $R_m = X$, so X is cyclic module. Since X is semi local hollow module, therefore X is hollow module by (Remark (i)).*

$\Longrightarrow$ *suppose that X is hollow module and cyclic module, so it is a finitely generated module and hence X has a maximal submodule contains each proper semi small submodule say A.*

Let M be a proper semi small submodule of X. If M is not contained in A then $M + A = X$, while X is semi local hollow module, so $A = X$ which is contradiction. This implies that every proper semi small submodule of X is contained in A. Thus X is semi local hollow module. □

Proposition 3.2. *Let X be an $R-$module, X is a semi local hollow module if and only if, X is hollow module and has a unique maximal sub module.*

Proof .

$\Longrightarrow$ *Suppose that X is semi local hollow module, so X is hollow module by (Remark (i), definition 2.1) so X has a unique maximal submodule.*

$\Longrightarrow$ *To consider X is hollow module, so that has a unique maximal submodule say A. We only have to show that X is a cyclic module. To consider $m \in X$ and $m \notin A$, so $R_m + A = X$ and since X is hollow module then A is semi small submodule of X and so $X = R_m$. Therefore X is a cyclic module and by (prop. 3.1) then X is semi local hollow module.* □

Proposition 3.3. *Let X be an $R-$module, X is a semi local hollow module if and only if, it is a cyclic module and every non-Zero factor module of X is indecomposable.*

Proof .

$\Longrightarrow$ *Suppose that X is semi local hollow module, so by (propo 3.1). X is hollow cyclic module and by [8]. Then every non- Zero factor module of X is indecomposable.*

$\Longrightarrow$ *Let X be cyclic module and every non-Zero factor module of X is indecomposable then by [8]. X is hollow module and by (prop. 3.1). Thus X is semi local hollow module.* □

Proposition 3.4. *Let X be an $R-$module, X is a semi local hollow module, if and only if, X is hollow module and Rad $X \neq X$.*

Proof .

$\Longrightarrow$ *Let X be semi local hollow module, then X is hollow and cyclic module, by (prop. 3.1). Since X is cyclic module, so X is finitely generated, hence Rad $X \neq X$.*

$\Longrightarrow$ *Let X is hollow module and Rad $X \neq X$, then Rad X is semi small sub module of X. Also by [6], Rad X is the unique maximal submodule of X and thus $X/Rad\ X$ simple module and hence cyclic. Implies that $X/Rad\ X =< x + Rad\ X >$. for some $x \in X$. We prove that $X = R_x$. To consider $y \in X$, so $y + Rad\ X \in X/Rad\ X$ therefore, there is $r \in R$, such that: $y + Rad\ X = r\,(x + Rad\ X) = rx + Rad\ X$.*

Implies that, $y - rx \in Rad\ X$, then $y - rx = w$, for some $w \in Rad\ X$. So $y = rx + w \in R_x + Rad\ X$. Hence $X = R_x + Rad\ X$. But Rad X is semi small submodule of X, then $X = R_x$. Thus X is a cyclic module and by (prop. 3.1) we get X is semi local hollow module. □

Proposition 3.5. *Let X be semi local hollow module, if and only if, Rad X is a semi small and maximal in X.*

Proof .

$\Longrightarrow$ *Suppose that Rad X is semi small and maximal submodule. To prove that X is semi local hollow*

module. We want to prove that Rad X is a unique maximal submodule in X. Suppose that M is another maximal submodule in X, then $X = M + Rad\ X$, while Rad X is a semi small submodule which implies $M = X$, which is contradiction. Thus Rad X is a unique maximal submodule in X. We claim every semi small submodule of X is contained in Rad X. Let A be a semi small submodule of X, if A is not containd in Rad X then $A + Rad\ X = X$. While Rad X is a semi small submodule of X, which implies that $A = X$, this contradiction. Therefore X is semi local hollow module.
$\Longleftarrow$ suppose that X is semi local hollow module, so by (Remark (i)), therefore X is hollow module and by [6]. Then Rad X is a maximal submodule. Since X is semi local hollow module. Thus Rad X is unique maximal submodule of X. Hence $Rad\ X + A = X$, for some proper submodule A of X. If Rad X is not semi small submodule of X. Then A is semi small submodule of X. Thus $Rad\ X = X$ which is contradiction, by [8]. Hence Rad X is semi small submodule of X. □

4. Semi Local Hollow Module and Some Other Module

This section, takes the relation between semi local hollow module and other modules such that amply supplemented, indecomposable and lifting modules.

Definition 4.1. *[8] A module X is said amply supplemented, if for every two submodule O, P of X, such that: $X = O + P$, there exists a supplement P_1 of O in X such that $P_1 \leq P$.*

Example: The Z-module Z_4 is amply supplemented, while the Z-module Z_{12} is not amply supplemented.

Proposition 4.2. *Every semi local hollow module is amply supplemented.*
Proof . *Let X be semi local hollow module, and to consider O is a unique maximal submodule of X. Since X is semi local hollow module, so we have $O + X = X$ and $O \cap X = O$ is a semi small submodule of X. Therefore X is amply supplemented.* □

Remark 4.3. *The converse of (prop. 4.2)) is not true in general, as given in this example;*
The Z-module Z_6 is amply supplemented while not semi local hollow module.

Definition 4.4. *[2] An R-module X is indecomposable if $X \neq 0$ and the only a direct summands of X are $< 0 >$ and X. Implies that X has no direct sum of two non-Zero submodule.*

Example : The simple module is indecomposable, while the Z-module is Z_6 is not indecomposable.

Proposition 4.5. *Every semi local hollow module is indecomposable.*
Proof . *Let X is semi local hollow module then there exists a unique maximal submodule A such that contains each semi-small submodule of X. Suppose that X is decomposable, so there are a proper submodules B and M such that B, M are submodules of A and $X = B \bigoplus M$. But X is semi local hollow module then either M is a semi small submodule of X with M is submodule of A; implies that $B = X$. Or, B is semi small submodule of X with B is submodule of A, implies that $M = X$; which is contradiction. Then X is indecomposable.* □

Proposition 4.6. *Let X a cyclic module; X is semi local hollow module; if and only if, every non-Zero factor module of X is indecomposable.*
Proof .
$\Longrightarrow$: *X/N non-Zero factor module of X. Since X is semi local hollow module therefore X/N is semi local hollow module by (Corollary 1-5). And by (pro. (3-5)) we get X/N is indecomposable.*

$\Longleftarrow$ *To consider A maximal submodule of X and to consider M is a submodule of A. Suppose that $X = M + B$; where B is a submodule of X, by ([6], Lemma (1-3-10), p. 34]). We get $X/(M \cap B) \cong \left(\frac{X}{M}\right) \bigoplus (X/B)$. while $X/(M \cap B)$ is indecomposable then either $X/M = 0$ or $X/B = 0$. Since M is a submodule of A, and A is a submodule of X. Hence M is a proper submodule of X. Then $X/M \neq 0$, therefore $X/B = 0$. Hence $X = B$. Therefore M is semi small submodule of X. Thus X is hollow module and since X is a cyclic module, so by (prop. 3.1). Thus X is semi local hollow module.* □

Definition 4.7. *[5] Let X be a module, X is said to be lifting module (or satisfied D_1) if for each submodule A of X there are submodule B and M of X where $X = B \bigoplus M$, B is submodule of A and $A \cap B$ is a small submodule of B.*

Example: The Z-module Z_6 is lifting module. While the Z-module Z_{12} is not lifting module.

Proposition 4.8. *Every semi-local hollow module is lifting module.*
Proof . *Let X be semi local hollow module, then there exists a unique maximal A of X contains all semi small submodule, then $X = X \bigoplus \{o\}$; where $\{o\}$ is a submodule of A, $A \cap X = A$ and since X is semi local hollow module. Therefor $A \cap X = A$ is semi small submodule of X. Thus X is lifting module.* □

Remark 4.9. *The converse of prop. 4.8 is not true in general, as for example: The Z-module Z_{10} is lefting module; while is not semi local hollow module.*

Proposition 4.10. *Let X be a cyclic indecomposable module; if X is lifting module, so X is semi-local hollow module.*
Proof . *Let A is a proper submodule of X; since X is lifting module, so $X = N + K$, where N is a submodule and $A \cap N$ is semi small submodule of N. While X is an indecomposable, thus $K = 0$ and hence $N = X$; which $A \cap X = A$, so A is semi small submodule of X. Hence X is hollow module and since X is cyclic module, so X is semi local hollow module. (by prop. 3.1).* □

Definition 4.11. *[9] Let A and M be submodule of X. A submodule A is said to be a supplement of M in X if it is minimal with respect to the property $X = A + M$.*

Proposition 4.12. *Let B a maximal submodule of module X. If M is a supplement of B in X, then M is semi local hollow module.*
Proof . *Suppose that M is a supplement of B and to consider M_1 is proper submodule of M with $M_1 + M_2 = M$; for some submodule M_2 of M. Now ; $B + M = X = B + M_1 + M_2 = X$ and M_1 is a submodule of B, since other wise $B.M_1 = X$ and B is maximal submodule of X; we get $M_1 = M$, which is a contradiction. Thus $B + M_2 = X$ and since B is maximal submodule of X, we get $M_2 = M$. Implies that M is hollow module. To show that M is a cyclic module. Let $m \in X$ and $m \notin B$ then $R_m + B = X$ and this implies that $R_m = M$ by minimality of M. And (by prop. (2-1)), thus M is semi local hollow module.* □

Definition 4.13. *[4] A submodule A of an R-module X is said to be coclosed in X if A/B is small submodule of X/B implies that $A = B$, for each submodule B of X contained in A.*

Example: $< 2 > = \{\overline{0},\ \overline{2},\ \overline{4}\}$ is coclosed submodule in the Z-module Z_6.

Proposition 4.14. *If X is semi local hollow module then each non-Zero coclosed submodule of maximal submodule of X is semi local hollow module.*

Proof . *Suppose that X is semi local hollow module and to consider A be a unique maximal submodule of X. Let N be a non-Zero coclosed submodule of A [3]. Suppose that M is a proper submodule of N. Since X is semi local hollow module thus M is semi small sbmodule of X contained in A. And hence N is coclosed submodule of X. Thus M is semi small submodule of N. Hence N is semi local hollow module.* □

Proposition 4.15. *Suppose that N a submodule of an R-module X. If N is semi local hollow module; so either N is semi small submodule of X, or coclosed submodule of X, while not both.*

Proof . *Assume that N is not coclosed submodule of X. To prove that N is a semi small submodule of X, then there is a proper submodule of X/K. While N is semi local hollow module, so N is hollow module (by Remark (i)). Then (by [8]), we get K is a semi small submodule of N and hence N is a semi small submodule of X (by [8]). Now, we want to prove N is not coclosed and N is semi small submodule of X, we must show that N is zero submodule of X. Since N is semi local hollow module then N is not zero submodule.* □

Proposition 4.16. *Let X be a cyclic module, and let $f : W \to X$ be a projective cover of X and then the following statements are equivalent:*

(1) X is semi local hollow module.
(2) X is hollow module.
(3) W is hollow module.
(4) W is indecomposable and supplemented.
(5) End (w) is local ring.

Proof .

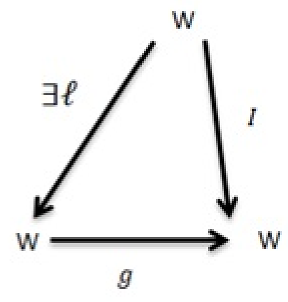
$(1) \to (2)$: *It's clear by (Remark (i)).*

$(2) \to (3)$: *To consider X hollow module and since $f : W \to X$ is an epimorphism; so $W/\ Kerf$ is isomorphism to X and therefore a hollow module and since kerf is small submodule of W; so W is hollow module [7].*

$(3) \to (4)$: *It's clear by [6].*

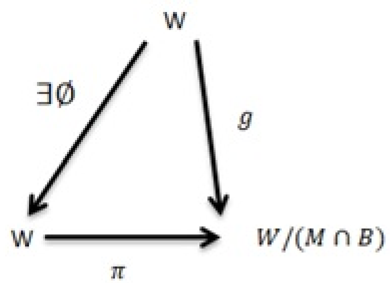
$(4) \to (5)$: *To consider $g : W \to W$ is a homomorphism we have two cases.*

Case 1: *g is onto; since W is projective module, consider this diagram.*



Where $I : W \to W$ is the identity homomorphism and there is a homomorphism, $\ell : W \longrightarrow W$; where $g \circ \ell = I$, implies that g has a right inverse; this implies that $W = \ker g \bigoplus \ell(W)$. But W is indecomposable by (4). Then $kerg = 0$, thus $W = \ell(W)$ then g is one-to-one. Hence g is an isomorphism.

Case 2: *g is not onto; we know that $W = g\,(W) + (I - g)\,(W)$; W is amply supplemented by [6], then there is a supplement B of $g(W)$ in $(I - g)(W)$.*
Implies that $W = g\,(W) + B$ and $g(W) \cap B$ is a semi small submodule of B, and there exists a supplement M of B in $g\,(w)$ implies that $W = M + B$ and $M \cap B$ is a semi small submodule of B. Now; M and B are natural supplement and hence $(M \cap B) = 0$. So $W = M \bigoplus B$, but W is indecomposable and $B \neq 0$, then $B = W$.
Now; B is a submodule of $(I - g)(W)$ this implies that $(I - g)\,(W) = W$. Implies that $(I - g)(W)$ is onto and by the previous argument $(I - g)$ is an isomorphism.
$(5) \rightarrow (1)$*: To consider that X is hollow module, we need only to show that W is hollow by [6].*
Define $g : W \rightarrow W/M \cap B$ as follows. For $m \in W$, $m = s + b$ for some $s \in M$ and $b \in B$. Set $g\,(m) = s + M \cap B$, and g is well-defined and homomorphism and since W is a projective module, there is a homomorphism $\varnothing : W \rightarrow W$ where is diagram is commutative.



Where $\pi : w \rightarrow w/M \cap B$ is the natural epimorphism. To prove that $\varnothing(w)$ is a submodule of M. To see this, let $w \in \varnothing(w)$ then there exists $y \in W$ such that $w = \varnothing(y)$. Now; $(\pi o \varnothing)\,(y) = g(y)$, where $y = s + b$, for some $s \in M$ and $b \in B$. Implies that $\varnothing\,(y) + M \cap B = s + M \cap B$ implies that $\varnothing\,(y) - s \in M \cap B$ is a submodule of M. Then $\varnothing(y) \in M$, and hence $\varnothing(w)$ is a submodule of M.
Similarly; one can show that $(I - \varnothing)(w)$ is a submodule of B. Now, $\varnothing \in End(w)$ and by (5); $End(w)$ is a local ring then $\varnothing$ or $(I - \varnothing)$ is onto, but $\varnothing$ is not onto, since other wise $\varnothing(w)$ is a submodule of M, which implies that $M = W$, which is contradiction.
Therefore $(I - \varnothing)$ is onto. Implies that $B = W$. Thus W is a hollow module. Since W is a hollow module implies that X is a hollow module and since X is cyclic module. Therefore X is semi local hollow module by (prop. 3.1). □

5. Conclusion

1. Each semi local hollow module is hollow module while hollow and cyclic module is semi local hollow module.
2. Epimorphic image of semi local hollow module is semi local hollow module.
3. Every semi local hollow module is amply supplemented .
4. Each semi local hollow module is lifting module.
5. Every semi local hollow module is indecomposable .

Open problems

1. Application of semi local hollow modules.
2. Some generalizations of semi local hollow modules.

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