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Directed Power Graphs

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Abstract

In this article, first we introduce six types of power graphs related to a graph (or directed graph), with the help of set theory. Then we show that these newly defined power graphs are pairwise distinct by a few examples. Finally, we discuss the relation between Eulerian being the base graph and these six power graph types. Moreover, we express the relation between pairwise Eulerian of these power graphs.

Keywords: directed Euler tour, directed Euler path, cycle, directed graph, connected directed graph, directed power graph, Eulerian power graph.

1. Introduction and Definitions

Alexander Treier in 2019 has introduced the series of graphs which were obtained from the series of B-graph with the help of the defined graph power set operation [5]. In a 2014 article, M. A. Shalu and S. Devi Yammini, discussed a subclass G_n^R (right power set graphs G_n^R) of chordal graphs and its complement graph class G_n^L, and they proved that the number of maximal in dependent sets in a subclass G_n^R of chordal graphs can be in polynomial time using Golomb'snonlinear recurrence relation. Also they discussed a superclass F_n (power set graphs) of G_n^L [4]. Melody and Renson in 2019, introduced the concept of power set graph of a simple graph. This graph is taken from the lattice diagram of a power set. Moreover, the characterization such has order, size, degree, dominance number and independence number have been investigated. [3]. But what we are saying here is very different. One of the most important modeling tools in social sciences is the analysis

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of interpersonal relationships and inter-community relationships. In fact, the basic question if the members of a population are related is whether it is possible to establish any inter-group relationship through interpersonal relationships or not.
A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function $\psi(G)$ that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and v_i and v_j are vertices such that $\psi_G(e) = \{v_i, v_j\}$, then e is said to join v_i and v_j, and the vertices v_i and v_j are called the ends of e. We denote the number of vertices and edges in G by $v(G) = |n|$ and $e(G) = |m|$; these two basic parameters are called the order and size of G, respectively [1] . The degree of a vertex v in a graph G denoted by $deg(v)$, is the number of edges of G incident with v ,each loop counting as two edges. In particular if G is a simple graph, $deg(v)$ is the number of neighbours of v in G [1]. Given a graph G, a walk in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, ..., v_{m-1}v_m$ also denoted by $v_0 \to v_1 \to v_2 \to ... \to v_m$, in which any two consecutive edges are adjacent. The number of edges in a walk is called its length. for example in Figure 1, $v \to x \to y \to z \to y \to w$ is a walk of length 5 from v to w. A walk in which all the edges are distinct is a trail. If in addition the vertices $v_0, v_1, ..., v_m$ are distinct (except possibly $v_0 = v_m$), then the trail is a path. A path or trail is closed if $v_0 = v_m$, and a closed path containing at least one edge is a cycle [6]. Note that any loop or pair of multiple edges is a cycle. (An edge with identical ends is called a loop [2]). We see that $v \to w \to x \to y \to z \to x$ is a trail, $v \to w \to x \to y \to z$ is a path, $v \to w \to x \to y \to z \to x \to v$ is a closed trail, and $v \to w \to x \to y \to v$ is a cycle. A graph that is in one piece, so that any two vertices are connected by a path, is a connected graph [6].

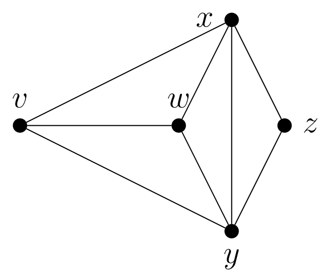

Figure 1: Connected graph G with $v(G) = |5|$ and $e(G) = |8|$.

A trail that traverses every edge of a graph is called an Euler trail, because Euler (1736) was the first to investigate the existence of such trails [1]. A tour of a connected graph G is a closed walk that traverses each edge of G at least once, and an Euler tour one that traverses each edge exactly once(in other word, a closed Euler trail) [1]. A connected graph G is Eulerian if there exists a closed trail containing every edge of G [6]. A directed graph, or digraph D,is an ordered pair $(V(D), E(D))$ consisting of a set $V(D)$ of vertices and a set $E(D)$,disjoint from $V(D)$, of arcs, together with an incidence function $\psi(D)$ that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D. If a is an arc and $\psi D(a) = (v_i, v_j)$, then a is said to join v_i to v_j; in other words v_i dominates v_j. The vertex v_i is the tail of a, and the vertex v_j its head; those are the two ends of a [1]. A directed Euler trail is a directed trail which traverses each arc of the digraph exactly once, and a directed Euler tour is a directed tour with this same property [1]. A digraph is Eulerian if it admits a directed Euler tour [2]. A simple cycle is a cycle that does not repeat any vertices or edges (except the first/last vertex) [2]. A vertex of degree zero is called an isolated vertex and a vertex of degree 1 is an end-vertex [6]. For a vertex v in a digraph D, the outdegree $od(v)$ of v

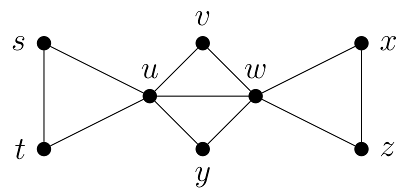

Figure 2: A graph with an Eulerian trail .

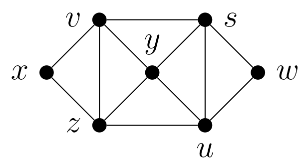

Figure 3: Eulerian graph.

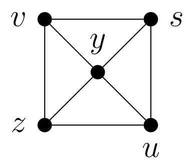

Figure 4: Non-Eulerian graph.

(or $deg^+(v)$) is the number of vertices of D to which v is adjacent, while the indegree $id(v)$ of v (or $deg^-(v)$) is the number of vertices of D from which v is adjacent[2]. For the digraph D of Figure 5,

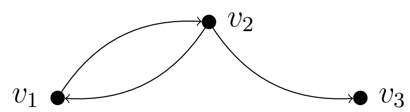

Figure 5: graph D.

$$od(v_1) = id(v_1) = 1; od(v_2) = 2, id(v_2) = 1; od(v_3) = 0, id(v_1) = 1$$

A digraph D is said to be connected (or weakly-connected) if it can not be expressed as the union of two disjoint digraphs, defined in the obvious way; This is equivalent to saying that the underlying graph of D is a connected graph suppose, in addition, that for any two vertices v and w of D there is a path from v tow, then D is called strongly-connected. It is clear that every strongly-connected digraph is connected ,but converse is not true[6].

Theorem 1.1 (Euler's theorem). *A connected digraph is eulerian if and only if it is even. In other words, a nontrivial connected digraph D is Eulerian if and only if $odv = idv$; $(\Sigma_{v \in V} d^+(v) = \Sigma_{v \in V} d^-(v))$, for every vertex v of D [2].*

2. Introduce of power graphs

Throughout this article, we restrict our attention to directed graphs. In this section, first we have introduced the concept of power graphs. Assume that $D = (V, E)$ is a graph. In this case, with the help of edges set E and given the definitions of E_i's for $1 \leq i \leq 6$, six power graphs $P_i(D)$ for $1 \leq i \leq 6$ are defined as $P_i(D) = (V_i, E_i)$, in which V_i will be power set V except ϕ ($V_i = 2^V \setminus \phi$ and $|V_i| = 2^n - 1$). In this section, E_i's for $1 \leq i \leq 6$ are introduced with an example that shows the differences between E_i's, before $P_i(D)$'s and their construction methods are analyzed.

Definition 2.1. *Assume that A and B are two subsets of V. Then, each E_i for $1 \leq i \leq 6$ is defined as follows:*

$$\begin{array}{lll} AE_1B \ \ if & \forall a \in A, \forall b \in B & ; \ \ aEb \\ AE_2B \ \ if & \exists a \in A, \exists b \in B & ; \ \ aEb \\ AE_3B \ \ if & \exists b \in B, \forall a \in A & ; \ \ aEb \\ AE_4B \ \ if & \forall a \in A, \exists b \in B & ; \ \ aEb \\ AE_5B \ \ if & \exists a \in A, \forall b \in B & ; \ \ aEb \\ AE_6B \ \ if & \forall b \in B, \exists a \in A & ; \ \ aEb \end{array}$$

The following relationships are established between the E_i's members:

$$E_1 \subseteq E_3 \subseteq E_4 \subseteq E_2$$
$$E_1 \subseteq E_5 \subseteq E_6 \subseteq E_2$$

Pay attention to this example:

Example 2.2. *In this example, graph D is as follows for $V = \{a, b, c\}, E = \{(a, b), (b, a)\}$. According*

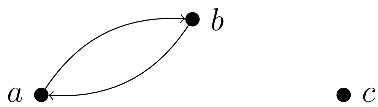

Figure 6: graph D.

to the definition of E_i's, each of the set members E_i's for $1 \leq i \leq 6$ is observed as follows:

$AE_1B \ \ if \ \ \forall a \in A, \forall b \in B \ \ ; \ \ aEb$

$E_1 = \{(\{a\}, \{b\}), (\{b\}, \{a\})\}$.

$AE_2B \ \ if \ \ \exists a \in A, \exists b \in B \ \ ; \ \ aEb$

$E_2 =\{(\{a\}, \{b\}), (\{a\}, \{a, b\}), (\{a\}, \{b, c\}), (\{a\}, \{a, b, c\}), (\{b\}, \{a\}), (\{b\}, \{a, b\}), (\{b\}, \{a, c\}) (\{b\}, \{a, b, c\}), (\{a, b\}, \{a\}), (\{a, b\}, \{b\}), (\{a, b\}, \{a, b\}), (\{a, b\}, \{b, c\}), (\{a, b\}, \{a, c\}), (\{a, b\}, \{a, b, c\}), (\{a, c\}, \{b\}), (\{a, c\}, \{a, b\}), (\{a, c\}, \{b, c\}), (\{a, c\}, \{a, b, c\}), (\{b, c\}, \{a\}), (\{b, c\}, \{a, b\}), (\{b, c\}, \{a, c\}), (\{b, c\}, \{a, b, c\}), (\{a, b, c\}, \{a\}), (\{a, b, c\}, \{b\}), (\{a, b, c\}, \{a, b\}), (\{a, b, c\}, \{a, c\}), (\{a, b, c\}, \{b, c\}), (\{a, b, c\}, \{a, b, c\})\}$.

$$AE_3B \quad if \quad \exists b \in B, \forall a \in A \quad ; \quad aEb$$

$$E_3 = \{(\{a\},\{b\}),(\{a\},\{a,b\}),(\{a\},\{b,c\}),(\{a\},\{a,b,c\}),(\{b\},\{a\}),(\{b\},\{a,b\}),(\{b\},\{a,c\}),(\{b\},\{a,b,c\})\}.$$

$$AE_4B \quad if \quad \forall a \in A, \exists b \in B \quad ; \quad aEb$$

$$E_4 = \{(\{a\},\{b\}),(\{a\},\{a,b\}),(\{a\},\{b,c\}),(\{a\},\{a,b,c\}),(\{b\},\{a\}),(\{b\},\{a,b\}),(\{b\},\{a,c\}),(\{b\},\{a,b,c\}),(\{a,b\},\{a,b\}),(\{a,b\},\{a,b,c\})\}.$$

$$AE_5B \quad if \quad \exists a \in A, \forall b \in B \quad ; \quad aEb$$

$$E_5 = \{(\{a\},\{b\}),(\{b\},\{a\}),(\{a,b\},\{a\}),(\{a,b\},\{b\}),(\{a,c\},\{b\}),(\{b,c\},\{a\}),(\{a,b,c\},\{a\}),(\{a,b,c\},\{b\})\}.$$

$$AE_6B \quad if \quad \forall b \in B, \exists a \in A \quad ; \quad aEb$$

$$E_6 = \{(\{a\},\{b\}),(\{b\},\{a\}),(\{a,b\},\{a\}),(\{a,b\},\{b\}),(\{a,b\},\{a,b\}),(\{a,c\},\{b\}),(\{b,c\},\{a\}),(\{a,b,c\},\{a\}),(\{a,b,c\},\{b\}),(\{a,b,c\},\{a,b\})\}.$$

In this example ,power graphs $P_1(D)$,...,$P_6(D)$ is shown in Figures 7 to 11.

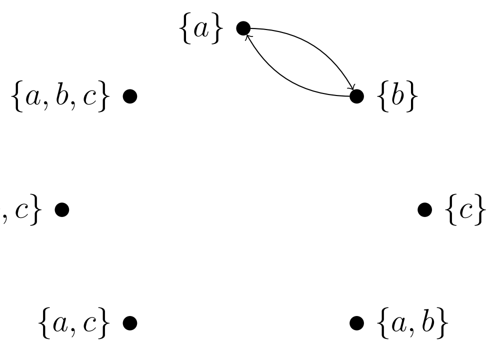

Figure 7: Power graph $P_1(D)$.

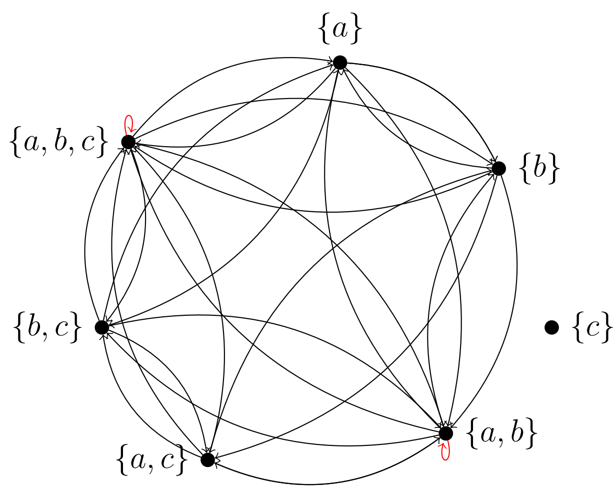

Figure 8: Power graph $P_2(D)$.

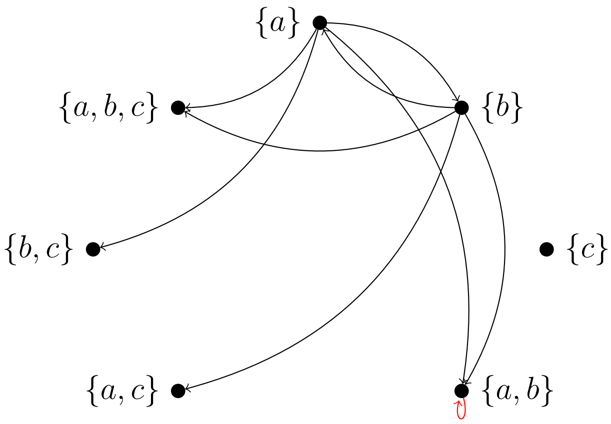

Figure 9: Power graph $P_3(D)$.

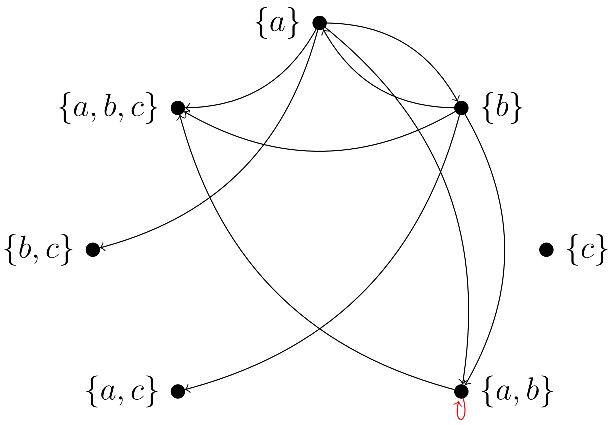

Figure 10: Power graph $P_4(D)$.

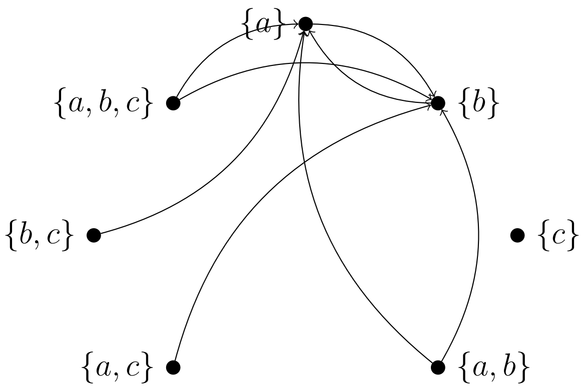

Figure 11: Power graph $P_5(D)$.

Now assume that $D = (V, E)$ is a graph. As discussed earlier, it is possible to define $P_i(D)$'s for $1 \leq i \leq 6$ based on graph D. In other words, each $P_i(D)$ can be constructed with respect to the adjacency matrix of graph D and its corresponding E_i. An example is given for constructing $P_i(D)$'s

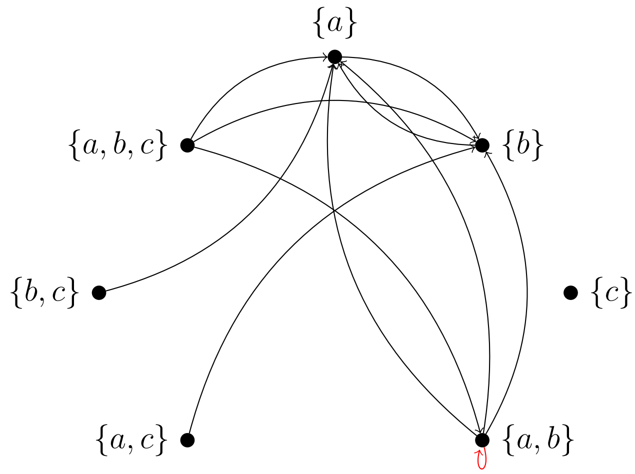

Figure 12: Power graph $P_6(D)$.

for $1 \leq i \leq 6$.

Example 2.3. *Consider graph $D = (V, E)$ that is shown in Figure 13.*

$$\mathbb{A} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

The goal is to create power graph $P_1(D) = (V_1, E_1)$ by graph D.

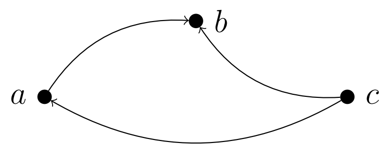

Figure 13: graph D with an adjacency matrix $\mathbb{A}$.

$V_1 = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
$E_1 : \quad AE_1B \quad if \quad \forall a \in A, \forall b \in B \quad ; \quad aEb$
$\Rightarrow E_1 = \{(\{a\}, \{b\}), (\{c\}, \{a\}), (\{c\}, \{b\}), (\{c\}, \{a, b\}), (\{a, c\}, \{b\})\}$.
If we decide to use adjacency matrix $\mathbb{A}$ to determine the set E_1, the rows and columns of $\mathbb{A}$ must be considered set A and set B, respectively. Since both surveys are general, the positions of elements are determined in the adjacency matrix of power graph $P_1(D)$ based on the positions of nonzero elements. For instance, the first element of 1 exists on the first row of the second column; therefore, there will be an edge from $\{a\}$ to $\{b\}$. The second nonzero element exists on the third row of the first column and results in edge $(\{c\}, \{a\})$. The next nonzero element exists on the third row of the second column and creates edge $(\{c\}, \{b\})$. Since there are two nonzero elements on the third row, there will also be edge $(\{c\}, \{a, b\})$; (note that if there are more than one nonzero elements like the number of m nonzero elements on each row, then 2, 3, ..., or m-fold combinations are taken into account). Considering

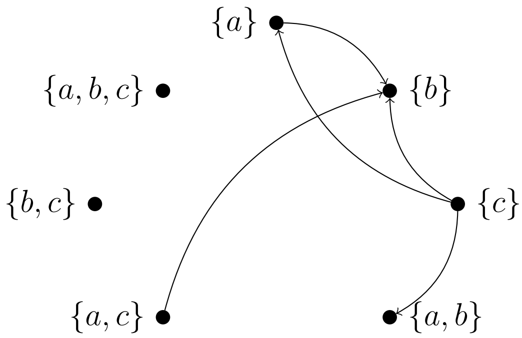

Figure 14: Power graph $P_1(D)$.

the fact that there are two nonzero elements on the second column, edge $(\{a,c\},\{b\})$ will exist in graph $P_1(D)$, ($P_1(D)$ is shown in Figure 14). Similarly, it is possible to obtain $P_i(D)$ for $2 \leq i \leq 6$ from adjacency matrix $\mathbb{A}$ based on the quantifiers used in E_i for $2 \leq i \leq 6$.

Assume that we have an industrial town and there is n people who work in this town and the number of its products is $2^n - 1$. In the following example, we considering a base graph between the people of the town and the relationship between people in an industrial town determines the edges of the graph D, for example a person a has relationship with a person b if a person a can buy products from the production where a person b works. Also each producer is considered as a set of people. Then, we want analyze the relationship between productions with the help of power graphs $P_i(D)$ for $1 \leq i \leq 6$. Note that, we have a relationship between humans, the existence of a loop in the base graph is meaningless.

Example 2.4. *Consider the set $V = \{a,b,c,d,e,f,g,h,k\}$ is the member of people in the industrial town and let*

$$E = \{(a,d),(a,f),(a,g),(b,e),(b,f),(b,g),(d,a),(d,c),(g,a),(g,b),(f,b),(e,f),(e,g),(k,h),(h,a)\}$$

be the relationships between people in this town. We consider a number of factories in this town. For example, we suppose that people a and b work in the Dairy production and as the same way, Sanitary ware production $\{c,d,e\}$, Lamp production factory $\{f,g\}$, Garment production $\{h,k\}$. Now given that the edges (E_i's for $1 \leq i \leq 6$), are defined between the two products, so we analyze relationships between a number of these productions.
According to definition E_1, the product A has a relation E_1 with the product B, whenever each member A buy from each member in the product B. Consider two factories Dairy and Lamp production. Given that anyone in the Dairy production can buy from anyone in the Lamp production, so the Dairy production has the relationship of E_1 with the Lamp production. In other word $(\{a,b\},\{f,g\}) \in E_1$. But for example, the dairy production has not the relationship of E_1 with the Sanitary ware production, since a person a in the dairy production can not buy from people c or e in the Sanitary ware production. Therefore $(\{a,b\},\{c,d,e\}) \notin E_1$.
But the product A has a relation E_2 with the product B, whenever there is at least a member in A so that can buy at least from a member in B. Pay attention that, the Garment production has the relationship of E_2 with the Dairy production factory, since by E_2 there is a person in the Garment production that this person can buy from a person in the Dairy production factory, so $(\{h,k\},\{a,b\}) \in E_2$.

Now consider two factories Garment and Lamp production. This is clearly that, $(\{h, k\}, \{g, f\}) \notin E_2$, *since there is no any person in the Garment Production so that can buy from a person in the Lamp production.*
Consider two factories Dairy and Garment. If we look at the definition of E_3*, we realize that* $(\{a, b\}, \{h, k\}) \notin E_3$. *To understand why the Dairy production can not the relationship of* E_3 *with the Garment factory, we pay attention to the these explanations that, by* E_3 *there must be someone in the Garment factory so that all of the people in the Dairy production can buy from the person in the Garment factory. While there is not such a person in the Garment factory. But Lamp factorie has the relation of* E_3 *with Dairy factorie. Since, for example there is a person like* b *in the Dairy factorie so that all of the people in the Lamp factorie can buy from* b *in the Dairy factorie. So* $(\{g, f\}, \{a, b\}) \in E_3$.
Via definition E_4*, the product* A *has a relation* E_4 *with the product* B*, whenever all of the members in* A *can buy from at least a member in* B*. Consider two factories Dairy and Garment. Since none of the people in the Dairy production can buy from at least a person in Garment factory, then the Dairy production has not the relationship of* E_4 *with the Garment factory. So* $(\{a, b\}, \{h, k\}) \notin E_4$. *Now consider two other factories like Dairy and Sanitary ware. the Dairy production has the relationship of* E_4 *with the Sanitary ware factory, whenever all of the people in the Dairy production can buy from a person in the Sanitary ware factory. According to this description, clearly* $(\{a, b\}, \{c, d, e\}) \in E_4$.
Also we have see by the definition E_5*, the factorie* A *has a relation* E_5 *with the factorie* B*, whenever there is at least one person in the factorie* A *so that this person can buy from all of the people in the factorie* B*. for instance, the Sanitary ware factorie has not the relation of* E_5 *with the Dairy production factorie, because there is no any one person in the Sanitary ware factorie so that this person can buy from all of the people in the Dairy factorie. Therefore* $(\{c, d, e\}, \{a, b\}) \notin E_5$. *While the Sanitary ware factorie has a relation* E_5 *with the Lamp production factorie, that means we have* $(\{c, d, e\}, \{f, g\}) \in E_5$. *Since there is at least one person in the Sanitary ware factorie so that can buy from all of the people in the Lamp production factorie.*
We say that the factorie A *has a relation* E_6 *with the factorie* B*, whenever for all of the person in the* B*, there is at least a person in* A*, so that to buy from them. According to the description given clearly,* $(\{c, d, e\}, \{a, b\}) \notin E_6$. *Now suppose that we have two factories Lamp and Dairy production. Observe that the Lamp factory has a relation* E_6 *with the Dairy production factorie, So* $(\{g, f\}, \{a, b\}) \in E_6$.

Example 2.5. *In this example, aims to theoretically show of being a donor and a receiver between blood groups(* O^+ *,* O^- *,* A^+ *,* A^- *,* B^+, B^-, AB^+ *and* AB^- *) in power graphs.*

$$\mathbb{A} = \begin{array}{c} \\ O^- \\ O^+ \\ A^- \\ A^+ \\ B^- \\ B^+ \\ AB^- \\ AB^+ \end{array} \begin{array}{c} \begin{array}{cccccccc} O^- & O^+ & A^- & A^+ & B^- & B^+ & AB^- & AB^+ \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Given the fact that group A *has Antigen* A *and Antibody* B *and that group* B *has Antigen* B *and Antibody* A *as well as group* AB *having Antigens* A *and* B *and group* O *having Antibodies* A *and*

B, it is obvious that group AB is a powerful receiver, whereas group O is a powerful donor. The eight existing blood groups are the members of V, that we in this example consider part of the graph $D = (V, E)$. Where in $V = \{AB^-, AB^+, B^-, B^+\}$ and
$E = \{(AB^-, AB^-), (AB^-, AB^+), (AB^+, AB^+), (B^+, B^+), (B^+, AB^+), (B^-, B^-), (B^-, B^+), (B^-, AB^-), (B^-, AB^+)\}$.

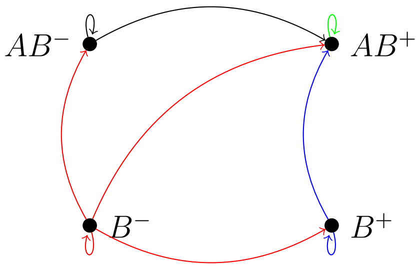

Figure 15: A part of the graph D.

We now see how a donor and a receiver between blood groups in this people, can be employed to analyze by definition of E_i's for $1 \leq i \leq 6$. We consider the set A as the family A and the set B as the family B.
The family of A has an relationship of E_1 with the family of B. This means that there are definitely donor of blood from all of the people existing in the family A to all of the people existing in the family B. For instance, the existence of an edge $e_1 = (\{B^-, B^+\}, \{B^+, AB^+\}) \in E_1$, that means all of the people in the family A have blood groups B^- and B^+, can donor of blood to all of the people in the family B that have blood groups B^+ and AB^+. Therefore, $A = \{B^-, B^+\}$ has an relationship of E_1 with $B = \{B^+, AB^+\}$. But if $A = \{B^+, AB^-\}$ and $B = \{B^-, B^+\}$, then the family of A has not an relationship of E_1 with the family of B. In other word, $e_2 = (\{B^+, AB^-\}, \{B^-, B^+\}) \notin E_1$, since everyone in the family A must be donor of blood to all of the people in the family B, while for example a person with blood group B^+ in A can not donor of blood to a person with blood group B^- in B. Let's consider another edeg like $e_3 = (\{B^+, AB^-\}, \{B^+, AB^+\})$. According to the similar to description, $e_3 \notin E_1$.
But by the definition of E_2, clear that there is at least a donor blood from one of the people existing in the family A to at least one of the people existing in the family B. Consider the edge $e_2 = (\{B^+, AB^-\}, \{B^-, B^+\})$. by E_2, this edge that means a person with a blood group B^+ in the family A, it can donor of blood to a person with blood group B^+ in the family B. So, $A = \{B^+, AB^-\}$ has the relationship of E_2 with $B = \{B^-, B^+\}$. Or if we consider the edge e_3, clearly $A = (\{B^+, AB^-\})$ has the relationship of E_2 with $B = \{B^+, AB^+\}$. Now suppose that $A = \{AB^+, B^+\}$ and $B = \{B^-, AB^-\}$. Since there is no one in the family A that can donor of blood to someone in the family B, then the family A has not the relationship of E_2 with B . In the other word $e_4 = (\{AB^+, B^+\}, \{B^-, AB^-\}) \notin E_2$. But the family $A = \{AB^-, B^+\}$ can not have a relationship E_3 with the family $B = \{AB^-, B^+\}$, in other word $e_5 = (\{AB^-, B^+\}, \{AB^-, B^+\}) \notin E_3$. Since by the definition E_3, there must be at least one of the people with blood group AB^- or B^+ of the family B that all of the people existing in the family A with blood groups AB^- and B^+ blood donor to an the existing person in the family B. While there is no such person in the family B. Or if consider the edge e_4, with similar explanations we see that, $e_4 = (\{AB^+, B^+\}, \{B^-, AB^-\}) \notin E_3$. Now we consider edge $e_6 = (\{AB^-, B^+\}, \{AB^+, B^-\}) \in E_3$. The definition of E_3 shows obviously that there is at least one people with blood group AB^+ in the family B so that all of the people with blood groups AB^- and

B^+ in the family A blood donor to an existing person in the family B with blood group AB^+. Also if consider edge e_1, it's clearly the family $A = \{B^-, B^+\}$ has the relationship of E_3 with the family $B = \{B^+, AB^+\}$. So $e_1 = (\{B^-, B^+\}, \{B^+, AB^+\}) \in E_3$.

Now, the definition of E_4 shows that all of the people existing in the family A can be blood donor to at least one of the people existing in the family B. We choose one of the edge as desired. For instance $e_3 = (\{AB^-, B^+\}, \{AB^+, B^+\}) \in E_4$, shows that each of people with blood groups AB^- and B^+, has definitely blood donor to at least one of the people with blood groups AB^+ and B^+. Follows by mentioned edge, a person has blood group AB^- in the family A can blood donor to a person has blood group AB^+ in the family B and also a person has blood group B^+ in the family A can blood donor to a person has blood group B^+ in the family B. But consider the edge e_4. According to the definition of E_4, there is no one person in the family $A = \{AB^+, B^+\}$ that can blood of donor to at least one of the people in the family $B = \{B^-, AB^-\}$ and then $e_4 = (\{AB^+, B^+\}, \{B^-, AB^-\}) \notin E_4$. Note that $e_5 = (\{B^+, AB^-\}, \{B^+, AB^-\}) \notin E_5$. In other word, the family A has no relation E_5 with the family B. In fact by definition of E_5 for all of the people existing in the family B there is at least one person in the family A, so that blood donor to all of the people in the family B. But, such a person does not exist in the family A. Suppose that $A = \{B^-, AB^+\}$ and $B = \{B^-, B^+, AB^-\}$ be two families. The family A can have a relationship E_5 with the family B, so $e_7 = (\{B^-, AB^+\}, \{B^-, B^+, AB^-\}) \in E_5$. Via E_5, there is a person with blood group B^- in the family A so that there are blood donor from this person to all of the people with blood groups B^- and B^+ and AB^- in the family B. Or if consider the edge $e_1 = (\{B^-, B^+\}, \{B^+, AB^+\})$, it should be noted that with similar explanations the family $A = \{B^-, B^+\}$ has the relationship of E_5 with the family $B = \{B^+, AB^+\}$.

According to definition E_6, for all of the persons in family B, there is at least one person in family A so that blood donor to people existing in the family B. Consider the edge $e_5 = (\{B^+, AB^-\}, \{B^+, AB^-\}) \in E_6$. The existence of this edge in $P_6(D)$ meanse that there is one person in the family A with blood group B^+ that blood donor to one person in family B with blood group B^+ and another person in the family A with blood group AB^- that blood donor to another person in the family B with blood group AB^-. Now for example, consider the edge $e_4 = (\{AB^+, B^+\}, \{B^-, AB^-\})$. This is clearly that the family $A = \{AB^+, B^+\}$ has not the relationship of E_6 with the family $B = \{B^-, AB^-\}$. Since for none of the people in the family B, there is no one people in the family A so that this person in A can blood donor to people existing in the family B.

Example 2.6. *Assume that we have the availability of information on direct flights between the US, Belgium, Sweden, and Denmark. The defined E_i's can now be employed to analyze the relationships of these four countries.*

$V = \{u, b, s, d\} = \{US, Belgium, Sweden, Denmark\}$;

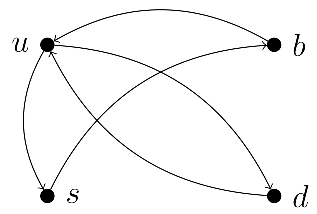

Figure 16: graph D.

$E = \{(u, s), (u, d), (b, u), (s, b), (d, u)\}$.

$$\begin{aligned} V_i &= \{v_1, v_2, ..., v_{15}\} \\ &= \{\{u\}, \{b\}, \{s\}, \{d\}, \{u, b\}, \{u, s\}, \{u, d\}, \{b, s\}, \{b, d\}, \{s, d\}, \{u, b, s\}, \{u, b, d\}, \{b, s, d\}, \\ &\quad \{u, s, d\}, \{u, b, s, d\}\}. \\ E_i &\subseteq \{(\{u\}, \{u\}), (\{u\}, \{b\}), ..., (\{u, b, s, d\}, \{u, b, s, d\})\}. \\ &AE_1B \, if \forall a \in A, \forall b \in B \quad ; \quad aEb \\ E_1 &= \{(\{u\}, \{s\}), (\{u\}, \{d\}), (\{u\}, \{s, d\}), (\{b\}, \{u\}), (\{s\}, \{b\}), (\{d\}, \{u\}), (\{b, d\}, \{u\})\} \end{aligned}$$

According to the definition of E_1, there are definitely direct flights from all of the countries existing in the set A to all of the countries existing in the set B. For instance, the existence of edge $(\{u\}, \{s, d\})$ in power graph $P_1(D)$ means that there are direct flights from the US to Sweden and Denmark. However, given the fact that $(\{s, d\}, \{u\}) \notin E_1$, it is possible to take indirect flights to travel from Sweden and Denmark to the US by air.

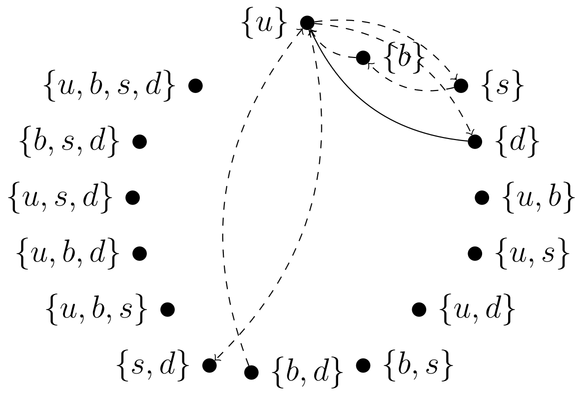

Figure 17: Power graph $P_1(D)$.

$$\begin{aligned} &AE_2B \qquad if \qquad \exists a \in A, \exists b \in B \quad ; \quad aEb \\ E_2 = &\{(\{u\}, \{s\}), (\{u\}, \{d\}), (\{u\}, \{u, s, d\}), .., (\{u, b\}, \{s\}), (\{u, b\}, \{d\}), (\{u, b\}, \{u, s, d\}), .. \\ &(\{b\}, \{u\}), (\{b\}, \{u, b\}), .., (\{u, b\}, \{u\}), (\{u, b\}, \{u, b\}), .., (\{b, s\}, \{u\}), (\{b, s\}, \{u, b\}), .. \\ &, (\{s\}, \{b\}), (\{s\}, \{b, s\}), ..., (\{u, s\}, \{b\}), (\{u, s\}, \{b, s\}), ..., (\{d\}, \{u\}), (\{d\}, \{u, s\}), ... \\ &, (\{u, d\}, \{u\}), (\{u, d\}, \{u, s\}), ..., (\{u, b, s, d\}, \{u, b, s, d\})\}. \end{aligned}$$

The definition of E_2, clearly shows that there is a direct flight from at least one of the countries of the set A to at least one of the countries of the set B. For instance, if edge $(\{u\}, \{b, s, d\})$ of graph $P_2(D)$ is considered, it is then fair to reason that there is at least a direct flight from the US to Belgium, Sweden, or Denmark, that this in example, there are direct flights from the US to Sweden and Denmark. However, if graph $P_2(D)$ is examined carefully, it will be revealed that there is no direct flight from either Belgium or Denmark to Sweden $((\{b, d\}, \{s\}) \notin E_2)$. Nevertheless, it is possible to travel from either Belgium or Denmark to Sweden on an indirect flight. We see the adjacency matrix of graph $P_2(D)$ in the following form.

$$\mathbb{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$AE_3B \;\; if \;\; \exists b \in B, \forall a \in A \;\; ; \;\; aEb$

$E_3 = \{(\{u\},\{s\}),.....,(\{b\},\{u\}),.....,(\{s\},\{b\}),.....,(\{d\},\{u\}),.....,(\{b,d\},\{u\}),......\}.$

According to the definition E_3, the set A has relation E_3 with the set B, if there is at least one country in the set B, so that we have direct flight from all of the existence in A to country in B. For example, if $A = \{u,d\}$ and $B = \{u,b,d\}$, then $(\{u,d\},\{u,b,d\}) \notin E_3$. In other word, the set A has not relation E_3 with the set B, since there is no any countries in the set B so that from countries US and Denmark in set A can be flight to the existence country in B.

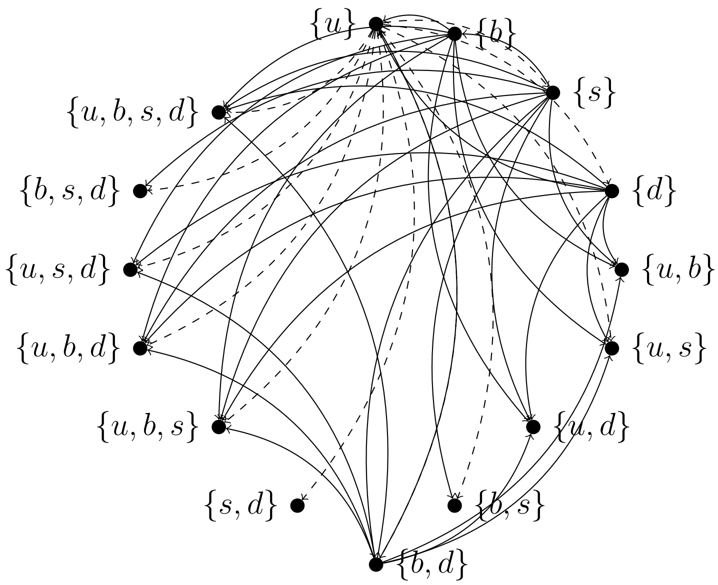

Figure 18: Power graph $P_3(D)$ with its adjacency matrix .

$$\mathbb{A} = \begin{array}{cc} & \begin{array}{ccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$AE_4B \quad if \quad \forall a \in A, \exists b \in B \quad ; \quad aEb$$

$$E_4 = \{(\{u\},\{s\}),(\{u\},\{d\}),(\{u\},\{s,d\}),...,(\{b\},\{u\}),(\{b\},\{u,b\}),...,(\{s\},\{b\}),...,(\{s\},\{u,b\}),.....,(\{d\},\{u\}),(\{d\},\{u,b\}),.....,(\{b,d\},\{u\}),.....,(\{u,d\},\{u,d\}),(\{u,d\},\{u,b,d\}),....\}.$$

The definition of E_4 shows obviously that all of the countries existing in the set A have direct flights to at least one of the countries existing in the set B. But we have edge $(\{u,d\},\{u,b,d\}) \in E_4$. The existence of this edge indicates that each of countries US and Denmark has definitely a direct flight to at least one of the countries US and Belgium and Denmark. In this example, country US can has direct flight to country Denmark and country Denmark can has direct flight to country US. Adjacency matrix of power graph $P_4(D)$ is shown in following form.

$$\mathbb{A} = \begin{array}{cc} & \begin{array}{ccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$AE_5B \qquad if \qquad \exists a \in A, \forall b \in B \;\; ; \;\; aEb$$
$$E_5 = \{(\{u\},\{s\}),(\{u\},\{d\}),(\{u\},\{s,d\}),.....,(\{b\},\{u\}),(\{u,b\},\{u\}),.....,(\{s\},\{b\}),(\{s,b\},\{b\}),.....,(\{d\},\{u\}),(\{u,d\},\{u\}),.....\}$$

Moreover, the definition of E_5 can be stated as follows. For all of the countries existing in B, there is at least one country in A, from which it is possible to have direct flights to all countries of B. for instance, $(\{u,d\},\{u,d\}) \notin E_5$, since if the existing country is considered US (in A), there is a direct flight from the US only to country Denmark (according to E, $(u,d) \in E$), where as there is no relationship between country US (in A) and country US (in B) ; (according to E, $(u,u) \notin E$) .

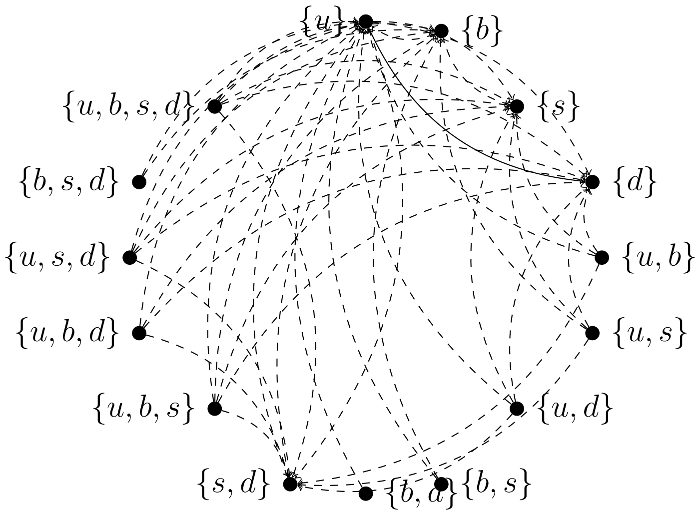

Figure 19: Power graph $P_5(D)$.

$$AE_6B \qquad if \qquad \forall b \in B, \exists a \in A \;\; ; \;\; aEb$$
$$E_6 = \{(\{u\},\{s\}),(\{u\},\{d\}),(\{u\},\{s,d\}),....,(\{b\},\{u\}),(\{u,b\},\{u\}),....,(\{s\},\{b\}),(\{s,b\},\{b\}),.....,(\{d\},\{u\}),(\{u,d\},\{u\}),.....,(\{u,d\},\{u,d\}),(\{u,b,d\},\{u,d\}),.....\}.$$

By the definition E_6, this example is analyzed in a way that there is at least one country of A having direct flights to all of the countries existing in B. Consider an edge like $(\{u,b\},\{s,d\})$ in power graph $P_6(D)$. The existence of this edge in power graph $P_6(D)$ means that at least one of US and Belgium has direct flight to both countries Sweden and Denmark. Also with similar explanations it is clear *that $(\{u,d\},\{u,d\}) \in E_6$. Adjacency matrix of power graph $P_6(D)$ is shown in following form.*

$$\mathbb{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

3. Eulerian Analysis of $P_i(D)$'s for $1 \leqslant i \leqslant 6$

In this section, in order to answer the following questions, we must analyze various examples and theorems, what Eulerian power graphs $P_i(D)$ for $1 \leqslant i \leqslant 6$, will result from either Eulerian or non-Eulerian graph D and vice versa.

3.1. Analyzing whether power graph $P_1(D)$ is Eulerian if connected graph D is Eulerian or not:

According to the definition of Eulerian, power graph $P_1(D)$ is Eulerian if firstly $P_1(D)$ has exactly one connected component. Note that by definition of E_1, power graph $P_1(D)$ has a connected component if at least one member of V like v_i is connected to all members of V from both right and left. In this case, an edge enters all vertices of $P_1(D)$ from vertex $\{v_i\}$ which an edge enters from all vertices in to vertex $\{v_i\}$. This makes power graph $P_1(D)$ consist of only one component; therefore, the power graph is connected. Otherwise, if a member like $v_i \in V$ is not connected to only one member of V like $v_j \in V$ (left or right), no edges enter $V = \{v_1, ..., v_n\}$ from any vertices of power graph $P_1(D)$. In this case, no edges enter other vertices from vertex V; hence, vertex V remains isolated, and power graph $P_1(D)$ becomes disconnected. Secondly, there is exactly one directed Euler tour in power graph $P_1(D)$. In fact, there is such a Euler tour in power graph $P_1(D)$ if power graph D is Eulerian.

Theorem 3.1. *Euler's theorem for power graph $P_1(D)$.*
Connected power graph $P_1(D)$ is Eulerian, if connected graph D is Eulerian.
Proof . *Assume that power graph $P_1(D)$ is a Eulerian graph. In this case, $P_1(D)$ has exactly one connected component. According to the above description, there is at least one vertex like $\{v_i\}$ in power graph $P_1(D)$ where it has two-way connections with all vertices. Consider the connection of vertex $\{v_i\}$ with vertex V. Since vertex V includes all vertices of graph D, two-way edges exist in this graph from all vertices to v_i. This shows that D is connected and has one directed Euler tour; therefore, D is Eulerian.* $\square$

However, this theorem is not conversely true necessarily. In other word, if D is Eulerian, then power graph $P_1(D)$ may be non-Eulerian. (See examples 3.4 to 3.6)

Corollary 3.2. *power graph $P_1(D)$ is Eulerian if both of the following states occurs:*

1. *power graph $P_1(D)$ is connected.*
2. *graph D is Eulerian.*

Corollary 3.3. *If graph D is not Eulerian, power graph $P_1(D)$ is obviously not Eulerian. (See Examples 3.10 and 3.11)*

Examples of Eulerian graphs in Figures 20,22,24 and their corresponding non-Eulerian power graphs are displayed in Figures 21,23,25.

Example 3.4. *In this example, graph D is as follows for $V = \{a, b, c\}, E = \{(a, c), (b, a), (c, b)\}$.*

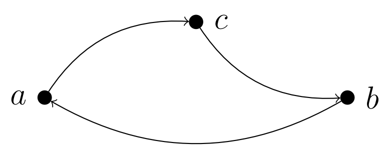

Figure 20: Connected graph D .

As observerd earlier, $E_1 = \{(\{a\}, \{c\}), (\{b\}, \{a\}), (\{c\}, \{b\})\}$. Power graph $P_1(D)$ is shown in Figure 21.

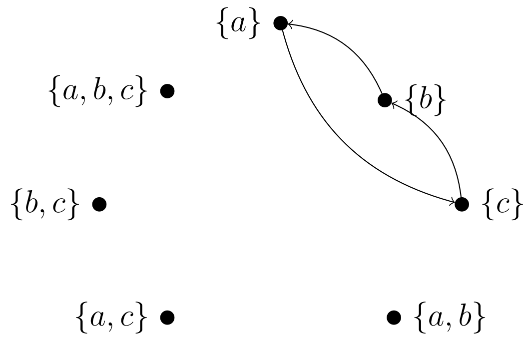

Figure 21: Unconnected power graph $P_1(D)$.

Example 3.5. *Consider vertex set $V = \{a, b, c\}$ and $E = \{(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$.*

$$E_1 =\{(\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{b, c\}), (\{b\}, \{a\}), (\{b\}, \{c\}), (\{b\}, \{a, c\}), (\{c\}, \{a\}), (\{c\}, \{b\}), (\{c\}, \{a, b\}), (\{a, b\}, \{c\}), (\{a, c\}, \{b\}), (\{b, c\}, \{a\})\}.$$

Power graph $P_1(D)$ is shown in Figure 23

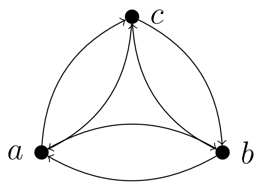

Figure 22: Connected graph D.

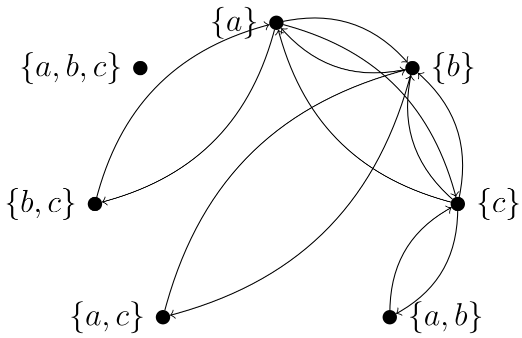

Figure 23: Unconnected power graph $P_1(D)$.

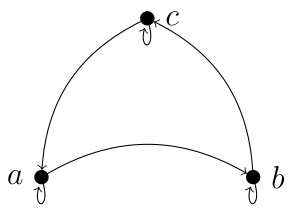

Figure 24: Connected graph D.

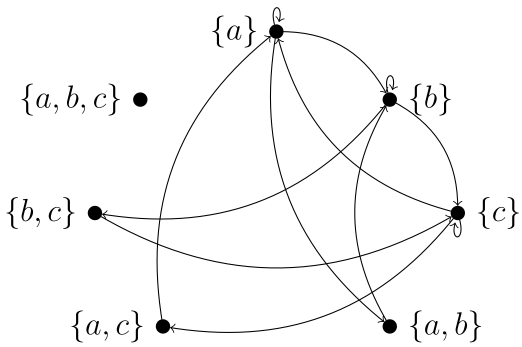

Figure 25: Unconnected power graph $P_1(D)$.

Example 3.6. *For the graph D in Figure 24 , we have $V = \{a, b, c\}, E = \{(a, a), (a, b), (b, b)$ $, (b, c), (c, a), (c, c)\}$. Also*

$$E_1 = \{(\{a\},\{a\}),(\{a\},\{b\}),(\{a\},\{a,b\}),(\{b\},\{b\}),(\{b\},\{c\}),(\{b\},\{b,c\}),(\{c\},\{a\}),(\{c\},\{c\}),(\{c\},\{a,c\}),(\{a,b\},\{b\}),(\{a,c\},\{a\}),(\{b,c\},\{c\})\}.$$

Power graph $P_1(D)$ is shown in Figure 25.

In the above examples, connected graph D is Eulerian because there is a directed Euler tour in every graph D that passes every edge exactly once. However, power graph $P_1(D)$ is not Eulerian in any of these examples. For instance, whereas we seek the Eulerian conditions in the connected graph, there is one isolated vertex in Figures 23 and 25, and several isolated vertices in Figure 21.
In the following examples, connected graph D is Eulerian. In these examples, its corresponding power graph is also Eulerian because there is one closed dirscted Euler trail in these . In other words, according to Euler's theorem, the summation of input degrees is equal to that of the output degrees in every vertex of $P_1(D)$.

Example 3.7. *Figure 26 it shown an example of a Eulerian graph, where in $V = \{a,b\}, E = \{(a,a),(a,b),(b,a)\}$. Also Eulerian power graph $P_1(D)$ is shown in Figure 27.*
$E_1 = \{(\{a\},\{a\}),(\{a\},\{b\}),(\{b\},\{a\}),(\{a\},\{a,b\}),(\{a,b\},\{a\})\}$.

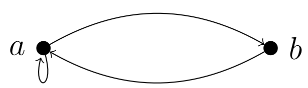

Figure 26: Connected graph D.

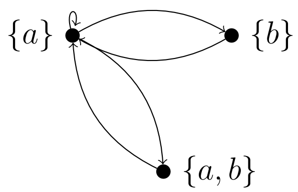

Figure 27: Connected power graph $P_1(D)$.

Example 3.8. *Consider vertex set $V = \{a,b,c\}$ and edge set $E = \{(a,a),(a,b),(a,c),(b,a),(c,a)\}$. As is shown by Figure 28, graph D is Eulerian. whose the set E_1 is*

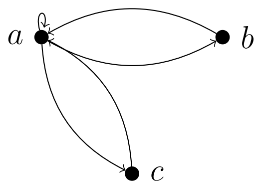

Figure 28: Connected graph D.

$$E_1 = \{(\{a\},\{a\}),(\{a\},\{b\}),(\{a\},\{c\}),(\{a\},\{a,b\}),(\{a\},\{a,c\}),(\{a\},\{b,c\}),(\{a\},\{a,b,c\}),(\{b\},\{a\}),(\{c\},\{a\}),(\{a,b\},\{a\}),(\{a,c\},\{a\}),(\{b,c\},\{a\}),(\{a,b,c\},\{a\})\}.$$

$P_1(D)$ shown in Figure 29, is Eulerian.

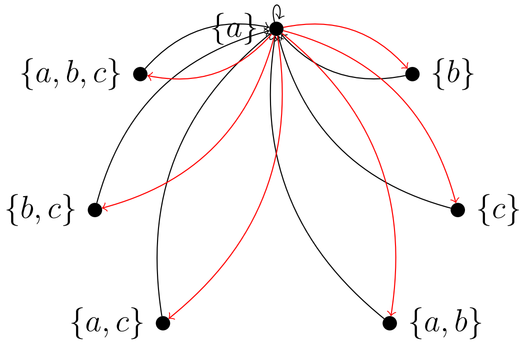

Figure 29: Connected power graph $P_1(D)$.

Example 3.9. *Suppose that we have* $V=\{a,b,c\}$ *and* $E=\{(a,a),(a,b),(a,c),(b,a),(b,c),(c,a),(c,b)\}$*, (Figure 30). Each of the set members* E_1 *is observed as follows:*

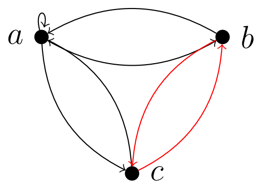

Figure 30: Connected graph D .

$$E_1 =\{(\{a\},\{a\}),(\{a\},\{b\}),(\{a\},\{c\}),(\{b\},\{a\}),(\{b\},\{c\}),(\{c\},\{a\}),(\{c\},\{b\}),(\{a\},\{a,b\}),\\ (\{a\},\{a,c\}),(\{a\},\{b,c\}),(\{a\},\{a,b,c\}),(\{b\},\{a,c\}),(\{c\},\{a,b\}),(\{a,b\},\{a\}),(\{a,b\},\{c\}),\\ (\{a,c\},\{a\}),(\{a,c\},\{b\}),(\{b,c\},\{a\}),(\{a,b,c\},\{a\})\}.$$

Eulerian power graph $P_1(D)$ *is shown in Figure 31.*

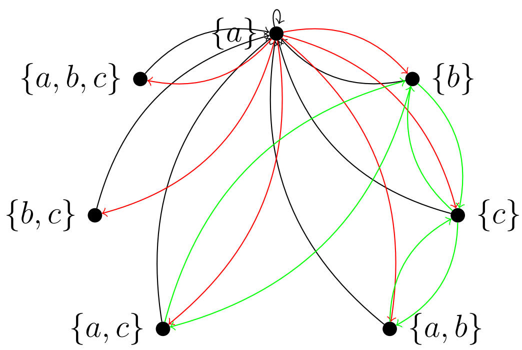

Figure 31: Connected power graph $P_1(D)$.

Example 3.10. *A connected graph* D *has order 2 and* $E=\{(a,a),(a,b),(b,b)\}$.
$E_1=\{(\{a\},\{a\}),(\{a\},\{b\}),(\{a\},\{a,b\}),(\{b\},\{b\})\}$*, (Figure 32).*
Also non-Eulerian power graph $P_1(D)$ *is shown in Figure 33.*

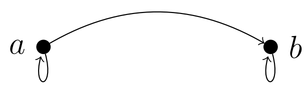

Figure 32: Connected graph D is not Eulerian.

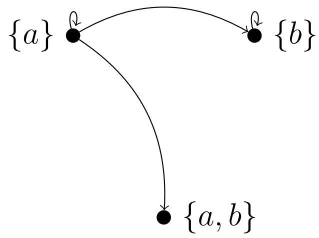

Figure 33: Connected power graph $P_1(D)$ is not Eulerian.

Example 3.11. *For the non-Eulerian graph D of this example (Figure 34), we have $V = \{a, b, c\}$ and $E = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, b)\}$.*

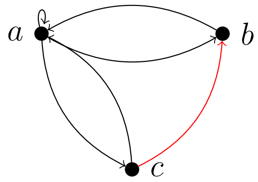

Figure 34: Connected graph D is not Eulerian.

$$E_1 = \{(\{a\}, \{a\}), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, \{b, c\}), (\{a\}, \{a, b, c\}), (\{b\}, \{a\}), (\{c\}, \{a\}), (\{c\}, \{b\}), (\{c\}, \{a, b\}), (\{a, b\}, \{a\}), (\{a, c\}, \{a\}), (\{a, c\}, \{b\}), (\{b, c\}, \{a\}), (\{a, b, c\}, \{a\})\}.$$

Non-Eulerian power graph $P_1(D)$ is shown in Figure 35.

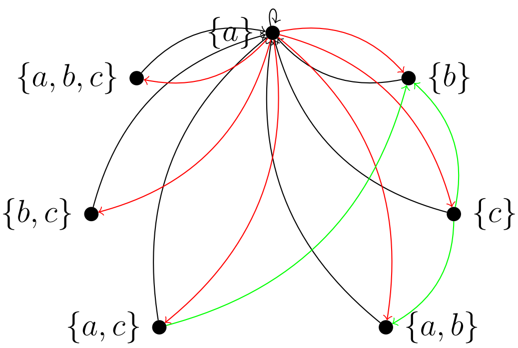

Figure 35: Connected power graph $P_1(D)$ is not Eulerian.

3.2. Analyzing whether power graph $P_2(D)$ is Eulerian if connected graph D is Eulerian or not:

In this section, first we determine the total number of edges that enter each vertex $P_2(D)$ and also the number of edges that exist of each vertex before analyzed the Eulerian of $P_2(D)$. We attention to that, in $P_2(D)$ with the number of $2^n - 1$ vertices, each member in the set V is in the number of 2^{n-1} vertices in the power graph $P_2(D)$. Figure 36, is presented a part of the Eulerian graph with n vertices.

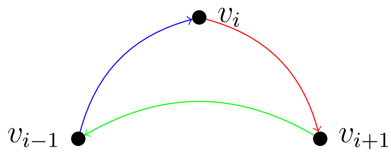

Figure 36: A part of the Eulerian graph D .

Where in $e_1 = (v_i, v_{i+1})$, $e_2 = (v_{i+1}, v_{i-1})$, $e_3 = (v_{i-1}, v_i)$, ... are in the set of edges E. Given that graph D is Eulerian, then we have $\Sigma_{v_1,...,v_n \in V} d^+(v_1, ..., v_n) = \Sigma_{v_1,...,v_n \in V} d^-(v_1, ..., v_n) \geq 1$. We assume that $\Sigma_{v_1,...,v_n \in V} d^+(v_1, ..., v_n) = \Sigma_{v_1,...,v_n \in V} d^-(v_1, ...$
$, v_n) = 1$. Via definition of E_2, one output edge from vertex v_i like e_1 in graph D, means that all vertices having a member of v_i are connected to all vertices that having a member of v_{i+1} in power graph $P_2(D)$. As it was mentioned above, there are the number of 2^{n-1} output edge from vertex $A = \{v_i\}$ to the number of 2^{n-1} vertices that having a member v_{i+1} $(*)$.
As the same way, one input edge to vertex v_i like e_3 in graph D, means that all vertices having a member of v_{i-1} are connected to all vertices that having a member of v_i, thus there are the number of 2^{n-1} output edges from the number of 2^{n-1} vertices that having a member v_{i-1} to vertex $A = \{v_i\}$ $(*')$. Therfore via $(*)$ and $(*')$ we have, $\Sigma_{v_i \in V} d^+(\{v_i\}) = \Sigma_{v_i \in V} d^-(\{v_i\}) = 2^{n-1}$. In the same way, Eulerian condition is established for all of the single-member vertices.
Now consider the vertex of two members like $A' = \{v_i, v_{i+1}\}$. According to the above description, for an edge e_1 in D there are the number of 2^{n-1} output edges from vertex A' to the number of 2^{n-1} vertices that having a member v_{i+1} and also for an edge e_2 in D, there are the number of 2^{n-1} output edges from vertex A' to the number of 2^{n-1} vertices that having a member v_{i-1}. Therefore the number of $2(2^{n-1})$ edges exist from vertex A'. In the same way, the number of $2(2^{n-1})$ edges enter to vertex A'. So $\Sigma_{v_i,v_{i+1} \in V} d^+(A') = \Sigma_{v_i,v_{i+1} \in V} d^-(A') = 2(2^{n-1})$. Thus Eulerian condition is established for all of the vertices with two members. In the same way, we have for vertex V with the number of n members $\Sigma_{v_1,...,v_n \in V} d^+(V) = \Sigma_{v_1,...,v_n \in V} d^-(V) = n(2^{n-1})$.
Note that, may there are multiple edges in one directions between some pairs of vertices two-member, three-member,..., n-member in $P_2(D)$, in obtaining, the number of input(output) edges for each vertex is calculated.
We now two examples are given to illustrate Euler's theorem for power graph $P_2(D)$. In the following examples graphs D are Eulerian. We determine whether their corresponding the power graphs are Eulerian? (see in Figures 38 and 41)
In these examples, considering to the lots of edges, we for a better understanding of Eulerian power graphs, the Eulerian condition is given for one of the vertices of $P_2(D)$; the results will be similar for other vertices.

Example 3.12. *Assumed that $V = \{a, b, c\}$ and $E = \{(a, b), (b, c), (c, a)\}$. As definition earlier,*

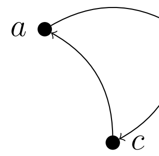

Figure 37: Connected graph D is simple cycle and no loop.

$$E_2 = \{(\{a\},\{b\}),(\{a\},\{a,b\}),(\{a\},\{b,c\}),(\{a\},\{a,b,c\}),(\{b\},\{c\}),(\{b\},\{a,c\}),(\{b\},\{b,c\}),(\{b\},\{a,b,c\}),(\{c\},\{a\}),(\{c\},\{a,b\}),\{c\},\{a,c\}),(\{c\},\{a,b,c\}),(\{a,b\},\{b\}),(\{a,b\},\{c\}),...,(\{a,b\},\{a,b,c\}),(\{a,c\},\{a\}),(\{a,c\},\{b\}),...,(\{a,c\},\{a,b,c\}),(\{b,c\},\{a\}),(\{b,c\},\{c\}),...,(\{b,c\},\{a,b,c\}),(\{a,b,c\},\{a\}),...,(\{a,b,c\},\{a,b,c\})\}.$$

Consider vertex $\{a,c\}$ of $P_2(D)$. By $(*)$ the number of $2(2^{n-1})=8$ edges exit from vertax $\{a,c\}$.

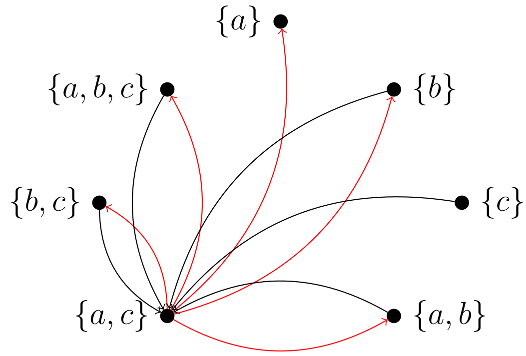

Figure 38: Connected power graph $P_2(D)$ is Eulerian.

Via $(*')$ the number of $2(2^{n-1})=8$ edges enter to vertax $\{a,c\}$. Therefore, $\Sigma_{a,c\in V} d^+(\{a,c\}) = \Sigma_{a,c\in V} d^-(\{a,c\}) = 8$. in the same way, is clearly the summation of input degrees is equal to that of the output degrees in every vertex of $P_2(D)$ and therefore $P_2(D)$ is Eulerian. As a reminder, there are multiple edges in one directions between pairs of vertices two-members, three-members in $P_2(D)$, so that the number of these input(output) edges is calculated for each vertex, but in the analysis of Eulerian conditions and drawing edges in $P_2(D)$, we consider multiple edges in one directions between two vertices only once. $P_2(D)$ is shown in Figure 38.

Example 3.13. *For the graph D of Figure 40, $V=\{a,b,c,d\}$ and $E=\{(a,b),(b,c),(c,d),(d,a)\}$.*

$$E_2 = \{(\{a\},\{b\}),(\{a\},\{a,b\}),...,(\{a\},\{a,b,c,d\}),(\{b\},\{c\}),(\{b\},\{a,c\}),...,(\{b\},\{a,b,c,d\})(\{c\},\{d\}),(\{c\},\{a,d\}),...,(\{c\},\{a,b,c,d\}),(\{d\},\{a\}),\{d\},\{a,b\}),...,(\{d\},\{a,b,c,d\}),...,(\{a,b,c,d\},\{a,b,c,d\})\}.$$

Eulerian conditions for the power graph $P_2(D)$ is drawn in the vertex $\{a,d\}$. In Figure 41, we see that $\Sigma_{a,d\in V} d^+(\{a,d\}) = \Sigma_{a,d\in V} d^-(\{a,d\}) = 16$, (Note that, the multiple edges in one directions between two vertices are drawn only once.) In the same way, Eulerian condition is confirm for all of the vertices. Therefore $P_2(D)$ is Eulerian.

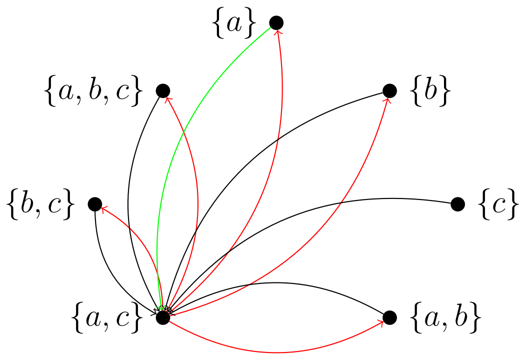

Figure 39: Connected power graph $P_2(D)$ is non-Eulerian.

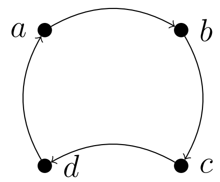

Figure 40: Connected graph D is simple cycle and no loop.

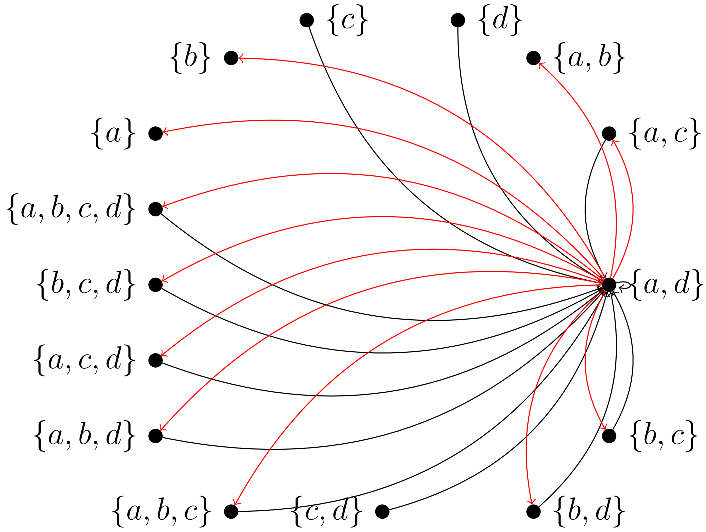

Figure 41: Connected power graph $P_2(D)$ is Eulerian.

Theorem 3.14. *Euler's theorem for power graph $P_2(D)$*
Consider a Eulerian graph D with $n \geq 3$ vertices.

1. *Let D with $n \geq 3$ vertices is a simple cycle graph and no loops or exactly n loops. Then $P_2(D)$ is Eulerian.*
2. *Let the number of loops in simple cycle graph D is l, that $1 \leq l \leq n-1$. Then power graph $P_2(D)$ is not Eulerian.*

Proof .

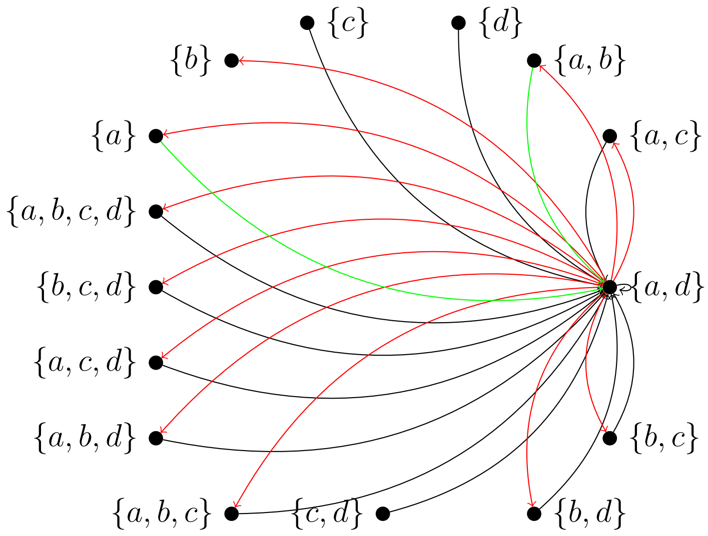

Figure 42: Connected power graph $P_2(D)$ is non-Eulerian.

1. *According to the theorem's assumption, graph D is Eulerian. Therefore, there is a directed Euler tour in the graph D and since D is a simple cycle graph we have, $v_i \to v_{i+1} \to \ldots \to v_{i-1} \to v_i$. first, we assumed D is with no loop. If a graph be Eulerian and simple cycle graph, all of its vertices would have the same conditions. In other word, $\Sigma d^+(v_i) = \Sigma d^-(v_i) = 1$. This reasoning is also true for all vertices of $P_2(D)$. Therefore, the summation of input degrees is equal to that of the output degrees in every vertex of $P_2(D)$ and power graph $P_2(D)$ is Eulerian. Now assumed that graph D has exactly n loops, so all of the connections one-way in $P_2(D)$ become two-way and being Eulerian remains.*
2. *Now if graph D has 1 to $n-1$ loops. Consider a vertex of the power graph with more than one member and fewer than n members, so that every member of vertex is a trail; (Eulerian conditions satisfy on the singleton vertices and the n-member vertex). Considering to the lots of edges, we analyze the Eulerian conditions for one of the vertices of $P_2(D)$ like vertex $B = \{v_i, v_{i+1}\}$. According to the definition of E_2, an edge leaves vertex $\{v_i\}$ and enters vertex $\{v_i, v_{i+1}\}$. Also, an edge leaves vertex $\{v_i, v_{i+1}\}$ and enters vertex $\{v_{i+1}\}$. (other connections between vertices and vertex B are two-way). Therefore, equation condition $\Sigma_{v_i,v_{i+1}\in V} d^+(B) = \Sigma_{v_i,v_{i+1}\in V} d^-(B)$ satisfy before the loop is added. First we add the loop (v_i, v_i) to graph D. Then, one new edge leaves vertex $\{v_i, v_{i+1}\}$ and enters vertex $\{v_i\}$ and causes $\Sigma_{v_i,v_{i+1}\in V} d^+(B) \neq \Sigma_{v_i,v_{i+1}\in V} d^-(B)$ in power graph $P_2(D)$. It is necessary to mention that in every vertex of $P_2(D)$ like B, it may be more than one new edge leaves vertex B or enters vertex B that causes the conditions of Euler's theorem are disturbed. Now if loop (v_{i+1}, v_{i+1}) is also added to the graph, with similar explanations on more vertices of the power graph does not satisfy Eulerian condition and as the same way, until we add the number of $n-1$ loops. Likewise, Euler's theorem conditions are disrupted when every loop ($n-1 \leqslant l \leqslant 1$) is added to graph D; thus power graph $P_2(D)$ become non-Eulerian. But when we add exactly n loops, all of the connections in $P_2(D)$ become two-way.*

□

Remark 3.15. *Note that in the simple cycle graph D with $n = 2$ vertices and with any number of loops, power graph $P_2(D)$ is Eulerian.*

We attention two examples of these graphs:

Example 3.16. *Let us suppose that $D = (V, E)$, where $V = \{a, b\}$, $E = \{(a, b), (b, a)\}$ and*

$$E_2 = \{(\{a\}, \{b\}), (\{a\}, \{a, b\}), (\{b\}, \{a\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a\}), (\{a, b\}, \{b\})\}.$$

In this example we have see $P_2(D)$ in Figure 44.

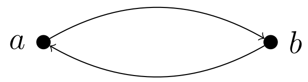

Figure 43: Connected graph D is Eulerian.

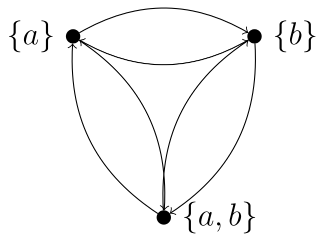

Figure 44: Connected power graph $P_2(D)$
is Eulerian.

Example 3.17. *Our second example $D = (V, E)$ where $V = \{a, b\}$, $E = \{(a, a), (a, b), (b, a)\}$ (see Figure 45) and $E_2 = \{(\{a\}, \{a\}), (\{a\}, \{b\}), (\{a\}, \{a, b\}), (\{b\}, \{a\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a\}), (\{a, b\}, \{b\})\}$. In this example, drawing of $P_2(D)$ is shown in Figure 46.*

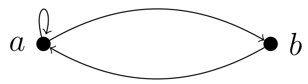

Figure 45: Connected graph D is Eulerian .

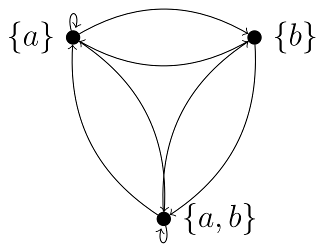

Figure 46: Connected power graph $P_2(D)$ is Eulerian.

Example 3.18. *In this example, the given graphs are Eulerian, but their corresponding power graphs are non- Eulerian. Considering to the lots of edges, the Eulerian condition is given for one of the vertices of $P_2(D)$ in the Figures 49 and 51; the results will be similar for other vertices.*

1. *Here is an example that graph D with $n = 6$ vertices is Eulerian, but its corresponding power graph is non-Eulerian. (Graph D is shown in Figur47)*

$$V = \{a, b, c, d, e, f\}$$
$$E = \{(a, c), (a, d), (b, a), (c, b), (c, f), (d, c), (d, e), (e, f), (f, d), (f, a)\}$$

Consider the vertex $\{c, d\}$ in power graph $P_2(D)$. The following edges enter to vertex $\{c, d\}$ and exits from $\{c, d\}$:

(a) *There are the number of $3(2^{n-1}) = 3(2^5) = 96$ input edges from vertices having a member a and a member d and a member f to vertex $\{c, d\}$ (*′). So $\Sigma_{c,d\in V} d^+\{c, d\} = 96$.*

(b) *There are the number of $4(2^{n-1}) = 4(2^5) = 128$ output edges from vertex $\{c, d\}$ to all of the vertices having a member b and a member f and a member c and a member e (*). Then $\Sigma_{c,d\in V} d^-\{c, d\} = 128$.*

Therefore, $\Sigma_{c,d\in V} d^+\{c, d\} \neq \Sigma_{c,d\in V} d^-\{c, d\}$, and $P_2(D)$ is non-Eulerian.

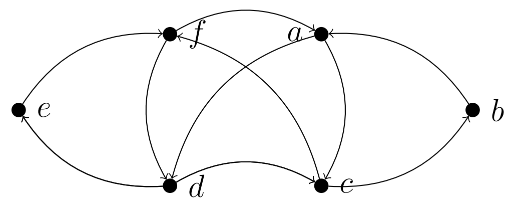

Figure 47: Connected graph D is Eulerian and cyclic.

2. *In this example consider $V = \{a, b, c, d\}$ and $E = \{(a, b), (b, c), (b, d), (c, b), (d, a)\}$, (see Figure 48). In Power graph $P_2(D)$ of Figure 49,the summation of input degrees is not equal to that of*

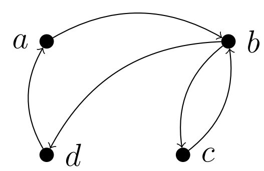

Figure 48: Connected graph D is Eulerian and cyclic.

the output degrees in some vertices. For example, $\Sigma_{a,c\in V} d^+(\{a, c\}) \neq \Sigma_{a,c\in V} d^-(\{a, c\})$.

3. *In the example, observe that power graph $P_2(D)$ is not Eulerian. Assume that $V = \{a, b, c, d\}$ and $E = \{(a, b), (b, c), (b, d), (c, b), (c, a), (d, c)\}$ (see Figure 50). It's corresponding power graph is shown in Figure 51.*

Theorem 3.19. *Let graph D with $n \geq 4$ is Eulerian and cyclic (not simple cyclic), then $P_2(D)$ is not Eulerian.*

Proof . *In Eulerian graph D, if $n = 2$, then D is simple cyclic and symmetric. If $n = 3$, then graph D is simple cyclic or symmetric. We have already mentioned that in simple cyclic graphs D, $P_2(D)$*

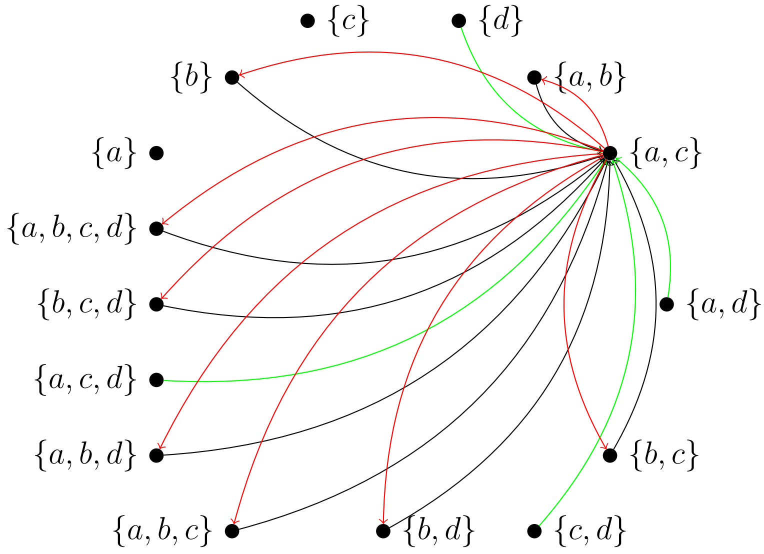

Figure 49: Connected power graph $P_2(D)$ is non-Eulerian.

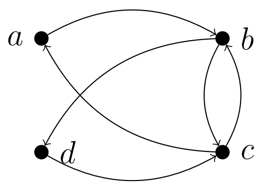

Figure 50: Connected graph D is Eulerian and cyclic.

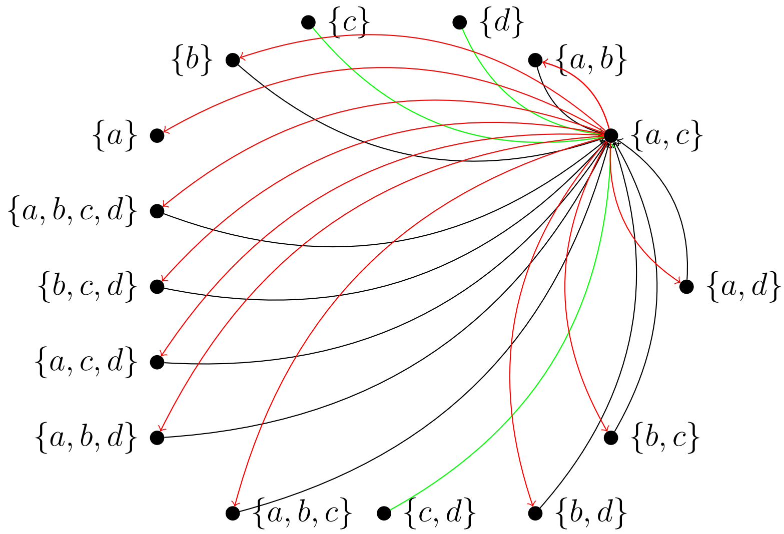

Figure 51: Connected power graph $P_2(D)$ is non-Eulerian.

is Eulerian. Also we will see in Theorem 3.22, that in symmetric graphs D, power graph $P_2(D)$ is Eulerian. So we assume that $n \geq 4$. We see part of the Eulerian graph with n vertices in Figure 52 .

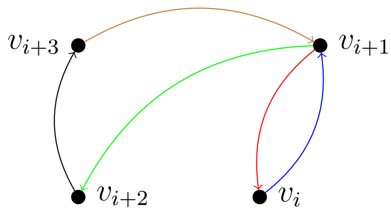

Figure 52: Part of the Eulerian graph D.

$E = \{e_1 = (v_i, v_{i+1}), e_2 = (v_{i+1}, v_i), e_3 = (v_{i+1}, v_{i+2}), e_4 = (v_{i+3}, v_{i+1}), e_5 = (v_{i+2}, v_{i+3}), ...\}$ *Now consider two members vertex like* $A'' = \{v_i, v_{i+2}\}$ *in* $P_2(D)$. *There are the number of* 2^{n-1} *input edges from all of the vertices that having member* v_{i+1} *to vertex* A'', *and there are the number of* 2^{n-1} *output edges from vertex* A'' *to all of the vertices that having member* v_{i+1}, *and all of the vertices that having member* v_{i+3}. *Thus, the number of* $2(2^{n-1})$ *edges exit from* A''. *We deduce that* $\Sigma_{v_i, v_{i+2} \in V} d^+(A'') \neq \Sigma_{v_i, v_{i+2} \in V} d^-(A'')$ *and then* $P_2(D)$ *is non-Eulerian.* □

Let's look at two examples here, in which the adjacency matrix is symmetric.

Example 3.20. *The graph* D *of Figure 53 has order 3 and size 4, where in matrix* $\mathbb{A}$ *shows that the symmetric matrix.*

$$\mathbb{A} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

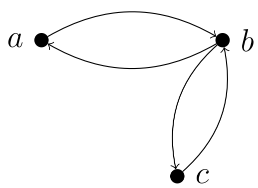

Figure 53: Connected graph D is Eulerian.

We now turn our attention to the drowing of power graph $P_2(D)$ *in Figure 54. We have drawn that the Eulerian condition on vertex* $\{b, c\}$.

Example 3.21. *Given an another example of a Eulerian graph (see Figure 55)that its matrix is symmetric. we have* $V = \{a, b, c, d\}$ *and*

$$\mathbb{A} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Consider a drawing of Eulerian condition of a power graph $P_2(D)$ *on vertex* $\{b, d\}$ *in the Figure 56.*

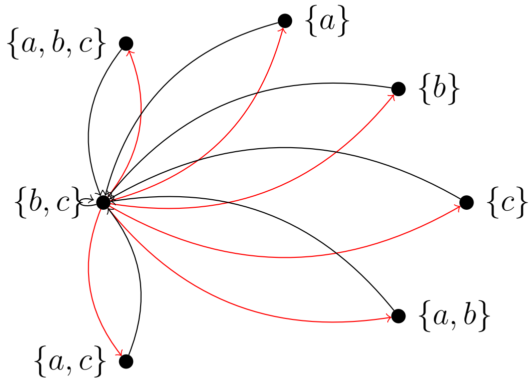

Figure 54: Connected power graph $P_2(D)$ is Eulerian.

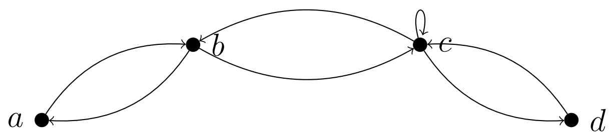

Figure 55: Connected graph D is Eulerian.

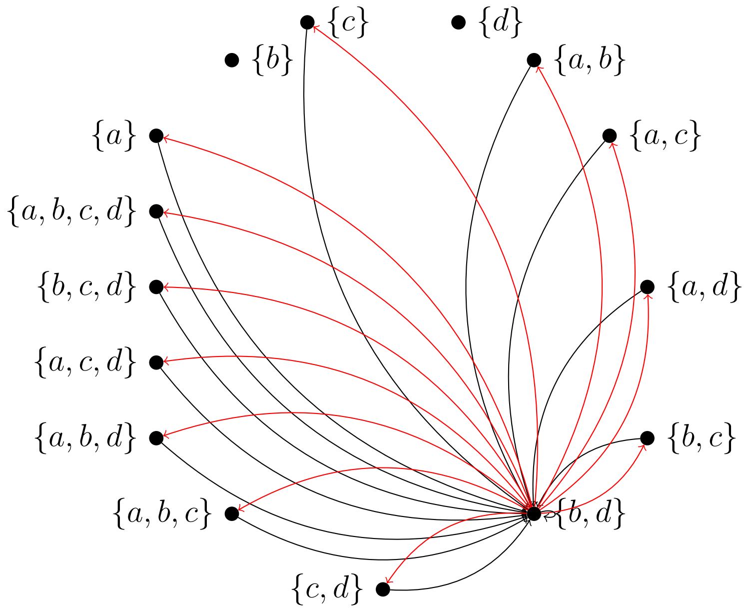

Figure 56: Connected power graph $P_2(D)$ is Eulerian.

Theorem 3.22. *Consider Eulerian graph D in which the adjacency matrix is symmetric. In that cace, $P_2(D)$ is Eulerian.*

Proof . *Since adjacency matrix in graph D is symmetric; In other word, there is edge (v_{i+1}, v_i) for every edge (v_i, v_{i+1}) in D. Thus by the definition of E_2, a two-way edge exists between two vertices in $P_2(D)$. In this case, , the summation of input degrees is equal to that of the output degrees in every vertex of $P_2(D)$ and power graph $P_2(D)$ is Eulerian.* $\square$

Example 3.23. *The graph D in the following Figure is example of non-Eulerian graph that its corresponding power graph is non-Eulerian and is shown in Figure 58.*

Figure 57: graph D is not Eulerian.

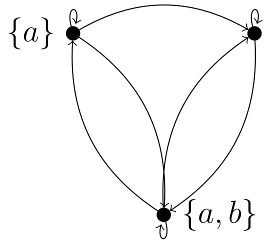

Figure 58: Connected power graph $P_2(D)$ is non-Eulerian.

Proposition 3.24. *Let D be a non-Eulerian graph. Then $P_2(D)$ is not Eulerian.*

Of course a special case which was discussed in example 3.25 an exception. Assumed that graph D is non-Eulerian, then its corresponding power graph is Eulerian.

Example 3.25. *The graphs D in this example are both non-Eulerian (see Figures 59 and 61), but their corresponding power graphs of $P_2(D)$, are Eulerian. (shown in Figures 60 and 62).*

1. *Give an example of a graph in which no two vertices are connected. we have $V = \{a, b\}$ and $E = \{(a, a), (b, b)\}$, (see Figure 59).*

Figure 59: Unconnected graph D is not Eulerian.

$$E_2 =\{(\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a\}), (\{a, b\}, \{b\}), (\{a, b\} , \{a, b\})\}$$

Eulerian power graph $P_2(D)$ is shown in Figure 60.

2. *In other example, $V = \{a, b, c\}$ and $E = \{(a, a), (b, b), (c, c)\}$, (see Figure 61).*
 $E_2 = \{(\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, \{a, b, c\}), (\{b\}, \{b\}), (\{a\}, \{a\})\}$.
 As we see in Figure 62, power graph $P_2(D)$ is Eulerian.

Theorem 3.26. *Assume that $E = I_v$. Then power graph $P_2(D)$ is Eulerian.*
Proof . *first we must say that power graph $P_2(D)$ is connected and then prove that it contains one Eulerian path. Given that $E = I_v$, so by the definition E_2, we can conclude that from each vertex*

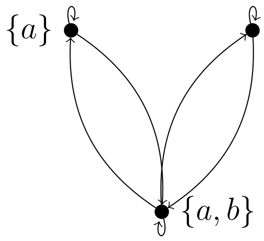

Figure 60: Connected power graph $P_2(D)$ is Eulerian.

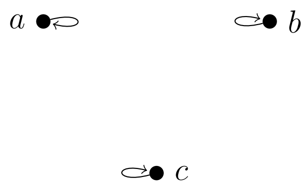

Figure 61: Unconnected graph D is not Eulerian.

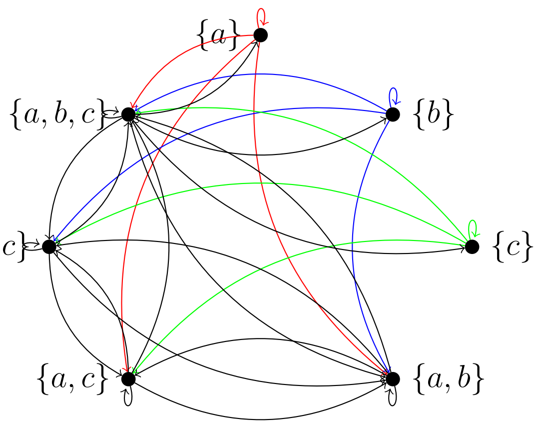

Figure 62: Connected power graph $P_2(D)$ is Eulerian.

in $P_2(D)$ an edge enters the vertex V and also there are an output edges from the vertex V to all of the vertices in $P_2(D)$; hence, power graph D_2 is evidently connected. At the same time, for each $(v_i, v_i) \in E$, there is a two-way relationship between all vertices that have $v_i \in V$ members, and that means, the summation of input degrees is equal to that of the output degrees in every vertex of $P_2(D)$. Then power graph $P_2(D)$ is Eulerian. $\square$

3.3. Analyzing whether power graph $P_i(D)$ is Eulerian (for $3 \leqslant i \leqslant 6$) if connected graph D is Eulerian or not:

We now illustrate theorem 3.28 with the following how many examples. The solution that we gave for example 3.27, show that there is only one solution for be Eulerian each of power graphs $P_i(D)$ that $3 \leqslant i \leqslant 6$. Let's consider some examples of this.

Example 3.27. *Which of the following graphs, its corresponding power graph is Eulerian? (for $3 \leqslant i \leqslant 6$)*

1. $D = (V, E)$, $V = \{a, b\}$ *and* $E = \{(a, a), (a, b), (b, a)\}$.
2. $D' = (V, E)$, $V = \{a, b, c\}$ *and* $E = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, a), (c, b), (c, c)\}$.
3. $D'' = (V, E)$, $V = \{a, b\}$ *and* $E = \{(a, a), (a, b), (b, a), (b, b)\}$.
4. $D''' = (V, E)$, $V = \{a, b, c\}$ *and* $E = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$.

Solution.

1. *Graph D is Eulerian, (Figure 63). Let $i = 4$. Therefore, $E_4 = \{(\{a\}, \{a\}), (\{a\}, \{b\}),$*

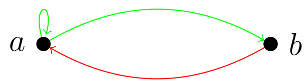

Figure 63: Eulerian graph D.

$(\{a\}, \{a, b\}), (\{b\}, \{a\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a\}), (\{a, b\}, \{a, b\})\}$.
Let's now turn our attention to the power graph $P_4(D)$ of Figure 64. According to the set of edges E, one member of V as $a \in V$ is connected to all members of V (including members $b \in V$ and itself from left). In this case, an edge enters all vertices $\{a\}$, $\{b\}$ and $\{a, b\}$ from vertex $\{a\}$. But one member of V as $b \in V$ is not connected itself. So, clearly by E_4 that no edges enter $\{b\}$ from vertex $\{a, b\}$ of power graph $P_4(D)$. While there is one edge from vertex $\{b\}$ to vertex $\{a, b\}$. Therefore,

$$\Sigma_{b \in V} d^+\{b\} \neq \Sigma_{b \in V} d^-\{b\}$$
$$\Sigma_{a,b \in V} d^+\{a, b\} \neq \Sigma_{a,b \in V} d^-\{a, b\}$$

Thus, $P_4(D)$ is not Eulerian. Or, pay attention to graph D' in Figure 65.

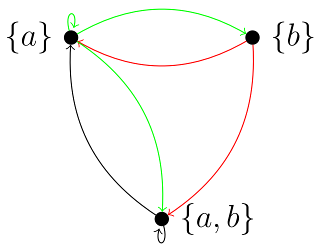

Figure 64: Power graph $P_4(D)$ is non-Eulerian.

2. *Graph D' is Eulerian. In this graph also, all members of V, are connected together, except one member of V as $b \in V$ that is not connected itself. So, by definition E_4, all vertices in power graph $P_4(D)$ have two-way connections except vertices having a member of $b \in V$. Therefore, is clearly no edges enter vertex $\{b\}$ from all of the vertices having member b. While there are edges from vertex $\{b\}$ to all of the vertices. Thus, the summation of input degrees is not equal to that of the output degrees in these vertices of $P_4(D)$. So $P_4(D)$ is not Eulerian. In Figure 66, we have analyzed that the Eulerian condition on vertex $\{b\}$.*

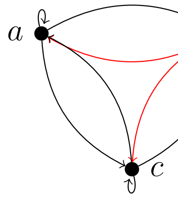

Figure 65: Eulerian graph D' .

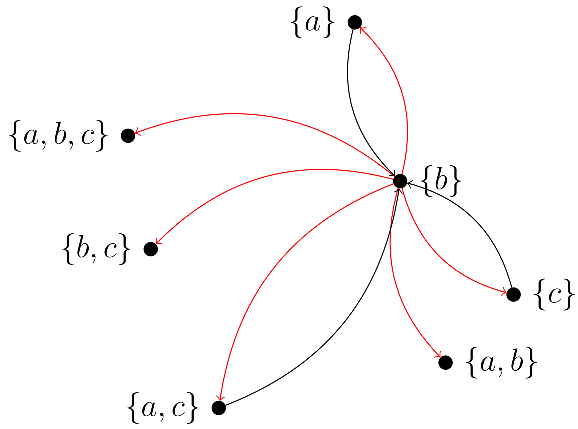

Figure 66: Power graph $P_4(D)$ is non-Eulerian.

3. *But in graph D if member $b \in V$ is also connected to itself, the graph D'' is obtained, then by E_4 there will be an edge from $\{a, b\}$ to $\{b\}$ in $P_4(D)$. Therefore,*

$$\Sigma_{b\in V} d^+\{b\} = \Sigma_{b\in V} d^-\{b\}$$
$$\Sigma_{a,b\in V} d^+\{a, b\} = \Sigma_{a,b\in V} d^-\{a, b\}$$

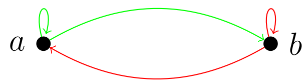

Figure 67: Graph D''.

This power graph is shown in Figure 68.

4. *Also, assumed that we add $(b, b) \in E$ to graph D'. The D''' is named for new graph (see in Figure 69). Therefore all vertices having a member of $b \in V$ are connected to vertex $\{b\}$ and the condition of Euler's theorem is established in power graph $P_4(D)$. (is shown in Figure 70)*

As we have seen, each member of V must be linked to all members of V in the set of edges E. Then $P_4(D)$ is definitely Eulerian.
Now assumed that $i = 3, 5, 6$. For power graphs $P_3(D)$, $P_5(D)$ and $P_6(D)$ are solotion in the same way as the power graph $P_4(D)$.

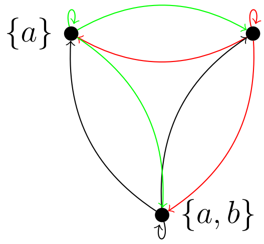

Figure 68: Power graph $P_4(D)$ is Eulerian.

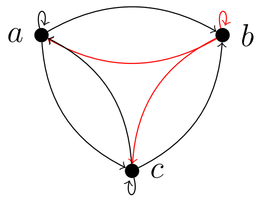

Figure 69: graph D'''.

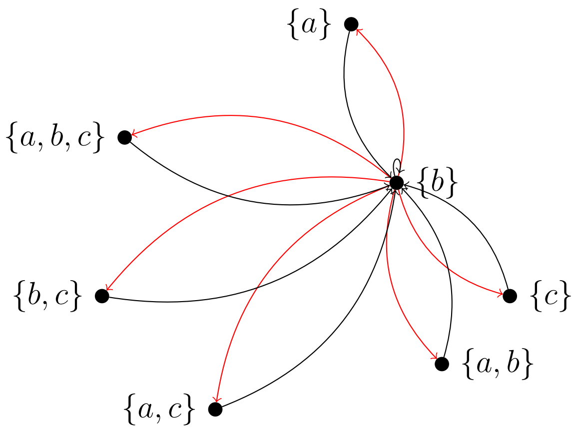

Figure 70: Power graph $P_4(D)$ is Eulerian.

Theorem 3.28. *Euler's theorem for power graph $P_i(D)$ that $3 \leq i \leq 6$.*
Assume that $P_i(D) = (V_i, E_i)$ is a power graph obtained from graph $D = (V, E)$. In that case, $P_i(D)$ is Eulerian if and only if $E = V \times V$.
(Note that if $E = V \times V$, then power graphs $P_1(D)$ and $P_2(D)$ are Eulerian).

The proof is given for one of the $P_i(D)$'s for $3 \leqslant i \leqslant 6$, the results will be similar for other power graphs.
Proof . *According to the assumption, graph D is Eulerian. Without missing out on the problem generality, assume that $E = V \times V \setminus (v_i, v_i)$. In this case, D is also clearly Eulerian. All vertices in power graph $P_4(D)$ have two-way connections except vertex $\{v_i\}$ and all vertices that have v_i members; however, there are edges from vertex $\{v_i\}$ to the vertices that have v_i members, and no edges enter vertex $\{v_i\}$ from those vertices. Hence, Euler's theorem conditions are not true on vertex $\{v_i\}$ and*

all vertices that have v_i members. In other word, power graph $P_4(D)$ shows:

$$\Sigma_{v_i \in V} d^+(\{v_i\}) \neq \Sigma_{v_i \in V} d^-(\{v_i\})$$
$$\Sigma_{v_i, v_j \in V} d^+(\{v_i, v_j\}) \neq \Sigma_{v_i, v_j \in V} d^-(\{v_i, v_j\})$$
$$.$$
$$.$$
$$.$$

Then $P_4(D)$ becomes non-Eulerian. however, as soon as (v_i, v_i) is added to E, edges enter vertex $\{v_i\}$ from all vertices that have v_i member. As a result, the summation of input and output degrees of vertex $\{v_i\}$ and those of vertices having v_i members will be equal, and power graph $P_4(D)$ becomes Eulerian. Now suppose that power graph $P_4(D)$ is Eulerian. Consider an edege $v_i \longrightarrow v_{i+1}$ of graph D. Clearly, there is an output edge from vertex $\{v_i\}$ to the number of 2^{n-1} vertices that having member v_{i+1} in power graph $P_4(D)$. Since $P_4(D)$ is Eulerian, so it must an edge exits from the other 2^{n-1} vertices in $P_4(D)$, and enters to the vertex $V = \{v_1, ..., v_n\}$. Of course, shows obviously that there is an input edge from vertex V to vertex $\{v_i\}$. By E_4, all the members in V are connected to v_i from left. This argument is for all of the vertices in power graph $P_4(D)$. Therefore all of the vertices V are connected together and then $E = V \times V$. □

Corollary 3.29. *If power graphs $P_i(D)$ for $3 \leqslant i \leqslant 6$ be Eulerian, then D is claerly Eulerian.*

Corollary 3.30. *If graph D is not Eulerian, is claerly none of the $P_i(D)$'s (for $3 \leqslant i \leqslant 6$) will be Eulerian.*

Corollary 3.31. *If any of the $P_i(D)$'s (for $3 \leqslant i \leqslant 6$) is not Eulerian, graph D may be Eulerian. Notice the graphs D and D' in example 3.27.*

4. Eulerian Analysis of the power graphs $P_i(D)$ and $P_j(D)$ for $1 \leqslant i, j \leqslant 6$; $i \neq j$

We see in this section that if power graph D_i be Eulerian, then which of the power graphs $P_j(D)$ are Eulerian? (for $1 \leqslant i, j \leqslant 6$; $i \neq j$). According to the theorems and results stated in the section 3, we can clearly get to the following results.

Firstly, we have assumed that $i = 1$ and $j = 2$.

Theorem 4.1. *If power graph $P_1(D)$ is Eulerian, then power graph $P_2(D)$ is Eulerian.*
Proof . *Let power graph $P_1(D)$ is Eulerian. Therefore, according to the conditions of being Eulerian for $P_1(D)$, the matrix adjacent to the graph D will be symmetric. So by theorem 3.22, power graph $P_2(D)$ is Eulerian.* □

Conversely this theorem is not true necessarily. For instance, we turn our attention to the examples 3.20 or 3.21. By the set of edges E, power graph $P_2(D)$ is Eulerian, but not $P_1(D)$. It must be noted that if power graph $P_1(D)$ is not Eulerian, power graph $P_2(D)$ can be Eulerian. In simple cycle graphs D with no loops (D is Eulerian), or symmetric graphs with no loops, $P_2(D)$ is definitely Eulerian; however, $P_1(D)$ is not Eulerian.

Corollary 4.2. *Assume that $P_2(D)$ is not Eulerian, so that $P_1(D)$ is definitely non-Eulerian.*

The following results for $i = 1$ and $j = 3, ..., 6$ is observed:
Assumed that $P_1(D)$ is Eulerian. Then according to the theorem 3.28, $P_j(D)$'s may not be Eulerian. But when $P_j(D)$'s are Eulerian, $P_1(D)$ is Eulerian. Now suppose that $P_1(D)$ is not Eulerian.It is claerly none of the $P_j(D)$'s will be Eulerian. But if any of the $P_j(D)$'s is not Eulerian, $P_1(D)$ may be Eulerian.
We have also $i = 2$ and $j = 3, ..., 6$. In this case, the results for $P_2(D)$ and $P_j(D)$'s, are exactly the same way as $P_1(D)$ and $P_j(D)$'s for $j = 3, ..., 6$.
For all $3 \leqslant i, j \leqslant 6$. As discussed earlier, power graphs $P_i(D)$ is Eulerian if $E = V \times V$; therefore, the fact that any of these power graphs is Eulerian can guarantee that other power graphs are Eulerian. Similarly, if they are not Eulerian, each of them states explicitly that others are not Eulerian, either.

5. Open problems

The abundance of unsolved power graph problems, a few of which are discussed in this section, indicates that analysis of this newly-emerged area with patience, accuracy, and enthusiasm as well as its many applications in medical, social, and economic sciences can lead to major breakthroughs in mathematics.

1. Assume that D is a simple graph. Under what conditions can $P_i(D)$'s also be simple for $1 \leqslant i \leqslant 6$.
2. Evidently, $deg(v) = id(v) + od(v) = deg^-_{(v)} + deg^+_{(v)}$ in graph D is true for every vertex in $P_i(D)$'s , $(1 \leqslant i \leqslant 6)$.
   (a) Determine the number of input/output edges of every vertex in $P_i(D)$'s for $1 \leqslant i \leqslant 6$ with respect to the corresponding graph D.
   (b) How are the total input/output degrees of every $P_i(D)$ related for $1 \leqslant i \leqslant 6$
   (c) If the input/output degrees of all members of V are a constant (k) in graph D, will there be an i, $(1 \leqslant i \leqslant 6)$ for which the input/output degrees of all members of $P_i(D)$ become a constant depending on k.
3. The maximum and minimum input/output degrees of D are shown as $\Delta^-(D)$, $\delta^-(D)$, $\Delta^+(D)$ and $\delta^+(D)$, respectively. How are the maximum and minimum input/output degrees of every $P_i(D)$ related for $1 \leqslant i \leqslant 6$.
4. A strongly directed graph is a directed graph with no loops in which there are no two edges in the same direction between two of its vertices. Determine how $P_i(D)$'s for $1 \leqslant i \leqslant 6$ can meet the condition for being a strongly directed graph.
5. Consider the power graphs in which there are no directed cycles. Prove that $\delta^-(P_i(D)) = 0$.
6. Evidently, being strongly connected is an equivalence relation on the set of vertices in graph D. Under what conditions is every $P_i(D)$ strongly connected for $1 \leqslant i \leqslant 6$.
   (Two vertices are strongly connected in D if one vertex is accessible through the other vertex. In other words, there should be a directed path between every two vertices.)
7. Shown as $D^{\leftarrow}$, the inverse version of graph D is a directed graph created by inverting the direction every arc. Which of the following equations is true for power graphs $(1 \leqslant i \leqslant 6)$.
   (a) $P_i(D) = P_i(D^{\Leftarrow})$
   (b) $\delta^-_{P_i(D)}(v) = \delta^+_{P_i(D^{\leftarrow})}(v)$
   (c) Determine the relationship between two strongly connected vertices in $P_i(D^{\leftarrow})$ related for $1 \leqslant i \leqslant 6$.
8. Determine the length of a directed path and that of a directed cycle in $P_i(D)$ for $1 \leqslant i \leqslant 6$.
9. How is the distance between two vertices in $P_i(D)$'s for $1 \leqslant i \leqslant 6$.

10. Assume that $D = (V, E)$ is a connected graph. Which $P_i(D)$'s is connected for $1 \leqslant i \leqslant 6$, and vice versa.
    (a) If graph D is unconnected, can any of $P_i(D)$'s be connected for $1 \leqslant i \leqslant 6$, and vice versa.
    (b) What is the necessary and sufficient condition for connectedness of every $P_i(D)$ for $1 \leqslant i \leqslant 6$.

    (The questions on the connectedness of power graphs were answered completely.)
11. If graph D is semi-Eulerian, which of $P_i(D)$'s will be semi-Eulerian for $1 \leqslant i \leqslant 6$. Under what conditions will every $P_i(D)$ be semi-Eulerian for $1 \leqslant i \leqslant 6$.
12. Assume that D is a Hamiltonian (semi-Hamiltonian) graph. Which of $P_i(D)$'s is Hamiltonian (semi-Hamiltonian) for $1 \leqslant i \leqslant 6$. On the contrary, which Hamiltonian $P_i(D)$'s can result in a Hamiltonian D.
    (a) If graph D is non-Hamiltonian, can any of $P_i(D)$ be Hamiltonian for $1 \leqslant i \leqslant 6$, and vice versa.
    (b) Analyze the necessary and sufficient condition for every $P_i(D)$ to be Hamiltonian (non-Hamiltonian) for $1 \leqslant i \leqslant 6$.
    (c) If power graph $P_i(D)$ is Hamiltonian, which $P_j(D)$ is Hamiltonian ($1 \leqslant i, j \leqslant 6$; $i \neq j$).
13. Analyze the homomorphism of power graphs for $1 \leqslant i \leqslant 6$.
14. How can every $P_j(D)$ be a complete power graph for $1 \leqslant i \leqslant 6$.
15. A directedness of a complete graph is called a tournament. Determine whether every tournament of a power graph for $1 \leqslant i \leqslant 6$ has a directed Hamiltonian path.
16. How can a tournament of power graphs for $1 \leqslant i \leqslant 6$ become a strongly connected tournament.
17. Consider the one-way directed graph D. What is the necessary and sufficient condition for every $P_i(D)$ for $1 \leqslant i \leqslant 6$. (The directed graph D is one-way if there is either a path from u to v or from v to u in D.)
18. Camion proved for the first time ever that every strongly connected tournament included a directed Hamiltonian cycle in D. Under what conditions can $P_i(D)$'s have such a directed Hamiltonian cycle for $1 \leqslant i \leqslant 6$.
19. Can the ancestral sets of power graphs be employed to analyze their relationships.
20. Can the ancestral graphs of power graphs be adopted to analyze their relationships.
21. What is the necessary and sufficient condition for every $P_i(D)$ to be a tree for $1 \leqslant i \leqslant 6$.
22. How the power graphs discussed in this article, are used in the social sciences.
23. If we consider the graph D as the personal preferences of the people in the set V, under what conditions which of the $P_i(D)$'s can be some kind of new preference on the power set V.
    This can have important applications in economics. For example in matching theory etc.
24. Is it possible to investigate subjects from microeconomics to macroeconomics with the help of the six definitions in this article?
25. Is it possible to analyze the properties of groups created in social networks with the help of the models presented in this article?

    Dear researchers, it is reiterated that all questions about the connectedness of power graphs mentioned in Item 10 were answered.

Conclusion

In this paper, we define power graphs related to a graph. The power graphs are new type of graphs based on six logical relationships, and the general conditions for Eulerian are presented as a few theorems. Given some applications of power graphs, efforts were made to introduce the area in which these graphs can be used. In addition to presenting other features of power graphs, future papers seek to discuss more accurate applications of these graphs in socioeconomic sciences.

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