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Fractional B-spline collection method for solving fractal-differential equations

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Abstract

This study used the fractional B-spline collocation technique to obtain the numerical solution of fractal-fractional differential equations. The technique was considered to solve the fractal-fractional differential equations (FFDEs) with $(0 < \gamma_i < 1, i = 1, 2, \cdots, N)$. In this suggested technique, the B-spline of fractional order was utilised in the collocation technique. The scheme was easily attained, efficient, and relatively precise with reduced computational work numerical findings. Via the proposed technique, FFDEs can be reduced for solving a system of linear algebraic equations using an appropriate numerical approach. The verified numerical illustrative experiments were presented will show the effectiveness of the technique proposed in this study in solving FFDEs in three cases of nonlocal integral and differential operators namely power law kernel, when the kernels are exponential and the generalization of Mittag-Leffler kernel. The approximate solution is very good and accurate to the exact solution.

Keywords: fractional B-spline , Linear fractional differential equations FDEs, Caputo-Fabrizio derivative C-F.
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1. Introduction

Over the past several decades, fractional calculus has been utilised to model various physical problems.
Fractional derivative models are mainly used because numerous systems present memory, history, or

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nonlocal effects that are hard to model with integer order derivative. The fundamental theory, some applications of fractional differential equations (FDEs) and fractional calculus have been reported in various studies (for example, [22, 12, 18, 2, 21, 14]).

Even though the majority of the initial studies are on the application of the Riemann–Liouville (R-L) or the Caputo fractional order derivative, recently, it has been highlighted that these derivatives' kernels possess a singularity which happens at the end point of an interval of definition. Consequently, various fractional derivatives definitions are reported in the literature (for example, [7, 1, 11, 20, 29, 17, 24, 19, 28]).

The key variation amongst the fractional derivatives would be the varying kernels that can be chosen to fulfil the needs of distinct applications. For instance, the key variation among the Caputo [12], C–F [11], and Atangana–Baleanu (A-B) [4] fractional derivatives is that these derivatives are defined using different laws, i.e. power low , Mittag–Leffler law, and low of exponential decay. In several latest publications, application of new operators of fractional order to real-life issues has been discussed. Among them, in their study of anomalous diffusion, Tateishi et al. [29] examined the traditional and novel fractional time-derivatives. Besides, Atangana et al. contrasted the C–F with A–B fractional derivatives in modelling delay differential equations of fractional order [5] and in modelling systems of chaotic [4]. The researchers discovered that the power law derivative of the L-R or the C–F provided noisy information because of its specific memory properties. Nevertheless, the C–FFD gave lower noise compared to the power law one, whereas the A–B fractional derivatives gave an exceptional description.

A few numerical approaches exist, which can be used to determine approximately solutions of DEs which models engineering problem [15, 27, 23, 9]. The collocation method is one of the available approaches.

In a study [30], a fractional orders of B-splines extension was given and it was demonstrated that every desirable property of B-splines of the integer order was carried over to the derivative of fractional case. It is possible to design the fractional splines to possess any order of smoothness. The researchers already numerically solved the linear and nonlinear time fractional derivative problems. They also demonstrated the efficiency and precision, while maintaining a low computational cost. This method's plus point is its simplicity that lowers the computational cost. Nonetheless, this method's level of precision is low, limiting its usability to quite a small area. In 2013, Jafari et al. [15] solved FDEs via the fractional B-spline method with the Caputo derivative. Meanwhile, Ramezani [26] solved nonlinear fractional order multi-term differential-time equations using fractional B-splines.

Other than that, Francesca [13] presented B-spline fractional collocation approach to numerically solving fractional predator-prey models.

The application of collocation approach using B-spline basis functions has become more common in solving engineering issues. The aim is to link the smoothness of fractional B-spline as approximate functions with the reduced cost of computation of collocation. This method is proven to be efficient and equal to other recognised approaches like finite element and finite difference methods. The fractional B-spline technique has several clear benefits than the finite element and finite difference approaches.

For example, compared to the finite element technique, the B-spline collocation approach is simpler and easier to use for various problems concerning differential equations.

This paper aims to investigate the application of the collocation technique with fractional B-spline basis function involving C-F fractional derivatives to solve fractional differential equations with initial condition. The goal is to determine the numerical solutions of fractional differential equations.

Consider the equation

$$D^{\gamma}\varphi(t)+\mathcal{L}[\varphi(t)]=\phi(t)\,,\quad m-1<\gamma<m \tag{1.1}$$

with initial condition:

$$\varphi^{(i)}(0)=c_i,\quad i=0,1,2,...,m-1. \tag{1.2}$$

where D^{γ} is a C-F fractional derivative, $\mathcal{L}$ is a linear operator, and $\phi(t)$ is a given function.
The organization of this paper is as follows: Section 2 presents a few famed definitions of derivative and integral in fractional calculus, while Section 3 will discuss the definitions of fractional B-spline collocation approach and some of its properties. Next, in Section 4, fractional B-splines involving C-F derivative is applied to fractional differential equations. Finally, Section 5 presents and verifies the effectiveness of this method.
To solve the considered fractional differential equation by numerical technique namely fractional B-spline collection method with C-F derivative we state the following definitions:

Definition 1. *Fractional calculus:*
The Caputo fractional derivative of order γ which defined as [25]:

$$\left({}_0^C D_t^{\gamma}\,y\right)(t)=\frac{d^m}{dt^m}\left({}_0D_t^{-(m-\gamma)}y\right)(t)=\frac{1}{\Gamma(m-\gamma)}\int_0^t\frac{y^m(x)}{(t-x)^{\gamma-m+1}}dx \tag{1.3}$$

where $m-1<\gamma<m$
The Caputo fractional derivative cannot always be an appropriate kernel $(t-x)^{-\alpha}$ to precisely define the memory effect in a real system due to the singularity in its kernel [25, 16], at the end point of the interval of integration. Recently, Caputo and Fabrizio [11] suggested a novel fractional derivative that has no singularity in its kernel. This fractional derivative's kernel possesses the form of an exponential function. In addition, Losada and Nieto [20] derived the fractional integral related to this fractional derivative. The C–F fractional operators' definitions and properties relevant to this study will be summarized in this section.
Consider $\mathbb{L}^2(\alpha,\beta)$ the space of square integrable functions on (α,β) as

$$H^1(\alpha,\beta)=\{v\mid v\in\mathbb{L}^2(\alpha,\beta)\ and\ v'\in\mathbb{L}^2(\alpha,\beta)\} \tag{1.4}$$

Let $v\in H^1(o,\beta)$, $\beta>0$, $0<\gamma<1$, then the fractional C-F derivative is:

$${}^{C-F}D_t^{\gamma}(v(t))=\frac{(2-\gamma)M(\gamma)}{2(1-\gamma)}\int_0^t\exp\Big[-\frac{\gamma(t-x)}{1-\gamma}\Big]v'(x)dx,\quad t\geqslant 0,\ 0<\gamma<1 \tag{1.5}$$

where $M(\gamma)$ which is a normalization function depending on $M(0)=M(1)=1$. while the C-F integral defined as:

$${}^{C-F}I_t^{\gamma}(v(t))=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}v(t)+\frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t v(x)dx. \tag{1.6}$$

This definition explained that the ${}^{C-F}I_t^{\gamma}v(t)$ is the mean between the $v(t)$ and its integral which is:

$$\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}+\frac{2\gamma}{(2-\gamma)M(\gamma)}=1, \tag{1.7}$$

therefore, $M(\gamma) = \frac{2\gamma}{2-\gamma}$, $0 < \gamma < 1$. *By this notation Losada and Niento [20] defined a new C-F derivative and integral:*

$$ {}^{C-F}D_t^{\gamma}(v(t)) = \frac{1}{(1-\gamma)} \int_a^t v'(x) \exp\Big[- \frac{\gamma(t-x)}{1-\gamma} \Big] dx, \quad t \geqslant 0. \tag{1.8} $$

$$ {}^{C-F}I_t^{\gamma}(v(t)) = (1-\gamma)v(t) + \gamma \int_0^t v(x)dx, \quad t \geqslant 0. \tag{1.9} $$

respectively.

2. Fractional B-spline collection method

In this section, we will present some definitions and theorems without prove.
A spline polynomial of degree p is a piecewise polynomial of order $p+1$ on the interval $[a, b]$ constrained in the steps as:
1- The partition of the interval $[a, b]$ as $a=\xi_1 \leq \xi_2 \leq ... \leq \xi_q = b$ are determined as knots of interpolation, there is one polynomials in every $[\xi_i, \xi_{i+1}]$ of degree$n$ to connected to other polynomials on the $[\xi_{i+1}, \xi_{i+2}]$:

$$ S^n(\xi) = \begin{cases} s_1(\xi) & , \ \xi_1 \leqslant \xi \leqslant \xi_2 \\ s_2(\xi) & , \ \xi_2 \leqslant \xi \leqslant \xi_3 \\ \vdots & \\ s_{q-1}(\xi) & , \ \xi_{q-1} \leqslant \xi \leqslant \xi_q \end{cases} \tag{2.1} $$

where $S^n(\xi)$ is a spline of degree p , and the polynomials in every interval are $s_i(\xi), i = 1, 2, 3, ..., q-1$.
2- Thus, the p-th order of $S^p(\xi)$ is bounded, evidence some isolated discontinuity at the knots. Every polynomial $s_i(\xi)$, $i = 1, 2, 3, ..., q-1$ has $(n-1)$ order continuity derivatives on the interval $[\xi_i, \xi_{i+1}]$.
Schoenberg's polynomial spline with uniform knots is B-spline basic functions which defined as:

$$ S^p(\xi) = \sum_{w \in \mathbb{Z}} c_w \beta^p(\xi - w) \tag{2.2} $$

$$ \beta^p = \frac{1}{p!} \sum_{w=0}^{p+1} (-1)^w \binom{p+1}{w} (\xi - w)_+^p \quad , \quad p \in \mathbb{N} \tag{2.3} $$

where the power function of one-side defined in :

$$ (\xi - w)_+^p = \begin{cases} (\xi - w)^p & , \ \xi \geqslant w \\ \\ 0 & , \ otherwise. \end{cases} \tag{2.4} $$

The fractional B-splines are defined by:

$$ \beta^{\gamma} = \frac{1}{\Gamma(\gamma+1)} \sum_{w \leq 0} (-1)^w \binom{\gamma+1}{w} (\xi - w)_+^{\gamma} \tag{2.5} $$

The fractional splines spaces with degree γ at scale a defined in:

$$ S_a^{\gamma} = \{ \ s_a | \ \exists c \in l^2, s(\xi) = \textstyle\sum_{k \in \mathbb{Z}} \ c_k \, \beta^{\gamma}(\frac{\xi - ak}{a}) \ \} \ . \tag{2.6} $$

So that, for an arbitrary $v \in \mathbb{L}^2(\mathbb{R})$ can be determined its least squares which approximates in S_a^γ for $\gamma > -\frac{1}{2}$. Moreover, for $\gamma > -1$ fractional splines $\beta^\gamma(\xi)$ are $\in \mathbb{L}^1(\mathbb{R})$.
suppose that $X = L^2(\mathbb{R})$, $\varphi(t) \in X$, we choose subspace of a finite dimensional of approximate solution $\varphi_n(t) \in X_n$ with $a = \frac{1}{2^n}$, $n \in \mathbb{N}$ which is closed to the exact solution $\varphi(t)$ with basis $\beta^\eta(\frac{a}{2^n} - k)$ then approximate solution can be written by applying fractional B-spline collection method as basis functions.:

$$\varphi_n(t) = \sum_{k \in \mathbb{Z}} c_k \beta^\eta(\frac{t}{2^n} - k). \tag{2.7}$$

consider $t \in [0, b]$, and $n \in \mathbb{N}$ we get

$$\varphi_n(t) = \sum_{k=1}^{q} c_k \beta^\eta(\frac{t}{2^n} - k), \; q \in \mathbb{R} \tag{2.8}$$

Now, by substituting the equation above in the linear differential fractional equation and solve it. Since $\varphi(t) \in [0, b]$ then the limit of k will be on $[0, b]$ by seeking the points $0 = t_0 < t_1 < t_2, ..., < t_q = b$. To determine $c_1, c_2, ..., c_q$ we have to solve the linear system which we get by applying

$$\varphi_n(t_i) = \sum_{k=1}^{q} c_k \beta^\eta(\frac{t_i}{2^n} - k), \; q \in \mathbb{R} \tag{2.9}$$

where $i = 0, 1, 2, ..q$.
with the definition of Atangana [3] derivative of order γ and substitution in equation (1.1) then

$$\sum_{i=1}^{q} c_i \{ \frac{1}{1-\gamma} \int_0^t \beta^\eta(2^n t - j)'(x) \exp(-\gamma \frac{t-x}{1-\gamma}) dx + \beta^\eta(2^n t - j) \} - \phi(t_i) = 0. \tag{2.10}$$

$$\sum_{i=1-2^n}^{q} c_i \{ \frac{1}{(1-\gamma)\Gamma(\eta+1)} \int_a^t \sum_{w \le 0} (-1)^w \binom{\eta+1}{w} (\frac{t_i}{2^n} - w)_t^\eta \; \exp(-\gamma \frac{t-x}{1-\gamma}) dx + \frac{1}{\Gamma(\eta+1)} \sum_{w \le 0} (-1)^w \binom{\eta+1}{w} (\frac{t_i}{2^n} - w)_t^\eta \} - \phi(t_i) = 0. \tag{2.11}$$

The absolute error $|\varphi(t) - \varphi^*(t)|$ is defined as:

$$E_n = \|\varphi(t) - \varphi_n(t)\|_2 = \sqrt{\int_0^b |\varphi(t) - \varphi_n(t)|^2 dt} \tag{2.12}$$

3. Numerical illustrative results

This section presents numerical results for solving fractal fractional differential equation by using fractional B-splines collection method. The examples show the accuracy of this method.

Example 1:
Consider the following FDE [8] :

$${}_0^F D_t^\Omega \, y(t) + \frac{d\, y(t)}{dt} + \frac{d^2\, y(t)}{dt^2} + y(t) = \phi(t), \qquad \Omega > 0 \tag{3.1}$$

we consider this problem with the FFP

$$ {}_0^{FFP}D_t^{\gamma,\Omega} y(t) + y'(t) + y''(t) + y(t) = \phi(t), \qquad \Omega, \gamma \in (0,1] \tag{3.2} $$

and consider $\phi(t)$ to be

$$ \phi(t) = -\frac{1}{4t^{\Omega+1}} + \frac{1}{2t^{\Omega}} + t^{\Omega} + \sqrt{t}. \tag{3.3} $$

We solve this example using fractional B-spline collection method on the interval $[0,1]$ with $\gamma = \frac{1}{2}$, $\eta = \frac{1}{2}$ and $a = \frac{1}{2}$
From the figure 1 , which explained approximate and exact solutions, and table 1 which described the absolute error we conclude that the validity applicable of considered method.

t	$\lvert\varphi(t) - \varphi_n(t)\rvert$ $\Omega = 0.5$	$\lvert\varphi(t) - \varphi_n(t)\rvert$ $\Omega = 0.02$
0	7.05106×10^{-16}	1.53925×10^{-15}
0.1	4.44089×10^{-16}	1.38778×10^{-15}
0.2	3.33067×10^{-16}	1.38778×10^{-15}
0.3	3.33067×10^{-16}	1.44329×10^{-15}
0.4	3.33067×10^{-16}	1.44329×10^{-15}
0.5	2.22045×10^{-16}	1.55431×10^{-15}
0.6	2.22045×10^{-16}	1.33227×10^{-15}
0.7	4.44089×10^{-16}	1.44329×10^{-15}
0.8	4.44089×10^{-16}	1.44329×10^{-15}
0.9	4.44089×10^{-16}	1.44329×10^{-15}

Now we consider equation 3.1 with the Mittag-Leffler kernel law in the form:

$$ {}_0^{FFM}D_t^{\gamma,\Omega} y(t) + y'(t) + y''(t) + y(t) = \phi(t) \qquad \Omega, \gamma \in (0,1] \tag{3.4} $$

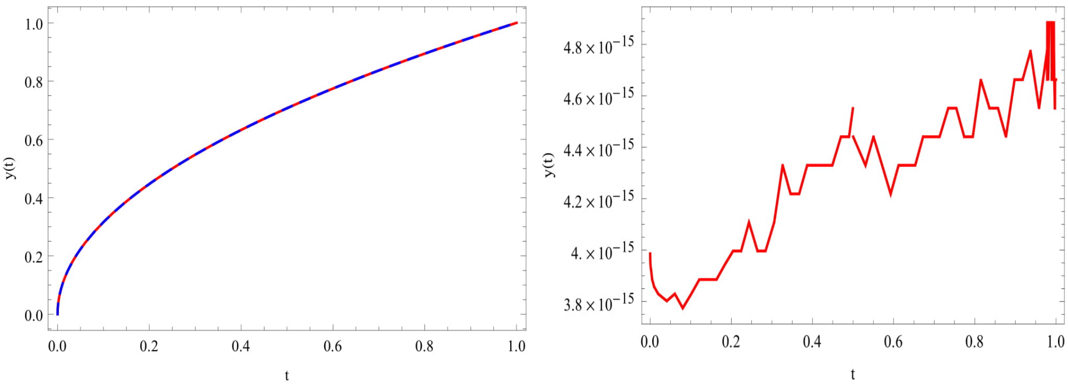
and consider $\phi(t)$ to be

$$ \phi(t) = -\frac{1}{4t^{\Omega+1}} + \frac{1}{2t^{\Omega}} + t^{\Omega} + 3.t^{\Omega}\,(\frac{-t\sqrt{\pi}}{12}(6 + t(3+t)) + \frac{t^{\Omega}}{15}(15 + 2t(5+2t))) \tag{3.5} $$

Example 2:
Consider the following C-FFDE :

$$ {}^{C-F}D^{\gamma} y(t) + \frac{y(t)}{\Gamma(\frac{1}{2})} = \frac{\sin(t)}{\sqrt{t}} + \frac{4}{3}\Big(- e^{-t} + \cos(t) + \sin(t)\Big) \tag{3.6} $$

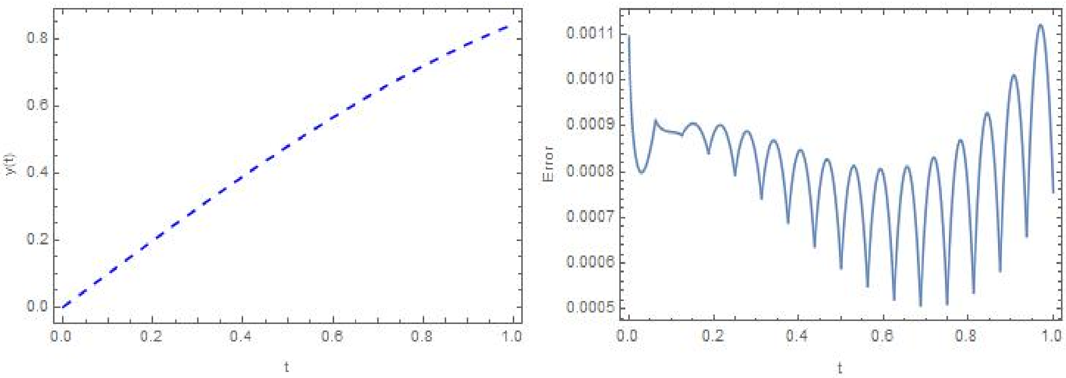
with
$y(0) = 0$ with the exact solution $y(t) = \sin t$
We solve this example using fractional B-spline collection method via C-F on the interval $[0,1]$ with $\gamma = 0.99$, $n = 4$ and $\eta = \frac{1}{2}$



(a) (b)

Figure 1: figure1:- a: Blue line: the exact solution , Dashed-line: the approximate solution. b: Comparison between the exact and approximate solutions.

t	$\lvert\varphi(t) - \varphi_n(t)\rvert$ $\Omega = 0.5$	$\lvert\varphi(t) - \varphi_n(t)\rvert$ $\Omega = 0.02$
0	3.69237×10^{-16}	1.53925×10^{-15}
0.1	2.77556×10^{-16}	1.38778×10^{-15}
0.2	2.77556×10^{-16}	1.38778×10^{-15}
0.3	1.11022×10^{-16}	1.44329×10^{-15}
0.4	2.22045×10^{-16}	1.44329×10^{-15}
0.5	2.22045×10^{-16}	1.55431×10^{-15}
0.6	2.22045×10^{-16}	1.33227×10^{-15}
0.7	0	1.44329×10^{-15}
0.8	0	1.44329×10^{-15}
0.9	-1.11022×10^{-16}	1.44329×10^{-15}



(a) (b)

Figure 2: figure2:- a: the approximate solution is very good and fit on the exact solution. b: Comparison between the exact and approximate solutions.

Example 3:

Consider the following C-FFDE:

$$^{C-F}D^{\gamma}\, y(t) + \frac{y(t)}{\Gamma(\frac{1}{2})} = \frac{e^t}{\sqrt{\pi}} + \frac{8\sinh(t)}{3} \tag{3.7}$$

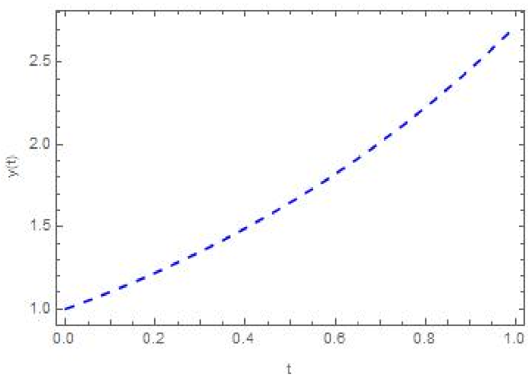
with:

$y(0) = 0$ with the exact solution $y(t) = e^t$

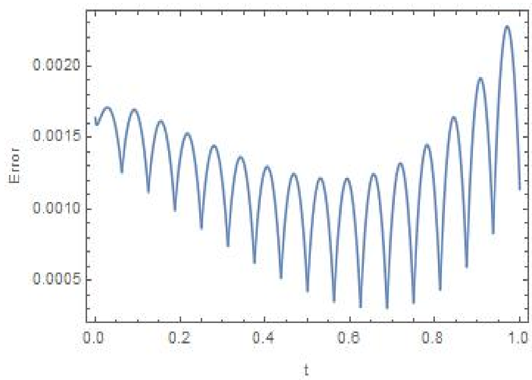
We solve this example using fractional B-spline collection method via C-F on the interval $[0, 1]$ with $\gamma = 0.99$, $n = 4$ and $\eta = \frac{1}{2}$

From the figure 3 , which explained approximate and exact solutions, and table 3 which described the absolute error we conclude that the validity applicable of considered method.

t	$\lvert\varphi(t) - \varphi_n(t)\rvert$
0	1.63621×10^{-3}
0.1	1.65954×10^{-3}
0.2	1.34392×10^{-3}
0.3	1.17312×10^{-3}
0.4	1.2757×10^{-3}
0.5	4.15682×10^{-4}
0.6	1.17317×10^{-3}
0.7	9.42558×10^{-4}
0.8	1.09238×10^{-3}
0.9	1.83431×10^{-3}



(a)



(b)

Figure 3: figure3:- a: the approximate solution is very good and fit on the exact solution.
b: Comparison between the exact and approximate solutions.

4. Conclusion

The collocation technique was utilised in this study to solve the fractional differential equations. Specifically, the fractional B-spline bases collocation approach was employed. The definition for the fractional derivative was based on C-F ($0 < \gamma_i < 1$, $i = 1, \ldots n$). There are two advantages of the method presented in this study. Firstly, the fractional B-spline collocation approach could be comprehensively documented via a very straightforward format. Secondly, is the application of fractional B-spline collocation approach involving C-F derivative to overcome the singularities for $t = x$ in the new kernel of the C-F over the Caputo operator. The attained numerical findings and figures have shown the efficiency and high accurateness of the suggested numerical approximate scheme. To test the scheme's accuracy, three numerical experiments with an acceptable agreement with the exact solutions are presented in this study.

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