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Quotient Spaces on Quasilinear Spaces

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Abstract

Trace of quotient spaces is usually seen wherever there is a study of a linear structure. In linear spaces, we use subspaces and their corresponding equivalence relation to define quotient spaces. With the same method, in this paper, we present two generalized structures of quotient space that are defined on quasilinear spaces. One of them is a quasilinear space and the other is a linear space. After that, we try to introduce norms on certain states of these spaces and examin some properties of them. We will also provide examples for better understanding throughout the process.

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1. Introduction

Aseev generalized the concept of linear spaces and introduced the notion of quasilinear spaces [2]. This was the beginning of an Extensive research that has continued to this day. We can refer to recent efforts to present the counterpart of classical Riesz lemma and an analogue of Hahn-Banach theorem in quasilinear spaces. For more details, see [7, 4].

By examining the results of linear spaces on quasilinear spaces and quasi-algebras [9], as well as introducing the generalized space of C^*-algebras, this path can be continued. For example, one of the topics that has attracted the attention of many mathematicians for decades is the issue of stability. The stability problems of several functional equations have been investigated by a number of authors. The interested reader can use [3, 10, 11, 12] to learn more.

One of the problems that arises is whether the question of Ulam [15] concerning the stability of group homomorphisms can be generalized to quasi-homomorphisms? As another idea, can significant

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results be achieved by changing the definition of a fuzzy normed vector space [1] and adding the following condition to it?

$N(x,t) \leq N(y,s)$ if $x \leq y$ and $t \leq s$.

However, it would not be easy to fulfill this goal. As we will see in this paper, defining quotient spaces requires initiative and new tools.

In linear spaces, a subspace and its transfers make a quotient space [13]. Our idea of defining quotient spaces on quasilinear spaces is the same. equivalence relations and equivalence classes will also be helpful here.

Consider the space of all nonempty convex closed bounded subsets of $\mathbb{R}$ denoted by $\Omega_c(\mathbb{R})$. Geometrically, $[-a+r, a+r]$ is $[-a, a]$ that has been displaced. So we can compress the whole space into all symmetric closed intervals around Zero. This unification of all transfers could occur in the space of all closed circles in $\mathbb{R}^2$, all closed cubes in $\mathbb{R}^3$, and etc. Closed intervals on $\mathbb{R}$ is an example of quasilinear spaces. According to the definition given in [16], $\mathbb{R}$ is a subspace of $\Omega_c(\mathbb{R})$. This example shows that the idea of constructing cosets in linear spaces can also be used in quasilinear spaces, so that $[-a, a] + \mathbb{R}$ will be a member of a quotient space. In section 3 we make this space in general way, and obtain some examples, theorems and results related to this new space.

But our attempt to build a new space from a quasilinear space does not end here. Suppose the space of all nonempty closed bounded subsets of a linear space E denoted by $\Omega(E)$. it is another example of a quasilinear space [2]. So, we see that a linear space get us a quasilinear space. But, is there a way to build a linear space from a quasilinear space?

The answer is yes. The space of all invertible elements of a quasilinear space is an obvious linear space that can be extracted from this space. Also, in section 4, we offer a way to build a non-trivial linear space from a quasilinear space by using the subtracting an element from itself.

2. QUASILINEAR SPACES

We suggest that the reader refer to [2] for a preliminary introduction. Only the necessary definitions and theorems will be stated here.

Definition 2.1. *A set X is said to be a quasilinear space if a partial order relation "$\leq$", an algebraic sum operation "$+$", and an operation of multiplication by real numbers "$\cdot$" are defined on it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:*

1. $x + y = y + x$;
2. $x + (y + z) = (x + y) + z$;
3. *There exists an element $0 \in X$, called Zero of X such that $x + 0 = x$;*
4. $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;
5. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;
6. $1 \cdot x = x$;
7. $0 \cdot x = 0$;
8. $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$;
9. $x + z \leq y + v$ *if* $x \leq y$ *and* $z \leq v$;
10. $\alpha \cdot x \leq \alpha \cdot y$ *if* $x \leq y$.

Lemma 2.2. *[2] In a quasilinear space X, Zero is minimal. In other words, $x = 0$ if $x \leq 0$.*

An element $x' \in X$ is called an additive-inverse of $x \in X$ if $x + x' = 0$. For simplicity, from now on, we use the word "inverse" instead of "additive-inverse". The inverse of x is unique, If it exists.

If $x \in X$ has an inverse, we call it regular, otherwise it is called singular. X_r and X_s denote the sets of all regular and singular elements in X, respectively [16, 6].

Lemma 2.3. *[2] Suppose that any element x in the quasilinear space X is regular. Then the partial order in X is determined by equality, and consequently, X is a linear space.*

Corollary 2.4. *[2] In a real linear space, equality is the only way to define a partial order such that conditions 1-10 hold.*

Similar to vector spaces, it will be assumed that $-x = (-1) \cdot x$ and $x - y$ means $x + (-y)$. However, here $-x$ is not necessarily the inverse of x.

Lemma 2.5. *[16] In a quasilinear space X, $x \in X_r$ if and only if $x - x = 0$.*

The following lemma shows that being minimal is not limited to Zero.

Lemma 2.6. *[16] In a quasilinear space every regular element is minimal.*

Definition 2.7. *Let X be a quasilinear space and $Y \subseteq X$. Y is called a subspace of X whenever Y is a quasilinear space with the same partial ordering and the same operations on X.*

Example 2.8. *X_r and $X_s \cup \{0\}$ are subspaces of X.*

Theorem 2.9. *[16] Y is a subspace of a quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \cdot x + \beta \cdot y \in Y$.*

Definition 2.10. *Suppose that X is a quasilinear space and $Y \subseteq X$. Y is a linear subspace of X whenever Y is a linear space with the same operations on X.*

Example 2.11. *X_r is a linear subspace of X.*

It is clear that a linear subspace of X is a subspace of X_r and vice versa.

3. QUASILINEAR QUOTIENT SPACES

Let N be a linear subspace of a quasilinear space X. So $n \in X_r$ for any $n \in N$. Let R be a relation on X such that

$$xRy \leftrightarrow x \in y + N \tag{3.1}$$

when $y + N = \{y + n : n \in N\}$. Then R is an equivalence relation on X.

$x = x + 0$ means xRx, and if $x = y + n$ thus $y = x - n$. Also, if xRy and yRz, then

$$x = y + n_1 \tag{3.2}$$

and

$$y = z + n_2 \tag{3.3}$$

for some $n_1, n_2 \in N$. Thus $x = y + n_1 = z + n_2 + n_1$. because $n_1 + n_2 \in N$ therefore xRz.

For every $x \in X$, the equivalence class of x is defined as follows

$$[x] = \{y : yRx\}. \tag{3.4}$$

Definition 3.1. *Let X be a quasilinear space, $s \in X$ and $S \subseteq X$. s is called a l-member of S and it is denoted by $s \in_{\leq} S$ if there exists $r \in S$ such that $s \leq r$. "l" is taken from the word "less". Likewise, t is called a g-member of S and it is denoted by $t \in_{\geq} S$ if there exists $r \in S$ such that $r \leq t$. "g" is taken from the word "greater".*

In the following, we will see how this concept can help us a lot. In [9], we have proposed to use this concept to define ideals in quasi-algebras.

Definition 3.2. *Let X be a quasilinear space and $y \in X$. y is a non-negative element of X if $y = 0$ or*

$$x + y \nleq x \tag{3.5}$$

for every $x \in X$. Also, $z \in X$ is a non-positive element of X if $z = 0$ or

$$x \nleq x + z \tag{3.6}$$

for every $x \in X$. $Y \subseteq X$ is a non-negative (non-positive) set if any element of it is non-negative (non-positive).

Example 3.3. $\mathbb{R}$ *is a non-positive linear subspace of* $\Omega(\mathbb{R})$.

Obviously, a non-negative linear subspace is also non-positive and vice versa. If $y \neq 0$ is a non-negative element of X then $y \nleq 0$.

Lemma 3.4. *Let X be a quasilinear space and N be a non-negative linear subspace of X. If $x \in_{\leq} y + N$ and $y \in_{\leq} x + N$. Then $x \in y + N$.*

Proof . There exist $r, s \in N$ such that $x \leq y + r$ and $y \leq x + s$. So $x \leq x + s + r$. Since N is a non-negative linear subspace, then $s + r = 0$. Thus $y + r = y - s \leq x$ and hence $x = y + r$. □

If N is a non-negative linear subspace of a quasilinear space X, then $X/N = \{[x] : x \in X\}$ is a quasilinear space, called the quasilinear quotient space of X modulo N. In this spac partial order, addition and multiplication by real numbers are defined by

1. $[x] \leq [y]$ if and only if $x \in_{\leq} y + N$;
2. $[x] + [y] = [x + y]$;
3. $\alpha \cdot [x] = [\alpha \cdot x]$.

Since N is a linear space, these operations are well defined. This means that if $[x] = [x']$ and $[y] = [y']$, thus

$$x = x' + n \tag{3.7}$$

and

$$y = y' + m \tag{3.8}$$

for some $n, m \in N$. If $[x] \leq [y]$, there exists $r \in N$ such that $x \leq y + r$. So

$$x' = x - n \leq y + r - n = y' + m + r - n \in y' + N, \tag{3.9}$$

and hence $[x'] \leq [y']$. Also, $x + y = x' + y' + n + m$, thus $(x + y)R(x' + y')$, and so

$$[x'] + [y'] = [x' + y'] = \{z : zR(x' + y')\} = [x + y] = [x] + [y]. \tag{3.10}$$

The part of scalar multiplication is proved in a similar way. Proofs of quasilinear properties are easy. We check some of them. If $[x] \leq [y]$ and $[y] \leq [x]$, there exist $r, s \in N$ such that $x \leq y + r$ and $y \leq x + s$. So $[x] = [y]$ by lemma 3.4. Also, since $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta x$, then

$$(\alpha + \beta) \cdot [x] = [(\alpha + \beta) \cdot x] \leq [\alpha \cdot x + \beta \cdot x] = [\alpha \cdot x] + [\beta \cdot x] = \alpha \cdot [x] + \beta \cdot [x]. \tag{3.11}$$

For the last proof, let $[x] \leq [y]$ and $[z] \leq [v]$. Then there exist $r, s \in N$ such that $x \leq y + r$ and $z \leq v + s$. So

$$x + z \leq y + v + r + s. \tag{3.12}$$

Hence $[x] + [z] = [x + z] \leq [y + v] = [y] + [v]$. Since $[x] + [0] = [x]$, then the origin of X/N is $[0] = N$.

Example 3.5. *Let E be a real normed linear space, then the quotient space $\Omega(E)/E$ is a quasilinear space. The equivalence class $[x]$, contains x and all transfers of x.*

3.1. NORMED QUOTIENT SPACES

In preliminary functional analysis, the quotient space inherits a norm from the main space. A normed quasilinear space can make such a contribution to its quotient space, too.

Definition 3.6. *Let X be a quasilinear space. A real function $\| \cdot \|_X : X \to \mathbb{R}$ is called a norm if the following conditions hold:*

1. $\|x\|_X > 0$ *if* $x \neq 0$;
2. $\|x + y\|_X \leq \|x\|_X + \|y\|_X$;
3. $\|\alpha \cdot x\|_X = |\alpha| \|x\|_X$;
4. *If $x \leq y$, then $\|x\|_X \leq \|y\|_X$;*
5. *If for any $\epsilon > 0$ there exists an element $x_\epsilon \in X$ such that $x \leq y + x_\epsilon$ and $\|x_\epsilon\|_X \leq \epsilon$, then $x \leq y$.*

A normed quasilinear space is a pair $(X; \| \cdot \|_X)$, where X is a non-zero quasilinear space and $\| \cdot \|_X$ is a given norm on it.

Let X be a normed quasilinear space. the Hausdorff metric on X is defined by

$$h_X(x, y) = \inf\{r \geq 0 : \exists a_1^r, a_2^r \in X : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\|_X \leq r\}. \tag{3.13}$$

Since $x \leq y + (x - y)$ and $y \leq x + (y - x)$, the equality $h_X(x, y)$ is well-defined and $h_X(x, y) \leq \|x - y\|_X$.

Theorem 3.7. *Let X be a normed quasilinear space. Let N be a non-negative closed linear subspace of X such that the following condition is satisfied for any $x, y \in X$:*
If $x + m \not\leq y$ for any $m \in N$, then there exists $\epsilon > 0$ such that $x + n \not\leq y + a$ for any $n \in N$ and $a \in \{z \in X : \|z\|_X < \epsilon\}$.
Then X/N is a normed quasilinear space.

Proof . Define $\| \cdot \|_{X/N} : X/N \to \mathbb{R}$ by

$$\|[x]\|_{X/N} = \inf\{\|x + n\|_X : n \in N\}. \tag{3.14}$$

If $\|[x]\|_{X/N} = 0$, then for any $\epsilon > 0$, there exists $n_\epsilon \in N$ such that

$$\|x + n_\epsilon\|_X < \epsilon. \tag{3.15}$$

Since $h_X(x, -n_\epsilon) \leq \|x + n_\epsilon\|_X$, thus there exists a sequence of elements of N that converges to x. Because N is closed in X hence $x \in N$ and so $[x] = N = 0_{X/N}$.

Proofs of norm conditions 2 and 3, are similar to linear spaces. See [8, 13]. Moreover, if $[x] \leq [y]$, then $x \leq y + r$ for some $r \in N$. For any $y + n$, since $x - r + n \leq y + n$ and $\|x - r + n\| \leq \|y + n\|$ Thus

$$\inf\{\|x + n\|_X : n \in N\} \leq \inf\{\|y + n\|_X : n \in N\}. \tag{3.16}$$

Finally, we examin the last property of quasilinear norm. Suppose that for any $\epsilon > 0$, there exists an alement $[c_\epsilon]$ such that $[x] \leq [y] + [c_\epsilon]$ and $\|[c_\epsilon]\|_{X/N} \leq \epsilon$. Because $[x] \leq [y + c_\epsilon]$, then

$$x \leq y + c_\epsilon + n_\epsilon \tag{3.17}$$

for some $n_\epsilon \in N$, and Since $\inf\{\|c_\epsilon + n\|_X : n \in N\} \leq \epsilon$, so there exists $n'_\epsilon \in N$ such that

$$\|c_\epsilon + n'_\epsilon\|_X \leq \epsilon. \tag{3.18}$$

Let $n = n'_\epsilon - n_\epsilon$ and $a = c_\epsilon + n'_\epsilon$ therefore $x + n \leq y + a$ concludes that $x \leq y + m$ for some $m \in N$. Hence $[x] \leq [y]$. □

Example 3.8. *$\Omega_c(\mathbb{R})/\mathbb{R}$ is a normed quasilinear space.*

According to Theorem 3.7 define $\|\cdot\| : \Omega_c(\mathbb{R})/\mathbb{R} \to \mathbb{R}$ by

$$\|[x]\|_{\Omega_c(\mathbb{R})/\mathbb{R}} = \inf\{\|x + r\|_\Omega : r \in \mathbb{R}\}. \tag{3.19}$$

In fact, $\|[x]\|_{\Omega_c(\mathbb{R})/\mathbb{R}} = \|y\|_\Omega$ when $y \in [x]$ is a symmetric interval around 0. Here, we resolve validity of the last norm condition in a different way. Suppose that for any $\epsilon > 0$, there exists an element $[c_\epsilon]$ such that $[x] \leq [y] + [c_\epsilon]$ and $\|[c_\epsilon]\|_{\Omega_c(\mathbb{R})/\mathbb{R}} \leq \epsilon$. Assume to the contrary that $[x] \nleq [y]$. Since $x, y \in \Omega_c(\mathbb{R})$, so there exist $a, b, a', b' \in \mathbb{R}$ such that $x = \{c \in \mathbb{R} : a \leq c \leq b\}$ and $y = \{c \in R : a' \leq c \leq b'\}$. Because $x \nsubseteq y + r$ for any $r \in \mathbb{R}$, then $a < a' + s \leq b' + s < b$ for some $s \in \mathbb{R}$. If $\|[c_\epsilon]\|_{\Omega_c(\mathbb{R})/\mathbb{R}} < \epsilon = \min\{a' + s - a, b - b' - s\}$, then there exists a symmetric interval $d_\epsilon \in [c_\epsilon]$ such that $d_\epsilon \subseteq (-\epsilon, \epsilon)$. Since

$$a < a' + s - \epsilon \leq b' + s + \epsilon < b, \tag{3.20}$$

then $[x] \nleq [y] + [c_\epsilon]$, a contradiction.

3.2. QUOTIENT MAP

Let us start this section by introducing the definition of a quasilinear operator. For more details see [2].

Definition 3.9. *Let X and Y be quasilinear spaces. A map $\Lambda : X \to Y$ is a quasilinear operator if it satisfies the following conditions:*

1. $\Lambda(\alpha \cdot x) = \alpha \cdot \Lambda(x)$;
2. $\Lambda(x + y) \leq \Lambda(x) + \Lambda(y)$;
3. *If $x \leq y$, then $\Lambda(x) \leq \Lambda(y)$.*

Let N be a non-negative linear subspace of a quasilinear X. For every $x \in X$, let $\pi(x)$ be the coset of N that contains x. Thus

$$\pi(x) = [x] \tag{3.21}$$

π is called the quotient map of X onto X/N.

Theorem 3.10. *π is a quasilinear operator of X onto X/N.*

Proof . $\pi(\alpha \cdot x) = [\alpha \cdot x] = \alpha \cdot [x] = \alpha \cdot \pi(x)$. Also

$$\pi(x+y) = [x+y] = [x] + [y] = \pi(x) + \pi(y) \tag{3.22}$$

and if $x \leq y$ then $[x] \leq [y]$, so $\pi(x) \leq \pi(y)$. Therefore π is a quasilinear operator. □

Theorem 3.11. *Let X be a normed quasilinear space. Let N be a non-negative closed linear subspace of X as in theorem 3.7, then the quotient map $\pi : X \to X/N$ is continuous.*

Proof . Suppose that $x_n \to x$, then for any $\epsilon > 0$ there exists an index M such that the following conditions hold for $n \geq M$:

$$x_n \leq x + a^{\epsilon}_{1,n}, x \leq x_n + a^{\epsilon}_{2,n}, \|a^{\epsilon}_{i,n}\|_X \leq \epsilon. \tag{3.23}$$

Consequently, $[x_n] \leq [x + a^{\epsilon}_{1,n}] = [x] + [a^{\epsilon}_{1,n}]$, $[x] \leq [x_n + a^{\epsilon}_{2,n}] = [x_n] + [a^{\epsilon}_{2,n}]$ and

$$\|[a^{\epsilon}_{i,n}]\|_{X/N} = \inf\{\|a^{\epsilon}_{i,n} + r\|_X : r \in N\} \leq \|a^{\epsilon}_{i,n}\|_X \leq \epsilon. \tag{3.24}$$

Hence $[x_n] \to [x]$. The theorem is proved. □

Definition 3.12. *Let X be a quasilinear space. We say it has the subtraction property of inequality, if we conclude $b \leq c$ from $a + b \leq a + c$ for any $a, b, c \in X$.*

Example 3.13. *Every real linear space has the subtraction property of inequality. In fact, since equality is the only way to define a partial order on a real linear space, the subtraction property of inequality is the subtraction property of equality.*

Example 3.14. *The subtraction property of inequality is stablished on $\Omega_c(\mathbb{R})$.*

Lemma 3.15. *Let X be a quasilinear space. If it has the subtraction property of inequality, then every linear subspace of it is non-negative.*

Proof . Suppose that N is a linear subspace of X and $x \leq x + n$ for some $x \in X$ and $0 \neq n \in N$. So $0 \leq n$ by subtraction property. Then $n = 0$ by lemma 2.6 and it is a contradiction. □

Theorem 3.16. *Let N be a closed linear subspace of a normed quasilinear space X such that the condition of theorem 3.7 is satisfied. Moreover, X has the subtraction property of inequality. Then the quotient map $\pi : X \to X/N$ is open.*

Proof . N is non-negative by lemma 3.15. Aalso, for any $\epsilon > 0$, suppose that $h_{X/N}([x],[y]) < \epsilon/4$. Then there exist $[a_1], [a_2]$ such that

$$[x] \leq [y] + [a_1], [y] \leq [x] + [a_2], \|[a_i]\|_{X/N} \leq \epsilon/4. \tag{3.25}$$

So $x \leq y + a_1 + n_1$,$y \leq x + a_2 + n_2$, $\|a_i + m_i\|_X \leq \epsilon/4$ for some $n_i, m_i \in N$. Thus

$$x \leq (y + n_1 - m_1) + (a_1 + m_1), (y + n_1 - m_1) \leq x + (a_2 + m_2) + n_2 + n_1 - m_1 - m_2. \tag{3.26}$$

So we conclude $-n_2 - n_1 + m_1 + m_2 \leq (a_2 + m_2) + (a_1 + m_1)$ from

$$x \leq (y + n_1 - m_1) + (a_1 + m_1) \leq x + (a_2 + m_2) + (a_1 + m_1) + n_2 + n_1 - m_1 - m_2. \tag{3.27}$$

Thus

$$\|n_2 + n_1 - m_1 - m_2\| \leq \|(a_2 + m_2) + (a_1 + m_1)\| \leq \epsilon/2. \tag{3.28}$$

Therefore $h_X(x, (y + n_1 - m_1)) < \epsilon$ by 3.26 and 3.28. □

Example 3.17. *The quotient map of* $\Omega_c(\mathbb{R})$ *onto* $\Omega_c(\mathbb{R})/\mathbb{R}$ *is an open, continuous quasilinear operator.*

Example 3.18. *Let* X *be the set of all* $B_{a,r} = \{x \in \mathbb{C} : \|x - a\| \leq r\} \in \Omega_c(\mathbb{C})$ *for* $a \in \mathbb{C}$ *and* $r \geq 0$. *Let* $A = B_{0,r_1}$,$B = B_{0,r_2}$ *and* $C = B_{0,r_1+r_2}$. *Clearly* $A + B \subseteq C$. *Suppose that* $r_1 \leq r_2$. *Let* $r_3 e^{i\theta} \in C$. *If* $0 \leq r_3 \leq r_1$, *then* $r_3 e^{i\theta} + 0 \in A + B$. *If* $r_1 \leq r_3 \leq r_1 + r_2$, *then* $r_1 e^{i\theta} + (r_3 - r_1)e^{i\theta} \in A + B$. *Hence* $A + B = C$. *Let* $D = B_{z_1,r_1}$,$E = B_{z_2,r_2}$. *Since*

$$D + E = B_{0,r_1} + z_1 + B_{0,r_2} + z_2 = C + z_1 + z_2 \tag{3.29}$$

therefore sum of both circles becomes a circle. So X *is a subspace of* $\Omega_c(\mathbb{C})$. *It is obvious that the quotient map of* X *onto* $X/\mathbb{C}$ *is an open, continuous quasilinear operator.*

4. LINEAR QUOTIENT SPACES

Similar to what has been done so far, To build Linear quotient spaces, The basis of our work is the using of equivalence relations and equivalence classes.

Definition 4.1. *Let* X *be a quasilinear space and* $M = \{a - a : a \in X\}$. *So* M *is a subspace of* X. *Let* Q *be a relation on* X *such that*

$$xQy \longleftrightarrow \exists n, m \in M : x + n = y + m. \tag{4.1}$$

Then Q *is an equivalence relation on* X. *It is not hard to see that this relation satisfies all the axioms of an equivalence relation. For every* $x \in X$, *the equivalence class of* x *is defined as follows:*

$$[x] = \{y : yQx\}. \tag{4.2}$$

Let Y *be the set of all* $[x]$. *Addition on* Y *is defined by*

$$[x] + [y] = [x + y]. \tag{4.3}$$

Then Y *is a Abelian group.*

Since M is a subspace of X, this operation is well-defined. It means that if $[x] = [x']$ and $[y] = [y']$, thus

$$x + n = x' + n' \tag{4.4}$$

and

$$y + m = y' + m' \tag{4.5}$$

for some $n, m, n', m' \in N$, then $x + y + n + m = x' + y' + n' + m'$, so $(x + y)Q(x' + y')$ and

$$[x'] + [y'] = [x' + y'] = \{z : zQ(x' + y')\} = [x + y] = [x] + [y]. \tag{4.6}$$

Also, since $0 + (a - a) = (a - a) + (0 - 0)$, then $[0] = [a - a]$ for every $a \in X$. So for evey $a \in X$, $[a] + [-a] = [0]$. Other parts of group axioms are proved easily.

Let X be a quasilinear space. If for every $x \in X$ and $\alpha, \beta \in \mathbb{R}$ there exists $a \in X$ such that the following condition holds:

$$(\alpha + \beta)x + a - a = \alpha x + \beta x. \tag{4.7}$$

So $[(\alpha + \beta)x] = [\alpha x + \beta x]$. Then Y is a linear space when multiplication by real numbers on Y is defined by $\alpha[x] = [\alpha x]$. We denote Y by X_{ext}.

Example 4.2. *Let X be a linear space, then X is a quasilinear space with partial order given by equality. It is obvious that $X_{ext} = X$.*

Example 4.3. *Let $X = \Omega_c(\mathbb{R})$. In fact $X = \{[a,b] : a,b \in \mathbb{R}\}$. Then since*

$$\begin{aligned} \alpha[a,b] + \beta[a,b] &= [\alpha a, \alpha b] + [\beta a, \beta b] \\ &= [\alpha a + \beta a, \alpha b + \beta b] \\ &= [(\alpha+\beta)a, (\alpha+\beta)b] \\ &= (\alpha+\beta)[a,b] \end{aligned} \tag{4.8}$$

when $\alpha, \beta \geq 0$, and

$$\begin{aligned} \alpha[a,b] + \beta[a,b] &= [\alpha b, \alpha a] + [\beta a, \beta b] \\ &= [\alpha b + \beta a, \alpha a + \beta b] \\ &= [(\alpha+\beta)a - \alpha a + \alpha b, (\alpha+\beta)b - \alpha b + \alpha a] \\ &= (\alpha+\beta)[a,b] + [\frac{-\alpha a + \alpha b}{2}, \frac{-\alpha b + \alpha a}{2}] - [\frac{-\alpha a + \alpha b}{2}, \frac{-\alpha b + \alpha a}{2}] \end{aligned} \tag{4.9}$$

when $\alpha \leq 0 \leq \beta$ and $|\alpha| \leq |\beta|$. So $X_{ext} = (\Omega_c(\mathbb{R}))_{ext}$ is a linear space. Geometrically, all closed intervals that have a similar center fall into an equvalence class.

Example 4.4. *Let $X = \Omega_c(\mathbb{R})$. $(X_s \cup \{\{0\}\})_{ext}$ is a linear space. In this example, no equivalence class, except Zero, contains a regular element of X. $[\{0\}]$ contains $\{0\}$ and all symmetric intervals around Zero.*

Theorem 4.5. *Let X be a quasilinear space which has the subtraction property of inequality, Q be the equivalence relation as defind in definition 4.1 and $x \in X_r$. if $xQ0$ then $x = 0$.*

Proof . Since $xQ0$, then

$$x + a - a = b - b \tag{4.10}$$

for some $a, b \in X$. Since $-(x + a - a) = -(b - b)$, thus

$$-x + a - a = b - b. \tag{4.11}$$

We conclude by combining 4.10 and 4.11 that $x + x + a - a = a - a$. So $2x = 0$ according to the subtraction property of inequality. Hence $x = 0$. □

Corollary 4.6. *Suppose that X and Q are defined as in theorem 4.5. Then two regular elements can not belong to an equivalence class.*

Proof . Let $x, y \in X_r$ such that xQy. Then

$$x + a - a = y + b - b \tag{4.12}$$

for some $a, b \in X$. So $x - y + a - a = b - b$. Since $x - y \in X_r$ thus $x - y = 0$ by theorem 4.5. □

Let X be a quasilinear space and X_{ext} be the linear space that is defind in definition 4.1. Then a seminorm on X_{ext} is defined by

$$\|[x]\|_{ext} = \inf\{\|y\|_X : \exists a \in X : y + a - a = x\}. \tag{4.13}$$

We prove that $\|[x]+[y]\|_{ext} \leq \|[x]\|_{ext} + \|[y]\|_{ext}$. Since

$$\{z : \exists a : z+a-a = x\} + \{z : \exists a : z+a-a = y\} \subseteq \{z : \exists a : z+a-a = x+y\}, \tag{4.14}$$

then

$$\{\|z+z'\|_X : \exists a,b : z+a-a = x, z'+b-b = y\} \subseteq \{\|z\|_X : \exists a : z+a-a = x+y\}. \tag{4.15}$$

Thus

$$\begin{aligned} &\inf\{\|z\|_X : \exists a : z+a-a = x+y\} \\ &\quad \leq \inf\{\|z+z'\|_X : \exists a,b : z+a-a = x, z'+b-b = y\} \\ &\quad \leq \inf\{\|z\|_X + \|z'\|_X : \exists a,b : z+a-a = x, z'+b-b = y\} \\ &\quad = \inf\{\|z\|_X : \exists a : z+a-a = x\} + \inf\{\|z\|_X : \exists a : z+a-a = y\}. \end{aligned} \tag{4.16}$$

Lemma 4.7. *Let X be a quasilinear space and $x \in X$. If $a_n + b_n \to x$ and $a_n \to 0$, then $b_n \to x$.*

Proof . Since $h_X(a_n + b_n, x) \to 0$ and $h_X(a_n, 0) \to 0$, then there exist $m \in \mathbb{N}$ and $t_1, t_2, t_3, t_4 \in X$ such that:

$$a_m \leq t_1, 0 \leq a_m + t_2 \tag{4.17}$$

and

$$a_m + b_m \leq x + t_3, x \leq a_m + b_m + t_4 \tag{4.18}$$

and $\|t_i\|_X \leq \dfrac{\epsilon}{2}$. Thus

$$b_m \leq a_m + b_m - a_m \leq x + t_3 - t_1, \|t_3 - t_1\|_X \leq \epsilon \tag{4.19}$$

and

$$x \leq a_m + b_m + t_4 \leq b_m + t_4 + t_1, \|t_4 + t_1\|_X \leq \epsilon. \tag{4.20}$$

□

Theorem 4.8. *Let X be a quasilinear space and $x \in X$. If $a_n - a_n \to x$, then $x = -x$.*

Proof . $h_X(a_n - a_n, x) \to 0$, hence

$$a_m - a_m \leq x + t_1, x \leq a_m - a_m + t_2 \tag{4.21}$$

for some $m \in \mathbb{N}$, $t_1, t_2 \in X$ and $|t_i|_X \leq \dfrac{\epsilon}{2}$. Then

$$-x \leq a_m - a_m - t_2 \leq x + t_1 - t_2, |t_1 - t_2|_X \leq \epsilon. \tag{4.22}$$

So $-x \leq x$. Thus $x = -x$. □

Theorem 4.9. *Let X be a quasilinear space and X_{ext} be a linear space. If for any convergent sequence $\{a_n - a_n\}$ there exists $a \in X$ such that*

$$a_n - a_n \to a - a, \tag{4.23}$$

then the seminorm $\|\cdot\|_{ext}$ is a norm.

Proof . Consider that $\|[x]\|_{ext} = 0$, so

$$\inf\{\|y\|_X : \exists b \in X : y + b - b = x\} = 0. \tag{4.24}$$

Then for every $n \in \mathbb{N}$, there exist $y_n, a_n \in X$ such that $y_n + a_n - a_n = x$ and $\|y_n\|_X < \frac{1}{n}$. Since $y_n \to 0$, so $a_n - a_n \to x$ by lemma 4.7. Then $x = a - a$ for some $a \in X$ by assumption. Therefor $[x] = [0]$. □

Example 4.10. *Let $X = \Omega_c(\mathbb{R})$ And $\{[a_n, b_n] - [a_n, b_n]\}$ be a convergent sequence in X. Then there exists $[c, d] \in X$ such that*

$$[a_n - b_n, b_n - a_n] \to [c, d] \tag{4.25}$$

So $a_n - b_n \to c$ and $b_n - a_n \to d$. Hence $c = -d$ and therefore the norm on X is defined by the following statement:

$$\|[[x_1, x_2]]\|_{ext} = \frac{|x_1 + x_2|}{2}. \tag{4.26}$$

5. Conclusion

Although it has been a long time since the quasilinear spaces have been defined, building quotient spaces on them has remained an open problem for years. The Benefits of quotient space and also the advantage of quasilinear spaces over linear ones (their greater conformity to engineering sciences), prompted us to define quotient spaces on quasilinear spaces.

We made our efforts to advance the definitions and theorems with the logical rhythm.This is why addition and scalar multiplication on the desired quotient space were defined in the simplest possible way and the closest thing that comes to mind. What challenged us was the definition of the partial order relation. The difficulty was that the definition had to satisfy all the properties of quasilinear spaces.

In this paper, we have shown that a quasi-linear space could be introduced by defining the notions of l-members and non-negative subspaces. By replacing $a \in A$, which means a is equal to an element of A, with $a \in_{\leq} A$, which means a is less than or equal to an element of A, generalization has happened well and generalized object satisfies our desires.

The notion of l-members can be used in all spaces with partial order structures. It can also be extended to l-subsets, l-subspaces and wherever the membership is presented.

A linear or quasilinear space without norm is like a soulless body. We tried to define the notion of norm by imposing some hypotheses. Also, by using the norm, the corresponding metric, and defining quotient map, we have discussed about the theorems of openness and continuity of this map.

Next, we have defined another type of quotient spaces. In contrast to previously defined quotient spaces, such spaces are linear. In this definition the subspace, which contains all members that are derived by subtraction of a, which a is an element of the space, from itself, plays a major role. it is worth noting that, the definition was presented in such a way that encompasses a wide range of examples and theorems. This issue also does not end without defining the norm . Then, as usual in this article, we gave examples to better understand the content.

The reader who really wants to work in this area should understand the motivation and the way of definitions, see the examples and analyze theorems in order to take steps to improve them, to continue them, or to use them in similar spaces.

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