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Various optical solutions to the (1+1)-Telegraph equation with space-time conformable derivatives

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Abstract

This paper presents a new sub-equation method based on an auxiliary equation which is implemented via the well-known generalized Kudryashov method, to construct new traveling waves to the Telegraph equation with time and space conformable derivatives. To illustrate its effectiveness, it was tested for seeking traveling wave solutions to the (1+1)-Telegraph equation with space-time conformable derivatives. With the help of Maple Software we derive some new solitary waves solutions. It can be concluded that the proposed method is an accurate tool for solving several kind of nonlinear evolution equations.

Keywords: (1+1)-Telegraph equation; Generalized Kudryashov method; Conformable derivative; Auxiliary equation; Traveling wave; Optical solutions.
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1. Introduction

Nowadays, differential equations with conformable derivative order become powerful tool for modeling nonlinear phenomena that are encountered in many fields, such as Physics, Mechanics, Engineering, etc. Finding accurate method for solving such problems has been undertaken by many

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researchers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. to this end a variety of powerful methods have been presented such as sine-cosine method [1, 2], homotopy perturbation method [3, 4], tanh-sech method [5, 6], homogeneous balance method [7, 8], F-expansion method [9, 10], Exp-function method [11, 12, 13], (G'/G)-expansion method [14, 15, 16], modified Kudryashov method [17, 18, 19], generalized Kudryashov method [20, 21, 22], double auxiliary equations method [27, 28, 29, 30], and so on.
Our study focusses on presenting a novel technique to solve partial differential equations with time and space conformable derivatives by combining the generalized Kudryashov method to the auxiliary equation technique. We implement this novel method to seek traveling waves for the space-time non linear Telegraph equation with conformable derivatives. The remainder of this paper is organized as follows. In section 2, we recall some basic definitions of conformable derivative and some of its useful mathematical properties that will be used throughout the paper. Section 3, deals with the the description of the new method. In section 4, we construct solitary wave solutions for the space-time non linear Telegraph Equation. In section 5, the behavior of the wave solutions of space-time non linear Telegraph Equation are displayed graphically and discussed in detail. In section 6 concluding remarks are given.

2. Preliminaries

Following the works by Khalil et al.[31] and Abdeljawad et al. [32, 33, 34, 35, 36], the Conformable derivative of order α with respect to x is defined as the following

$$D_t^\alpha(f(t)) = lim_{\epsilon->0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}, \quad \text{for all} \quad t>0 \quad 0<\alpha\leq 1. \tag{2.1}$$

Here, we recall some useful properties of Conformable derivative:

$$D_t^\alpha f\left(g\left(t\right)\right) = f'_g[g\left(t\right)]D_t^\alpha(g\left(t\right), \tag{2.2}$$

$$D_t^\alpha\left(f\left(t\right)g\left(t\right)\right) = f\left(t\right)D_t^\alpha g\left(t\right) + g(t)D_t^\alpha f\left(t\right), \tag{2.3}$$

$$D_t^\alpha(t^u) = u.t^{u-\alpha}, \tag{2.4}$$

$$D_x^\alpha\left(c\right) = 0, \text{where } c \text{ is a constant}, \tag{2.5}$$

$$D_t^\alpha(f)\left(t\right) = t^{1-\alpha}.\frac{df}{dt}. \tag{2.6}$$

3. Description of the method

We present here the main steps of the proposed method. Consider a general nonlinear equation with conformable derivatives as the following

$$P(u, u_x, u_y, u_z, \ldots, D_t^\delta, D_x^\alpha, D_y^\beta, D_z^\gamma, \ldots) = 0. \tag{3.1}$$

Where u is a function of independent variables, $(x, y, z, \ldots, t)$, $D_t^\alpha, D_x^\alpha, D_y^\beta$ and D_z^α are the Conformable derivatives of u with respect to t, x, y and z and P is polynomial in u.
To seek the traveling wave solutions u of Eq. 3.1 explicitly, we will follow the next three steps:

- Step 1. Using the wave transformation

$$U(x,y,z,\ldots,t)=U(\eta),\eta=\frac{kx^{\alpha}}{\alpha}+\frac{ly^{\beta}}{\beta}+\frac{hz^{\gamma}}{\gamma}+\cdots-\frac{vt^{\delta}}{\delta}. \tag{3.2}$$

Where k,l,h and v are constants to be determined later, Eq.(3.2) converts Eq.(3.1) to an ordinary differential equation with new variable η

$$H(u,u',u'',u''',\ldots)=0 \tag{3.3}$$

where the prime stands for differentiation with respect to η.

- Step 2. Assume that the exact solution of Eq.(3.3) can be written as

$$u(\eta)=\frac{\sum_{i=0}^{N}\ a_i\left(\frac{h(\eta)}{g(\eta)}\right)^i}{\sum_{j=0}^{M}\ b_j\left(\frac{h(\eta)}{g(\eta)}\right)^j}. \tag{3.4}$$

Here $a_i\,(i=0,...,N)$, $b_j\,(j=0,...,M)$ are constants to be determined later, $(a_N,b_M)\neq(0,0)$, and $\left(\frac{h(\eta)}{g(\eta)}\right)$ is the solution of

$$\left(\frac{h(\eta)}{g(\eta)}\right)'=A\left(\frac{h(\eta)}{g(\eta)}\right)^2+B\left(\frac{h(\eta)}{g(\eta)}\right)+C, \tag{3.5}$$

where $g(\eta)=\exp(\eta)\,(h(\eta))'$. Eq. (3.5) has the following set of solutions

1. Family1 : When $\Delta=B^2-4AC>0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right)=\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\eta)-\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}. \tag{3.6}$$

k_1 is a constant.

2. Family 2 : When $\Delta=B^2-4AC<0$, then

$$\left(\frac{h(\eta)}{g(\eta)}\right)=\frac{-2C[1+\tan(\frac{\sqrt{-\Delta}}{2}\eta)\tan(\frac{\sqrt{-\Delta}}{2}k_1)]}{B+B\tan(\frac{\sqrt{-\Delta}}{2}\eta)\tan(\frac{\sqrt{-\Delta}}{2}k_1)+\sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\eta)-\tan(\frac{\sqrt{-\Delta}}{2}k_1)]}. \tag{3.7}$$

k_1 is a constant.

3. Family 3: When $\Delta=B^2-4AC=0, AC>0$, then

$$\left(\frac{h(\eta)}{g(\eta)}\right)=-\frac{C(\eta-k_1)}{\sqrt{AC}(\eta-k_1)-1}, \tag{3.8}$$

k_1 is a constant.

4. Family 4: When $C=0, B\neq0$, then

$$\left(\frac{h(\eta)}{g(\eta)}\right)=-\frac{B\exp(B\eta)}{A\exp B\eta+Bk_1}, \tag{3.9}$$

k_1 is a constant.

5. Family 5: When $B=0, C=0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right) = -\frac{1}{A\eta + k_1}, \tag{3.10}$$

k_1 is a constant.

- Step 3: To compute the positive integer number N and M in Eq. (3.4) we balance the highest order linear term and the highest order nonlinear term occurring in Eq.(3.3) This completes the determination of the value of N and M.

  Substituting Eq.(3.4) along with Eq.(3.5) into Eq. (3.3), we calculate all the necessary derivatives $u', u'', \ldots$ As a result of this substitution,we get then a polynomial of $\left(\frac{h(\eta)}{g(\eta)}\right)$, setting the coefficients of the same power of $\left(\frac{h(\eta)}{g(\eta)}\right)$ to zero, we obtain a system of algebraic equations.Solving this system we get the unknown parameters $A, B, C, a_i\,(i=0,\ldots,N)$, $b_j,(j=0,\ldots,M)$, $k, l, h, \ldots$ and v. We finally obtain the exact solutions of Eq.(3.3). Substituting these results into the solutions of Eq(3.5) and using Eq.(3.2) we get the exact solutions of Eq. (3.1).

4. Derivation of new traveling waves to the Telegraph equation

In this section we seek the exact solutions of the Telegraph Equation through above described method. Such equation with time-space conformable derivatives can be written as

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + D_t^{\alpha}u + \gamma u + \beta u^3 = 0. \tag{4.1}$$

Where α is a parameter describing the order of derivation in conformable sense. When $\alpha = 1$, Eq. (4.1) reduces to the nonlinear Telegraph equation. Using the complex transform

$$\eta = \frac{kx^{\alpha}}{\alpha} - \frac{ct^{\alpha}}{\alpha} \tag{4.2}$$

, Eq.(4.1) is reduced to the following ordinary differential equation

$$(c^2 - k^2)u'' - cu' + \gamma u + \beta u^3 = 0. \tag{4.3}$$

Balancing the order of u'' and u^3 in Eq. (4.3), we obtain $N=2$ and $M=1$. Then the solution can be expressed as

$$u(\eta) = \frac{a_0 + a_1\left(\frac{h(\eta)}{g(\eta)}\right) + a_2\left(\frac{h(\eta)}{g(\eta)}\right)^2}{b_0 + b_1\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{4.4}$$

Substituting Eq.(4.4) and its derivatives using Eq.(3.5) into Eq.(4.3), collecting the coefficients of each power of $\left(\frac{h(\eta)}{g(\eta)}\right)^i$, $(i=0,\ldots,N)$, $(i=0,\ldots,M)$,

and set them to zero, we obtain a system of algebraic equations. Solving this system of algebraic equations with the aid of Maple, we obtain the following sets:

- Set 1:

$$a_0=0, a_1=\pm\tfrac{\sqrt{2}b_1}{4}\sqrt{\tfrac{-2\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}, a_2=\pm\tfrac{\sqrt{2}b_1}{8C}\sqrt{\tfrac{-2\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}(B+\tfrac{\Delta}{\sqrt{\Delta}}), b_0=0,$$
$$k=\pm\sqrt{\tfrac{9\gamma^2-2\gamma}{4\Delta}}, c=\tfrac{3\gamma}{\sqrt{4\Delta}}, \Delta=B^2-4AC. \tag{4.5}$$

Substituting these result into Eq (4.4) we have :

$$u_1(\eta)=\pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C+(B+\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right) \tag{4.6}$$

where
$\eta=\pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha}$ The exact solution $u_1(\eta)$ exists under the constraint condition $\Delta>0$.

- Set 2:

$$a_0=\pm\tfrac{b_0}{2}\sqrt{(\tfrac{-\gamma}{\beta\Delta})}[B-\tfrac{\Delta}{\sqrt{\Delta}}], a_1=\pm\tfrac{1}{2}\sqrt{(\tfrac{-\gamma}{\beta\Delta})}[b_1B+2b_0A-\tfrac{\Delta b_1}{\sqrt{\Delta}}], a_2=\pm Ab_1\sqrt{\tfrac{\gamma}{-\beta\Delta}},$$
$$k=\pm\tfrac{1}{2}\sqrt{\tfrac{9\gamma^2-2\gamma}{\Delta}}, c=\tfrac{3\gamma}{2\sqrt{\Delta}}, \Delta=B^2-4AC. \tag{4.7}$$

substituting these results in Eq (4.4), we have :

$$u_2(\eta)=\pm\frac{1}{2}\sqrt{\frac{-\gamma}{\beta\Delta}}\left(B-\frac{\Delta}{\sqrt{\Delta}}+2A\frac{h(\eta)}{g(\eta)}\right). \tag{4.8}$$

where $\eta=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 3:

$$a_0=\pm\tfrac{Cb_1}{4A}\sqrt{\tfrac{A\gamma}{\beta C}}, a_1=\pm\tfrac{b_1}{4A}\tfrac{\sqrt{\frac{\gamma A}{C\beta}}}{\frac{1}{\sqrt{-4AC}}}, a_2=\mp\tfrac{b_1}{4}\sqrt{\tfrac{A\gamma}{C\beta}}, b_0=0, B=0,$$
$$k=\pm\tfrac{1}{4}\sqrt{\tfrac{9\gamma^2-2\gamma}{-4AC}}, c=\tfrac{3\gamma}{4\sqrt{-4AC}}. \tag{4.9}$$

Substituting these results in Eq (4.4), we have :

$$u_3(\eta)=\mp\frac{1}{4}\sqrt{\frac{A\gamma}{\beta C}}\frac{\left(-\frac{C}{\sqrt{\Delta}}-\frac{h(\eta)}{g(\eta)}+\frac{A}{\sqrt{\Delta}}\left(\frac{h(\eta)}{g(\eta)}\right)^2\right)}{\frac{A}{\sqrt{\Delta}}\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{4.10}$$

Where $\eta=\pm\frac{1}{4}\sqrt{\frac{9\gamma^2-2\gamma}{-4AC}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{4\sqrt{-4AC}}\frac{t^\alpha}{\alpha}$.

- Set 4:

$$a_0 = \pm b_0 \frac{\sqrt{(\frac{-\gamma}{\beta\Delta})}}{4B}[B^2 - 2\frac{\Delta B}{\sqrt{\Delta}} + \Delta], a_1 = \pm \frac{b_0 A\sqrt{(\frac{-\gamma}{\beta\Delta})}[\frac{-\Delta}{\sqrt{\Delta}} + B]}{B},$$

$$a_2 = \pm \frac{b_0 A^2}{B}\sqrt{(\frac{-\gamma}{\beta\Delta})}, b_1 = \frac{2b_0 A}{B}, \Delta = B^2 - 4AC, \quad (4.11)$$

$$k = \pm\sqrt{\frac{9\gamma^2 - 2\gamma}{16\Delta}}, c = \frac{3\gamma}{\sqrt{16\Delta}}.$$

Substituting these results in Eq (4.4), we have :

$$u_4(\eta) = \pm\frac{1}{4}\sqrt{(\frac{-\gamma}{\beta\Delta})}\left(\frac{[B^2 - 2\frac{B\Delta}{\sqrt{\Delta}} + \Delta] + A[4B - 4\frac{\Delta}{\sqrt{\Delta}}]\left(\frac{h(\eta)}{g(\eta)}\right) + 4A^2\left(\frac{h(\eta)}{g(\eta)}\right)^2}{B + 2A\left(\frac{h(\eta)}{g(\eta)}\right)}\right) \quad (4.12)$$

where $\eta = \pm\sqrt{\frac{9\gamma^2 - 2\gamma}{16\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{16\Delta}}\frac{t^\alpha}{\alpha}$.

- Set5 :

$$a_0 = \pm b_0\sqrt{\frac{-\gamma}{\beta}}, a_1 = 0, a_2 = 0, b_1 = \frac{2Bb_0}{C}, A = 0,$$

$$k = \pm\frac{\sqrt{9\gamma^2 - 2\gamma}}{2B}, c = \frac{-3\gamma}{2B}, BC \neq 0 \quad (4.13)$$

Substituting these result into Eq (4.4) we have :

$$u_5(\eta) = \frac{\pm\sqrt{\frac{-\gamma}{\beta}}}{1 + \frac{2B}{C}\left(\frac{h(\eta)}{g(\eta)}\right)} \quad (4.14)$$

where $\eta = \pm\frac{\sqrt{9\gamma^2 - 2\gamma}}{2B}\frac{x^\alpha}{\alpha} + \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 6:

$$a_0 = \pm b_0\sqrt{\frac{-\gamma}{B}}, a_1 = \pm\frac{Bb_0}{C}\sqrt{\frac{-\gamma}{B}}, , a_2 = 0, b_1 = \frac{-b_0 B}{C}, A = 0,$$

$$k = \pm\frac{\sqrt{9\gamma^2 - 2\gamma}}{2B}, c = \frac{3\gamma}{2B}. \quad (4.15)$$

substituting these result into Eq (4.4) we have :

$$u_6(\eta) = \pm\sqrt{\frac{-\gamma}{\beta}}\left(\frac{C + B\frac{h(\eta)}{g(\eta)}}{-C + B\frac{h(\eta)}{g(\eta)}}\right) \quad (4.16)$$

where $\eta = \pm\frac{\sqrt{9\gamma^2 - 2\gamma}}{2B}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 7:

$$\begin{aligned}&a_0=\pm\tfrac{\sqrt{2}b_0}{2(B^2-\Delta)}\left(\sqrt{\tfrac{-2\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}(B-\tfrac{\Delta}{\sqrt{\Delta}})\right),a_1=\pm\tfrac{\sqrt{2}b_0}{4C}\left(\sqrt{\tfrac{-2\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}\right)\\&a_2=0,b_1=\tfrac{b_0}{2C}\left(B-3\tfrac{\Delta}{\sqrt{\Delta}}\right)\\&k=\pm\tfrac{1}{2}\sqrt{\tfrac{9\gamma^2-2\gamma}{\Delta}},c=\tfrac{3\gamma}{2\sqrt{\Delta}},\Delta=B^2-4AC.\end{aligned}\tag{4.17}$$

Substituting these result into Eq (4.4) we have :

$$u_7(\eta)=\pm\sqrt{\frac{-\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}\left(\frac{2BC-2\frac{\Delta C}{\sqrt{\Delta}}+(-\Delta+B^2)\left(\frac{h(\eta)}{g(\eta)}\right)}{(B^2-\Delta)\left(2C+(B-3\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right)}\right)\tag{4.18}$$

where $\eta=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 8:

$$\begin{aligned}&a_0=\pm b_0\sqrt{\tfrac{-\gamma}{\beta}},a_1=\pm\tfrac{Ab_0}{B}\sqrt{\tfrac{-\gamma}{\beta}},a_2=0,C=0,\\&k=\pm\tfrac{1}{2}\sqrt{\tfrac{9\gamma^2-2\gamma}{B^2}},c=\tfrac{-3\gamma}{2B}.\end{aligned}\tag{4.19}$$

substituting these results in Eq (4.4), we have :

$$u_8(\eta)=\pm b_0\sqrt{\frac{-\gamma}{\beta}}\left(\frac{1+\frac{A}{B}\left(\frac{h(\eta)}{g(\eta)}\right)}{b_0+b_1\left(\frac{h(\eta)}{g(\eta)}\right)}\right)\tag{4.20}$$

where $\eta=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{B^2}}\frac{x^\alpha}{\alpha}+\frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 9:

$$\begin{aligned}&a_0=\pm\tfrac{Cb_1}{B}\sqrt{\tfrac{-\gamma}{\beta}},a_1=0,,a_2=0,b_0=0,A=0,\\&k=\pm\tfrac{\sqrt{9\gamma^2-2\gamma}}{2B},c=\tfrac{-3\gamma}{2B}.\end{aligned}\tag{4.21}$$

Substituting these results in Eq (4.4), we have

$$u_9(\eta)=\frac{\pm\frac{C}{B}\sqrt{\frac{-\gamma}{\beta}}}{\left(\frac{h(\eta)}{g(\eta)}\right)}.\tag{4.22}$$

where $\eta=\pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha}+\frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 10:

$$a_0 = \pm\frac{1}{2}\sqrt{-\frac{2\gamma[2C^2b_1^2+2b_0b_1C\frac{\Delta}{\sqrt{\Delta}}+b_0^2\Delta+2b_0^2CA-2BCb_0b_1-Bb_0^2\frac{\Delta}{\sqrt{\Delta}}]}{\beta\Delta}},$$

$$a_1 = \pm\sqrt{2}\sqrt{-\frac{\gamma(2C^2b_1^2+2\frac{b_0b_1C\Delta}{\sqrt{\Delta}}+b_0^2\Delta+2b_0^2CA-2b_0b_1BC-\frac{B}{\sqrt{\Delta}}b_0^2\Delta)}{\beta\Delta}}\frac{(B+\frac{\Delta}{\sqrt{\Delta}})}{4C}, a_2 = 0, \tag{4.23}$$

$$k = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}, c = \frac{3\gamma}{2\sqrt{\Delta}}, \Delta = B^2 - 4AC.$$

Substituting these results in Eq (4.4), we have :

$$u_{10}(\eta) = \pm\frac{\left(2C + (B+\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right)}{4C(b_0+b_1\left(\frac{h(\eta)}{g(\eta)}\right))}\sqrt{\frac{-\gamma([B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}}]b_0^2+4C^2b_1^2+4b_0b_1(\frac{C\Delta}{\sqrt{\Delta}}-BC))}{\beta\Delta}} \tag{4.24}$$

where $\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha}$

- Set 11:

$$a_0 = \frac{Ca_1}{B}, a_2 = 0, b_0 = 0, b_1 = \pm a_1\sqrt{\frac{-\beta}{\gamma}}, A = 0,$$

$$k = \pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}, c = \frac{3\gamma}{2B}. \tag{4.25}$$

Substituting these results in Eq (4.4), we have :

$$u_{11}(\eta) = \pm\frac{\frac{C}{B}+\left(\frac{h(\eta)}{g(\eta)}\right)}{\sqrt{\frac{-\beta}{\gamma}}\left(\frac{h(\eta)}{g(\eta)}\right)} \tag{4.26}$$

where $\eta = \pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 12:

$$a_0 = \pm\sqrt{\frac{-\gamma[(\Delta+2CA-B\frac{\Delta}{\sqrt{\Delta}})b_0^2+2Cb_0b_1(\frac{\Delta}{\sqrt{\Delta}}-B)+2C^2b_1^2]}{2\beta\Delta}},$$

$$a_1 = \pm\sqrt{\frac{-2\gamma[(\Delta+2CA-B\frac{\Delta}{\sqrt{\Delta}})b_0^2+2Cb_0b_1(\frac{\Delta}{\sqrt{\Delta}}-2B)+2C^2b_1^2]}{\beta\Delta}}\frac{(B+\frac{\Delta}{\sqrt{\Delta}})}{4C},, a_2 = 0, \tag{4.27}$$

$$k = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}, c = \frac{3\gamma}{2\sqrt{\Delta}}.$$

Substituting these results in Eq (4.4), we have :

$$u_{12}(\eta) = \pm\sqrt{\frac{-\gamma[(\Delta+B^2-2B\frac{\Delta}{\sqrt{\Delta}})b_0^2+4b_0b_1C(\frac{\Delta}{\sqrt{\Delta}}-B)+4C^2b_1^2]}{\beta\Delta}}\left(\frac{2C+(B+\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)}{4C(b_0+b_1\left(\frac{h(\eta)}{g(\eta)}\right))}\right). \tag{4.28}$$

where $\eta = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 13:

$$a_0 = 0, a_2 = 0, b_0 = \frac{B}{A\gamma}(b_1\gamma + a_1\sqrt{-\beta\gamma}), C = 0,$$
$$k = \pm\frac{1}{2B}\sqrt{9\gamma^2 - 2\gamma}, c = \frac{3\gamma}{2B}. \tag{4.29}$$

Substituting these results in Eq (4.4), we have

$$u_{13}(\eta) = \frac{a_1\left(\frac{h(\eta)}{g(\eta)}\right)}{\frac{B}{A\gamma}(b_1\gamma + a_1\sqrt{-\beta\gamma}) + b_1\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{4.30}$$

Where $\eta = \pm\frac{1}{2B}\sqrt{9\gamma^2 - 2\gamma}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

In particular, the new exact solution of the Telegraph Equation with conformable space-time derivatives (4.1) with the help of Eq. (3.6) to Eq. (3.10) as follows:

When $C = 0, B \neq 0$,

$$u_{1.1}(\eta) = \pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C + (B + \frac{\Delta}{\sqrt{\Delta}}) - \frac{B\exp(B\eta)}{A\exp B\eta + Bk_1}\right),$$
$$\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha} \tag{4.31}$$

The solution $u_{1,1}(\eta)$ exists under the constraint condition $\Delta > 0$.
When $\Delta = B^2 - 4AC > 0$,

$$u_{1.2}(\eta) = \pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C + (B + \frac{\Delta}{\sqrt{\Delta}})\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\eta)-\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}\right),$$

$$\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha} \tag{4.32}$$

The other solutions can be derived in an analogous way.

5. Graphical illustrations and discussion

For the sake of briefness, we present here the profiles of some obtained solutions. The figures has been plotted by the Matlab software. In Figs.1-5, the 3D profiles and the contour plots of those solutions are given for some selected parameters satisfying the above mentioned calculations for each case. We underline that the resulting solutions were substituted in the studied equation to verify their correctness of the method. All computations were done using Maple17 software.

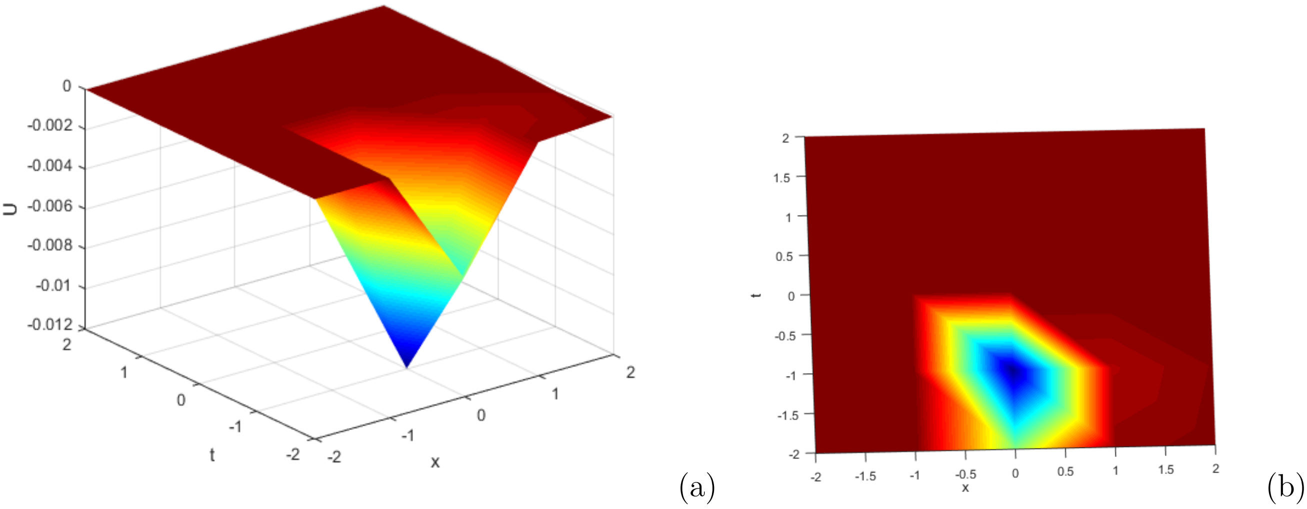

Figure 1: (a)Profile of the solution $u_1(x, y)$ for $A = 2$, $B = 5$, $C = 2$, $\alpha = 0.5$, $\beta = -1$, $\gamma = 1$ and $w_1 = 1$. (b) Contour plots corresponding to $u_1(x, y)$.

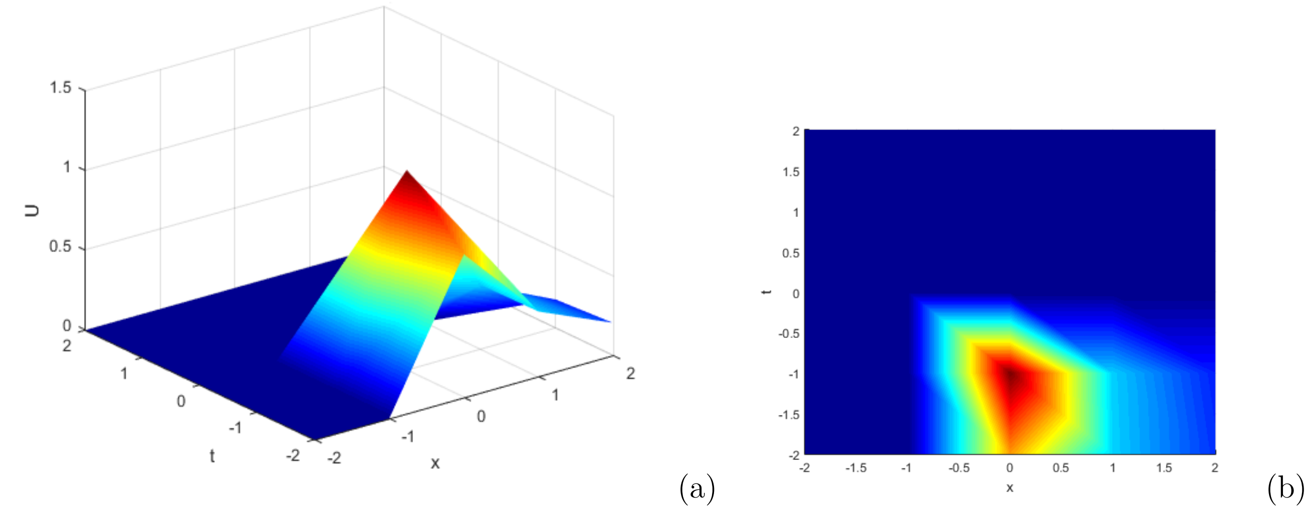

Figure 2: Profile of the solution $u_2(x, y)$ for $A = -2; B = -1; C = 5; \alpha = \frac{1}{2}; \beta = -\frac{1}{3}; \gamma = \frac{1}{2}$ and $w_1 = 0$. (b) Contour plots corresponding to $u_2(x, y)$.

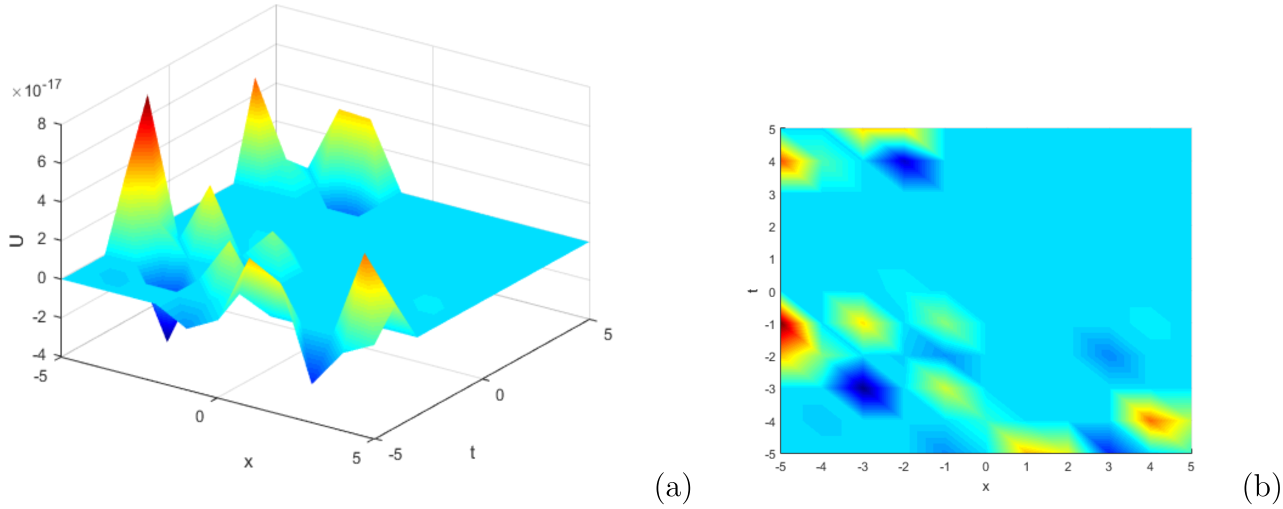

Figure 3: Profile of the solution $u_3(x,y)$ for $A=1$, $B=0$, $C=-8$, $\alpha=\frac{1}{3}$, $\beta=-2$, $\gamma=3$ and $w_1=8$. (b) Contour plots corresponding to $u_3(x,y)$.

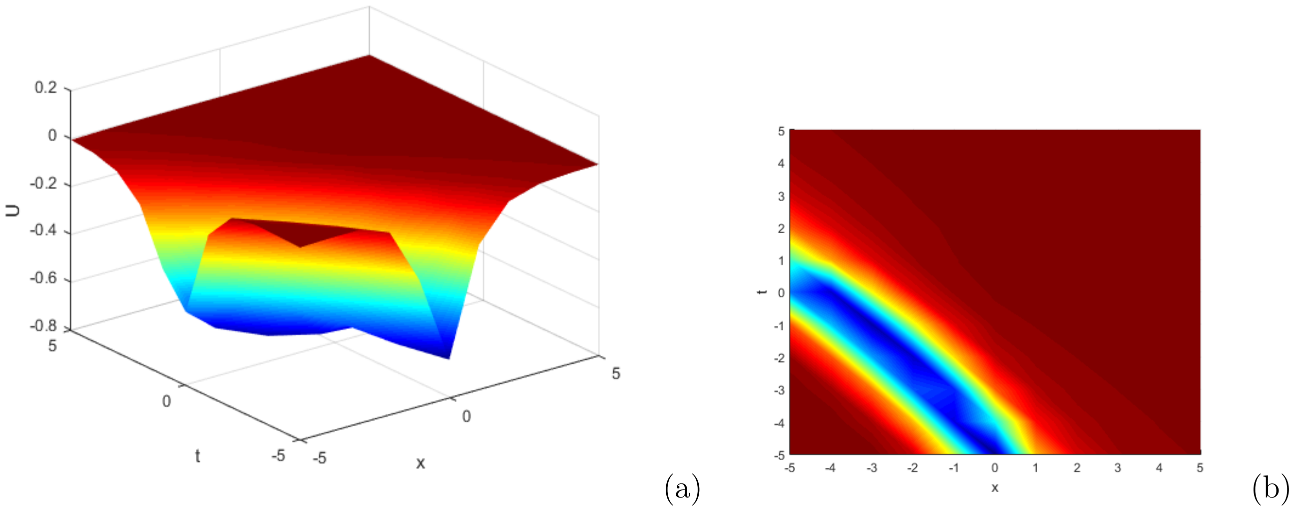

Figure 4: Profile of the solution $u_4(x,y)$ for $A=3$, $B=4$, $C=1$, $\alpha=0.9$, $\beta=0.25$, $\gamma=1$ and $w_1=1$. (b) Contour plots corresponding to $u_4(x,y)$.

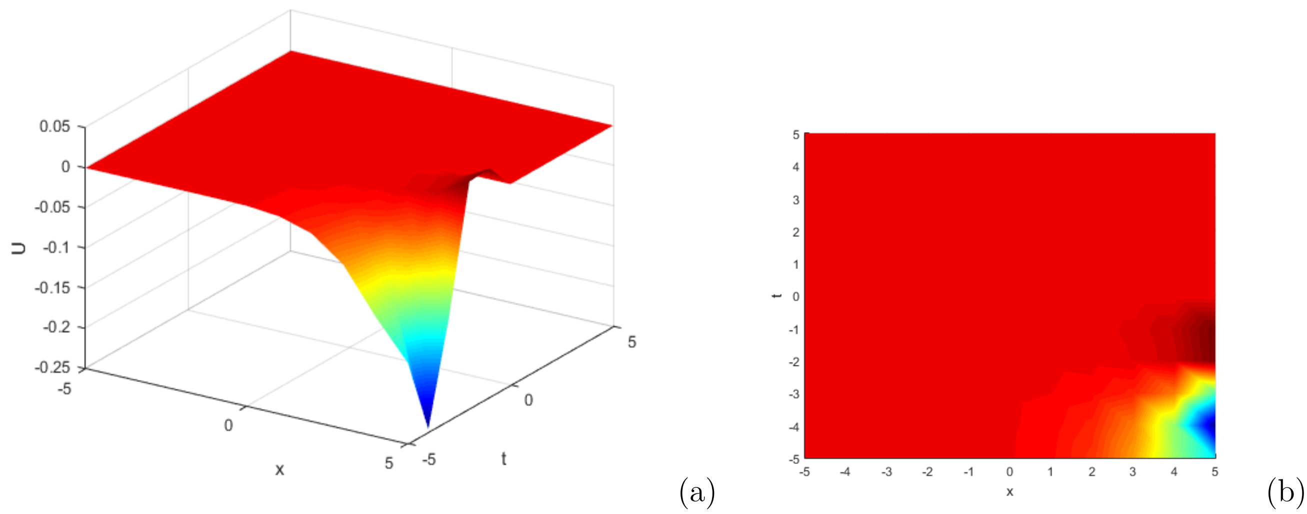

Figure 5: Profile of the solution $u_5(x, y)$ for $A = 0$, $B = -2$, $C = 3$, $\alpha = 0.9$, $\beta = 3$, $\gamma = -1$ and $w_1 = 5$. (b) Contour plots corresponding to $u_5(x, y)$.

The others solutions are also hyperbolic, trigonometric, exponential or rational solutions. They have similar profiles to the formers.

6. Conclusion

In the present paper, a new method was proposed using the auxiliary equation and the general Kudryashov method. Its accuracy has been tested by applying it successfully to the Telegraph equation with time-space conformable derivatives. It can be seen through this study, that this method is a powerful mathematical technique for finding traveling wave solutions for the partial differential equations. This method could be applied to the nonlinear evolution equations which arising in many fields of science. We derived various solitary waves for the (1+1)-Telegraph equation with space-time conformable derivatives, namely : hyperbolic, trigonometric, exponential and rational solutions. Some of the results have already been reported in the literature and some of them are new. The outcomes of this work would be beneficial to understand the behaviors of wave propagation in nonlinear science with beta derivatives or M-derivatives.

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