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A numerical method for solving variable order fractional optimal control problems

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Abstract

This study is devoted to introducing a computational technique based on Bernstein polynomials to solve variable order fractional optimal control problems (VO-FOCPs). This class of problems generated by dynamical systems describe with variable order fractional derivatives in the Caputo sense. In the proposed method, the Bernstein operational matrix of the fractional variable-order derivatives will be derived. Then, this matrix is used to obtain an approximate solution to mentioned problems. With the use of Gauss-Legendre quadrature rule and the mentioned operational matrix, the considered VO-FOCPs are reduced to a system of equations that are solved to get approximate solutions. The obtained results show the accuracy of the numerical technique.

Keywords: Fractional optimal control problems, Variable order, Bernstein polynomials, Operational matrix.
2010 MSC: 65K10; 26A33.

1. Introduction

In recent years, fractional calculus (FC) has been used by scientist and researchers to development mathematical models of real-world phenomena [1, 13, 27]. Moreover, the concept of VO fractional operator recently has been introduced in [21]. Many phenomena in science such as engineering and mathematical physics problems have been modeled using VO differential equations [7, 18, 21, 22, 28, 30, 31, 32, 34, 36, 37]. VO derivative is a novel and extended of arbitrary order derivative that

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might be stated in terms of space, time or more variables. Many dynamical processes may be better described using VO fractional derivatives. These new operators are more prospect and practical than the classical fractional ones because of their advantage of the memory property. So, the VO fractional differential equations (VO-FDEs) are receiving more important that need a solution.

Therefore, various research have been offered to obtain numerical solution of VO-FDEs [3, 4, 5, 6, 8, 9, 11, 12, 14, 20, 33, 35, 38, 41].

Fractional optimal control problems (FOCPs) have bben investigated by different type of fractional derivatives, such as the Riemann-Liouville and Caputo fractional derivatives. Also, to solve FOCPs, various methods are used [2, 10, 19, 23, 24, 25, 26, 29, 39]. Similarly, variable-order (VO) fractional optimal control problems (VO-FOCPs) might be investigated by different type of variable-order fractional derivatives, such as Caputo, Riemann-Liouville or Atangana-Baleanu derivatives. In these problems, VO-FDEs is considered as VO-FOCPs to describe the dynamics system. So far, few numerical methods presented to solve the mentioned problems [15, 16, 17, 40].
In this work, we use the Bernstein polynomials (BPs) to introduce a numerical technique for solvin the following type of VO-FOCPs:

$$\min \quad \mathbb{J} = \int_0^1 \chi(t, x(t), u(t))dt, \tag{1.1}$$

under

$$ {}_0^cD_t^{\alpha(t)}x(t) = \rho(t, x(t)) + a(t)u(t), \quad m-1 < \alpha(t) \leq m, \quad m \in \mathbb{N}, \tag{1.2}$$

and the initial conditions:

$$x(0) = b_0,\ x'(0) = b_1,\ \cdots,\ x^{(m-1)}(0) = b_{m-1}, \tag{1.3}$$

where $a(t) \neq 0$ and χ are smooth and ${}_0^cD_t^{\alpha(t)}x(t)$ is $\alpha(t)$ order caputo derivative. Here, we use BPs to obtain numerical solutin of the VO-FOCPs.

We expand the the VO fractional derivative of state variable $x(t)$ by using the BPs. Then using operational matrix of VO fractional integral we achieve a nonlinear algebraic equation with unknown coefficients which will be determined

The rest of this study is organized as follows: In Section 2, we present a brief review of VO calculus, BPs and the VO operational matrix. The proposed scheme is presented in Section 3, to obtain numerical solution of VO-FOCPs. In Section 4, we investigate the efficiency and applicability of the given technique by few numerical examples. Finally, in Section 5 the conclusion is summarized.

2. Preliminaries

Here, we briefly review definitions of the VO fractional operators [5, 6, 7]. Then for VO fractional integral we obtain operational matrix based on BPs.

2.1. VO fractional calculus

Definition 2.1. *The $\alpha(t)$-order integral is defined by*

$${}_0I_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha(t)}}d\tau.$$

Definition 2.2. *The $\alpha(t)$-order Caputo derivative is given by*

$$
{}_0^cD_t^{\alpha(t)}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha(t))}\int_0^t (t-\tau)^{m-\alpha(t)-1}f^{(m)}(\tau)d\tau, & m-1<\alpha(t)<m, \\ \\ \frac{d^mf(t)}{dt^m}, \quad \alpha(t)=m, & m\in\mathbb{N}. \end{cases}
$$

For the the above definitions, we have the following properties:

$$
{}_0^cD_t^{\alpha(t)}f(t) = {}_0I_t^{m-\alpha(t)}\frac{d^mf(t)}{dt^m}, \quad m-1<\alpha(t)\le m,
$$

$$
{}_0^cD_t^{\alpha(t)}t^w = \begin{cases} \frac{\Gamma(w+1)}{\Gamma(w-\alpha(t)+1)}t^{w-\alpha(t)}, & m-1<\alpha(t)\le m, \quad w\ge m, \\ \\ 0, \quad w<m. \end{cases}
$$

2.2. Bernstein polynomials

The BPs of degree j are defined on the interval $[0,1]$ as [19]:

$$
B_{j,n}(t) = \binom{n}{j}t^j(1-t)^{n-j}, \quad 0\le j\le n.
$$

The above polynomials can be rewritten as

$$
\begin{aligned}
B_{j,n}(t) &= \binom{n}{j}t^j(1-t)^{n-j} = \binom{n}{j}t^j\left(\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}t^k\right) \\
&= \sum_{k=0}^{n-j}(-1)^k\binom{n}{j}\binom{n-j}{k}t^{j+k}; \qquad j=0,1,2,\dots,n.
\end{aligned}
$$

We present the BPs in the matrix form as follow:

$$
\phi(t) = AT_n(t), \tag{2.1}
$$

where $\phi(t)=[B_{0,n}(t),B_{1,n}(t),\cdots,B_{n,n}(t)]^T$, $T_n=[1,t,\dots,t^n]^T$ and A is a matrix:

$$
A = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1-0} & \dots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ \ddots & & & \vdots \\ 0 & (-1)^i\binom{n}{i} & \dots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & (-1)^n\binom{n}{n} \end{bmatrix},
$$

and $|A| = \Pi_{i=0}^n\binom{n}{i}$, so it is easy to see that A is invertible.

2.3. Approximation of function

A function $h(t)\in L^2[0,1]$, it might be approximated be BPs as:

$$
h(t) = \sum_{j=0}^n c_jB_{j,n} = C^T\phi(t),
$$

here

$$C^T\langle \phi, \phi\rangle = \langle h, \phi\rangle, \tag{2.2}$$

where $\langle \phi, \phi\rangle$ is called dual matrix of ϕ which is indicated by Q, we have

$$\langle h, \phi\rangle = \int_0^1 h(t)\phi(t)^T dt = [\langle h, B_{0,n}\rangle, \langle h, B_{1,n}\rangle, \dots, \langle h, B_{n,n}\rangle],$$

and

$$\langle \phi, \phi\rangle = Q = \int_0^1 \phi(t)\phi(t)^T dt. \tag{2.3}$$

In view of (2.2) and (2.3), we have

$$C^T = \left(\int_0^1 h(t)\,\phi(t)^T dt\right) Q^{-1},$$

Here $C^T = [c_0, c_1, \dots, c_n]$ is called Bernstein coefficients.

2.4. Operational matrix of VO fractional integration

order $\alpha(t)$ We can be approximated the variable order fractional integral by BPs as

$${}_0I_t^{\alpha(t)}\phi(t) \simeq \mathbb{I}^{\alpha(t)}\phi(t),$$

where $\mathbb{I}^{\alpha(t)}$ is operational matrix for VO fractional integral. By replacing Eq. (2.1) instead of $\phi(t)$, we can construct $\mathbb{I}^{\alpha(t)}$ as follow:

$${}_0I_t^{\alpha(t)}\phi(t) = {}_0I_t^{\alpha(t)}AT_n(t) = A\ {}_0I_t^{\alpha(t)}T_n(t).$$

It yields

$$\begin{aligned}
{}_0I_t^{\alpha(t)}\phi(t) &= A\left[\frac{\Gamma(1)}{\Gamma(\alpha(t)+1)}t^{\alpha(t)} \quad \frac{\Gamma(2)}{\Gamma(\alpha(t)+2)}t^{\alpha(t)+1} \quad \cdots \quad \frac{\Gamma(n)}{\Gamma(\alpha(t)+n+1)}t^{\alpha(t)+n}\right]^T \\
&= A\begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\alpha(t)+1)}t^{\alpha(t)} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\alpha(t)+2)}t^{\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(\alpha(t)+n+1)}t^{\alpha(t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\
&= AWA^{-1}\phi(t),
\end{aligned}$$

that

$$W = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\alpha(t)+1)}t^{\alpha(t)} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\alpha(t)+2)}t^{\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(\alpha(t)+n+1)}t^{\alpha(t)} \end{bmatrix},$$

consequently, operational matrix of VO fractional integral of Bernstein polynomials can be constructed as:

$$\mathbb{I}^{\alpha(t)} = AWA^{-1}.$$

3. The Method

In this section, we present a computational technique to obtain numerical solution of VO-FOCPs (2.2). First, in the Eq. (1.2), we focus to calculate the control variable $u(t)$. In this regard, we replace the resultant $u(t)$ into (1.1), which yields

$$\min \quad \mathbb{J} = \int_0^1 \chi \left(t, x(t), \frac{1}{a(t)} ({}_0^c D_t^{\alpha(t)} x(t) - \rho(t, x(t))) \right) dt. \tag{3.1}$$

Now we approximate ${}_0^c D_t^{\alpha(t)} x(t)$ in terms of the BPs to solve the above-mentioned problem as:

$${}_0^c D_t^{\alpha(t)} x(t) \simeq X^T \phi(t), \tag{3.2}$$

where X^T is an unknown vector as following

$$X^T = [x_1, \cdots, x_n].$$

We obtain the following relation by using operational matrix, the moment property of the VO fractional integral and initial conditions:

$$x(t) \simeq X^T \mathbb{I}^{\alpha(t)} \phi(t) + \sum_{i=0}^{m-1} b_i \frac{t^i}{i!}.$$

Assume that

$$\sum_{i=0}^{m-1} b_i \frac{t^i}{i!} \simeq B^T \phi(t).$$

Therefore we get

$$x(t) \simeq (X^T \mathbb{I}^{\alpha(t)} + B^T) \phi(t). \tag{3.3}$$

By replacing Eqs. (3.2) and (3.3) into Eq. (3.1), we have an unconstrained optimization problem:

$$\min \quad \mathbb{J}[X] = \int_0^1 \chi \left(t, (X^T \mathbb{I}^{\alpha(t)} + B^T)\phi(t), \frac{1}{a(t)} (X^T \phi(t) - \rho(t, (X^T \mathbb{I}^{\alpha(t)} + B^T)\phi(t))) \right) dt.$$

we achieve

$$\min \quad \mathbb{J}[X] = \int_0^1 \chi_n(t, X) dt. \tag{3.4}$$

We can compute the numerical solution of the above equation (3.4) by using Gauss-Legendre quadrature rule as

$$\mathbb{J}[X] = \frac{1}{2} \sum_{j=1}^{\ell} \omega_j \chi_n (\frac{\tau_j + 1}{2}, X),$$

where $\{\omega_j\}_{j=1}^{\ell}$ are the corresponding Christoffel numbers and $\{\tau_j\}_{j=1}^{\ell}$ are the nodes of Gauss-Legendre. We obtain the necessary conditions for the extremum as

$$\frac{\partial \mathbb{J}[X]}{\partial x_i} = 0, \qquad i = 1, \cdots, n. \tag{3.5}$$

Finally by calculating system (3.5), we can obtain the desired optimal solution. Consequently we can calculate $x(t)$ from the Eq. (3.3).

4. Test examples

In this section, we apply the proposed method in Section 3 to obtain numerical solution of few examples.

Example 1. *We consider the following VO-FOCP:*

$$\min \quad \mathbb{J}[X] = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt,$$

subject to the dynamical system

$${}_0^cD_t^{\alpha(t)}x(t) = -x(t) + u(t), \quad 0 < \alpha(t) \leq 1,$$

with the I.C.

$$x(0) = 1.$$

For this problem, the exact solutions of $x(t)$ and $u(t)$ when $\alpha(t) = 1$ are given as

$$\begin{aligned} x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \\ u(t) &= (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t), \end{aligned}$$

where,

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} \simeq -0.98.$$

Now, we solve the above problem by the presented method with $n = 3$ for the following VOs

$$\begin{aligned} &\alpha_1(t) = 1 - 0.05t, && \alpha_4(t) = 0.95 + 0.04\sin t, \\ &\alpha_2(t) = 1 - 0.15t, && \alpha_5(t) = 0.85 + 0.04\sin t, \\ &\alpha_3(t) = 1 - 0.25t, && \alpha_6(t) = 0.75 + 0.04\sin t. \end{aligned}$$

Figs. 1 and 2 demonstrate the numerical results for the state variable $x(t)$ and control variable $u(t)$ for some variable orders $\alpha(t)$ with $n = 3$respectively.

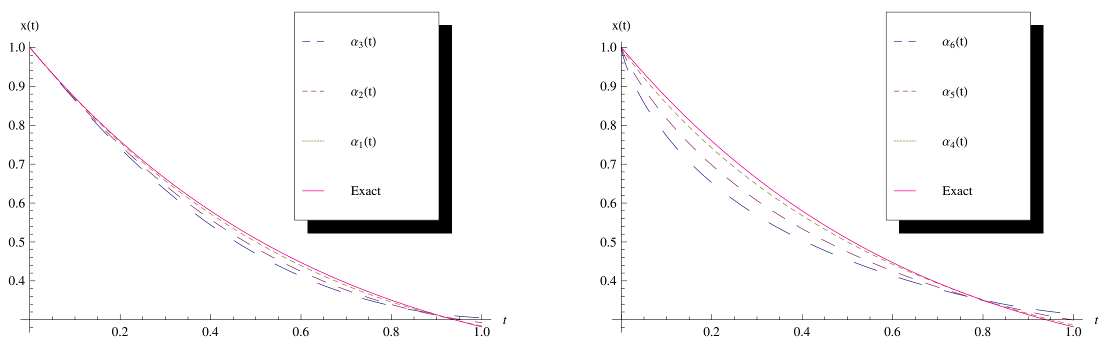


Figure 1: Numerical solutions of $x(t)$ for various values of $\alpha(t)$ for Example 1.

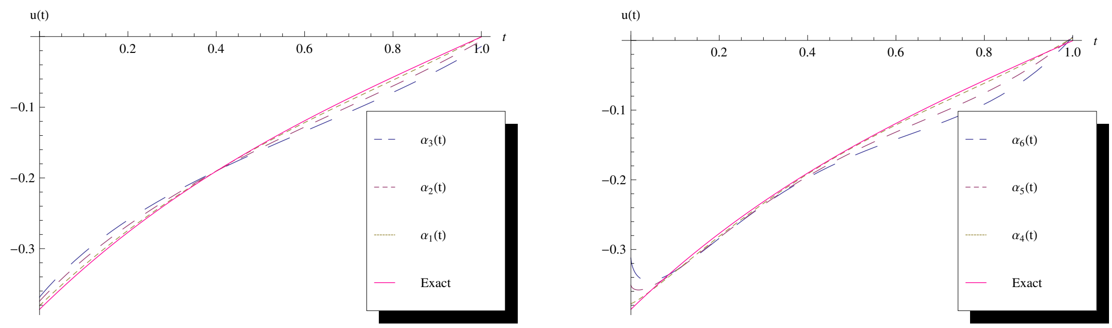


Figure 2: Numerical solutions of $u(t)$ in Example 1 for various values of $\alpha(t)$.

Example 2. *Consider VO-FOCP problem is given as:*

$$\min \quad \mathbb{J}[X] = \frac{1}{2}\int_0^1 \left[(x(t) - t^{\alpha(t)})^2 + (u(t) - t^{\alpha(t)} - \Gamma(\alpha(t)+1))^2\right] dt,$$

under the variable order fractional dynamical system

$$ {}_0^c D_t^{\alpha(t)} x(t) = -x(t) + u(t), \quad 0 < \alpha(t) \leq 1,$$

and the initial condition is given by

$$x(0) = 0.$$

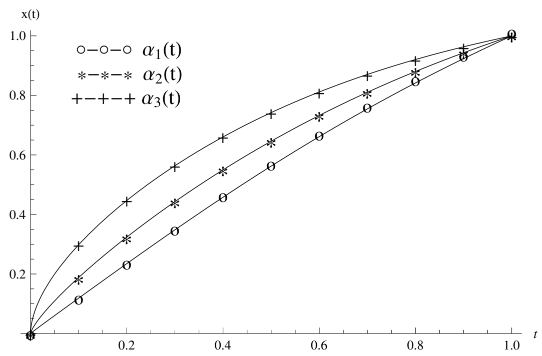
For this problem, the values $u(t) = t^{\alpha(t)} + \Gamma(\alpha(t)+1)$ and $x(t) = t^{\alpha(t)}$ are the minimizing solutions for the control and state variables, respectively, and the performance index $\mathbb{J}[X]$ has the minimum value of 0. The above problem is now solved by the presented technique with $n = 4$ and $n = 8$ for the following VOs

$$\begin{aligned}
\alpha_1(t) &= 0.95 - 0.25 \sin t, \\
\alpha_2(t) &= 0.75 - 0.25 \sin t, \\
\alpha_3(t) &= 0.55 - 0.25 \sin t.
\end{aligned}$$

The numerical solutions for the state variable $x(t)$ and the control variable $u(t)$ with some variable orders $\alpha(t)$ and $n = 3$ are compared in Fig. 3. The approximate values of the performance index J with $n = 3$ and different values of $\alpha(t)$ are reported in Table 1. Also, for various values of $\alpha(t), n$, Table 2 compares the absolute errors of $x(t)$ and $u(t)$.

Table 1: The approximate values of J at different choices of $\alpha(t)$ where $n = 3$ for Example 2.

$\alpha(t)$	J
$\alpha_1(t)$	3.33408×10^{-5}
$\alpha_2(t)$	4.92959×10^{-6}
$\alpha_3(t)$	2.64258×10^{-7}



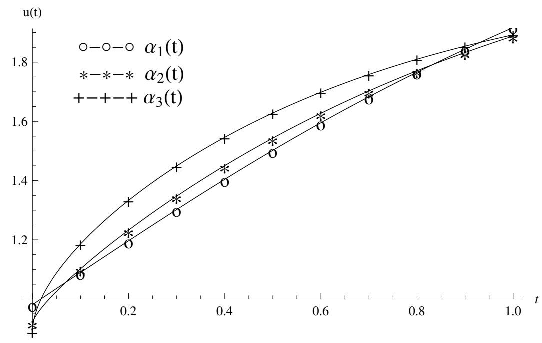


Figure 3: The state and control variables (left and right, respectively) with different values of $\alpha(t)$ and $n = 3$ for Example 2.

Table 2: The absolute errors $x(t)$ and $u(t)$ with different values of $\alpha(t), n$ for Example 2.

	e_x			e_u		
t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$
0.1	2.16193×10^{-4}	2.61456×10^{-4}	1.39036×10^{-4}	2.94412×10^{-3}	1.12052×10^{-3}	9.18998×10^{-5}
0.3	2.77463×10^{-3}	1.71445×10^{-3}	2.75113×10^{-4}	3.60358×10^{-3}	1.42273×10^{-3}	4.11697×10^{-5}
0.5	6.44952×10^{-3}	3.11097×10^{-3}	2.38837×10^{-5}	3.81326×10^{-3}	1.43818×10^{-3}	2.29627×10^{-4}
0.7	9.6158×10^{-3}	3.74622×10^{-3}	6.07569×10^{-4}	3.68555×10^{-3}	1.52123×10^{-3}	3.3346×10^{-4}
0.9	1.1687×10^{-2}	3.75459×10^{-3}	1.3259×10^{-3}	2.07634×10^{-3}	9.76752×10^{-4}	4.02544×10^{-4}

Example 3. *Finally, consider the following VO-FOCP:*

$$\min \quad \mathbb{J}[X] = \int_0^1 \left[e^t \left(x(t) - 1 + t - t^4 \right)^2 + (t^2+1) \left(u(t) + t^4 + 1 - t - \frac{24\,\Gamma(4-\alpha(t))}{\Gamma(5-\alpha(t))} \right)^2 \right] dt,$$

subject to the variable order fractional dynamical system

$${}_0^cD_t^{\alpha(t)} x(t) = x(t) + u(t), \quad 1 < \alpha(t) \leq 2,$$

and the initial conditions

$$x(0) = -x'(0) = 1.$$

In this problem, the minimizing solutions for the control and state variables are $u(t) = \frac{24\Gamma(4-\alpha(t))}{\Gamma(5-\alpha(t))} - t^4 + t - 1$ and $x(t) = 1 - t + t^4$. Now we solve the above problem by the presented technique with $n = 3$ for the following VOs

$$\alpha_1(t) = 1.95 - 0.01t^2, \qquad \alpha_4(t) = 1.5,$$

$$\alpha_2(t) = 1.95 + 0.05t^2, \qquad \alpha_5(t) = 1.5 + \frac{\cos^2(t)e^{t^2}}{60},$$

$$\alpha_3(t) = 1.95 + 0.15t^2, \qquad \alpha_6(t) = 1.5 - \frac{\cos(t)}{40}.$$

For the state variable $x(t)$ and control variable $u(t)$, the behavior of the numerical solutions are plotted in Figs. 4 and 5 for some different values of $\alpha(t)$. The approximate values of the performance index J with $n = 6$ and different values of $\alpha(t)$ are presented in Table 3. The absolute errors for $x(t)$ and $u(t)$ are listed in Table 4 and 5 respectively for the various $\alpha(t)$.

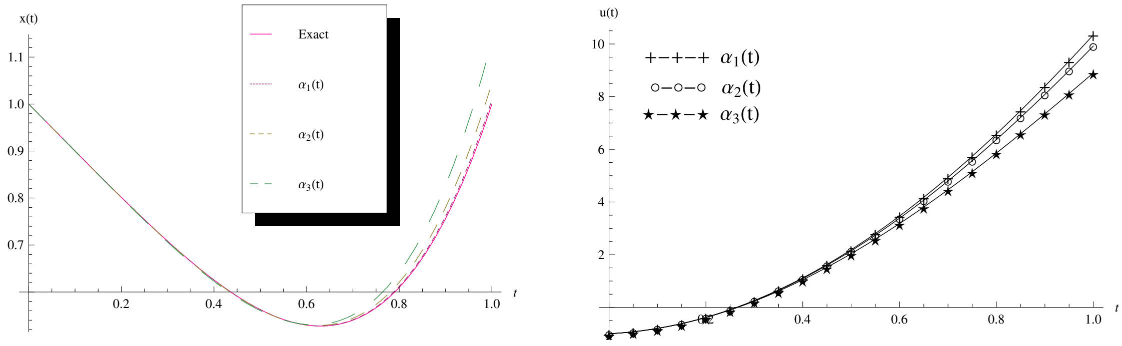


Figure 4: The result obtained of $x(t)$ and $u(t)$ (left and right, respectively) with various values of $\alpha(t)$ when $n = 3$ for Example 3.

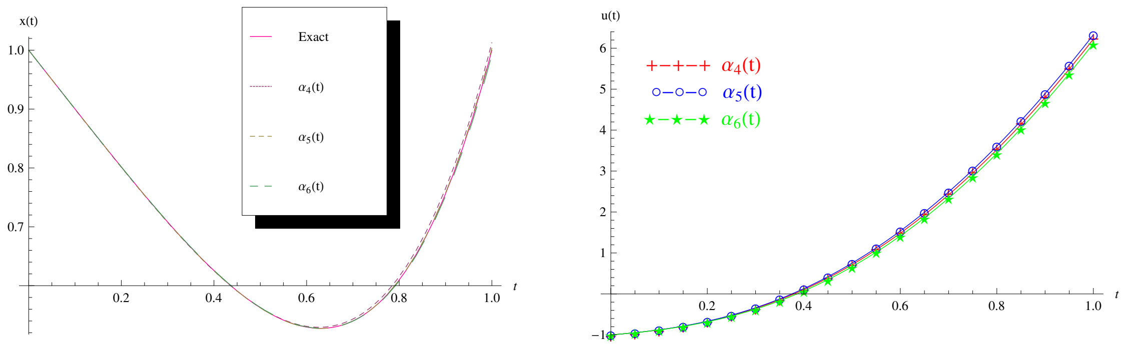


Figure 5: The result obtained of $x(t)$ and $u(t)$ (left and right, respectively) with various values of $\alpha(t)$ when $n = 6$ for Example 3.

Table 3: The approximate values of J at different choices of $\alpha(t)$ where $n = 6$ for Example 3.

$\alpha(t)$	J
$\alpha_1(t)$	1.78909×10^{-5}
$\alpha_2(t)$	4.51859×10^{-4}
$\alpha_3(t)$	4.15617×10^{-3}
$\alpha_4(t)$	1.38712×10^{-3}
$\alpha_5(t)$	1.86238×10^{-6}
$\alpha_6(t)$	3.32223×10^{-6}

Table 4: The absolute errors of $x(t)$ when $n = 6$ for Example 3.

t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\alpha_4(t)$	$\alpha_5(t)$	$\alpha_6(t)$
0.1	1.74459×10^{-5}	8.9255×10^{-5}	2.81288×10^{-4}	1.97715×10^{-4}	2.34129×10^{-5}	7.77156×10^{-5}
0.3	1.23087×10^{-4}	6.36534×10^{-4}	2.06031×10^{-3}	6.13259×10^{-4}	1.09893×10^{-4}	3.59669×10^{-4}
0.5	1.16221×10^{-4}	6.43910×10^{-4}	2.41061×10^{-3}	1.71158×10^{-4}	2.11404×10^{-4}	5.22689×10^{-4}
0.7	8.23214×10^{-4}	4.01755×10^{-3}	1.12622×10^{-2}	3.26721×10^{-3}	1.13484×10^{-4}	21398×10^{-4}
0.9	4.73353×10^{-3}	2.36688×10^{-3}	7.08222×10^{-2}	9.01625×10^{-3}	3.123243×10^{-3}	3.69785×10^{-3}

Table 5: The absolute errors of $u(t)$ with $n = 6$ for Example 3.

t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\alpha_4(t)$	$\alpha_5(t)$	$\alpha_6(t)$
0.1	2.72617×10^{-5}	1.37901×10^{-2}	4.3473×10^{-2}	6.58226×10^{-3}	1.00962×10^{-3}	2.74918×10^{-3}
0.3	1.83846×10^{-3}	9.38169×10^{-3}	3.02168×10^{2}	6.99660×10^{-3}	7.28313×10^{-4}	2.05295×10^{-3}
0.5	1.03098×10^{-3}	5.43587×10^{-3}	1.81322×10^{-2}	2.61349×10^{-2}	4.51044×10^{-4}	1.43165×10^{-3}
0.7	4.49031×10^{-4}	2.38228×10^{-3}	8.38128×10^{-3}	4.05599×10^{-2}	2.82732×10^{-4}	7.98673×10^{-4}
0.9	3.46879×10^{-5}	3.32401×10^{-4}	1.42755×10^{-3}	4.29872×10^{-2}	1.86392×10^{-5}	2.62964×10^{-4}

5. Conclusion

In this paper, we considered the solutions of VOFOCPs by using BPs. First, we constructed the operational matrix of the VO fractional integral based on BPs. Then using the mentioned matrix and Gauss-Legendre quadrature rule, the given equation has been transformed into a system of algebraic equations. With solving this system, we obtained the optimal solution of considered problems. Furthermore, some examples are presented to show the accuracy with this method.
For computations, the *Mathematica* was used in this article.

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