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Generalized p-Laplacian systems with lower order terms

Farah Balaadich*, Elhoussine Azroul

University of Sidi Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mehraz, B.P. 1796 Atlas, Fez-Morocco

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Abstract

This work is devoted to studying the existence of solutions to systems of p-Laplacian type. We prove the existence of at least one weak solution, under some assumptions, by applying Galerkin's approximation and the theory of Young measures.

Keywords: Generalized p-Laplacian systems, Weak solutions, Young measures, Sobolev spaces, Galerkin method.
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1. Introduction

Let Ω be a bounded open domain of $\mathbb{R}^n$, $n \geq 2$. Let $\mathbb{M}^{m\times n}$ denotes the set of real m by n matrices equipped with the inner product $A : B = \sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ij}$. The set $\mathbb{M}^{m\times n}$ can be reduced as $\mathbb{R}^{mn}$ topology, which is, if $A \in \mathbb{M}^{m\times n}$, then $|A|$ is the norm of A when regarded as a vector in $\mathbb{R}^{mn}$. In [2] the following generalized p-Laplacian system was considered

$$\begin{cases} -\operatorname{div}\big(\Phi(Du - \Theta(u))\big) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f belongs to the dual space $W^{-1,p'}(\Omega; \mathbb{R}^m)$ of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ (p' is the conjugate exponent of p) and $\Phi(A) = |A|^{p-2}A$ being the p-Laplacian operator, for every $m \times n$ matrix $A \in \mathbb{M}^{m\times n}$ and for

*Corresponding author
Email addresses: balaadich.edp@gmail.com (Farah Balaadich), elhoussine.azroul@gmail.com (Elhoussine Azroul)

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some exponent $p > 1$. The term $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ was assumed to be a Lipschitz continuous function depending on solution itself and satisfy

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(\xi) - \Theta(\eta)| \leq c|\xi - \eta| \tag{1.2}$$

for all $\xi, \eta \in \mathbb{R}^m$ and c is a positive constant satisfying

$$c < \frac{1}{\operatorname{diam}(\Omega)} \Big(\frac{1}{2}\Big)^{\frac{1}{p}}. \tag{1.3}$$

The authors do not require the conditions of type Leray-Lions to prove their result. Moreover, they used the theory of Young measures as technical tools to prove the existence of weak solutions.

In this paper, for a function $u : \Omega \to \mathbb{R}^m$, we consider the following system

$$\begin{cases} -\operatorname{div}\big(\Phi(Du - \Theta(u))\big) + |u|^{p-2}u + g(x, u, Du) & = f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

which is a Dirichlet problem. Here, Θ satisfies (1.2) for some $p \in (1, \infty)$, $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$ and the constant c in (1.2) is related now to p by the following:

$$c < \Big(\frac{p}{2}\Big)^{\frac{1}{p}}.$$

The choice of c will serve us to get the coercivity of T defined in Section 3. The nonlinearity $g : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ satisfies the following conditions:
(H)(Continuity) g is a Carathéodory function, i.e. $x \mapsto g(x, s, A)$ is measurable for all $(s, A) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(s, A) \mapsto g(x, s, A)$ is continuous for almost every $x \in \Omega$. Moreover, we assume that g satisfies one of the following conditions:

(a) There exists $0 \leq d \in L^{p'}(\Omega)$ such that

$$|g(x, s, A)| \leq d(x) + |s|^{p-1} + |A|^{p-1},$$

$$g(x, s, A) \cdot s \geq 0.$$

(b) The function g is independent of the third variable, or, for almost every $x \in \Omega$ and all $s \in \mathbb{R}^m$, the mapping $A \mapsto g(x, s, A)$ is linear.

As example of a problem to which the present result can be applied, we give

$$-\operatorname{div}\big(\Phi(Du - \Theta(u))\big) + |u|^{p-2}u + \alpha(u)(d(x) + |Du|^{p-1}) = f,$$

where $0 \leq d \in L^{p'}(\Omega)$ and $\alpha(.)$ satisfy the classical sign condition and bounded from below by a positive constant.

In view of [17], our problem (1.4) is a nonlinear degenerate and singular elliptic system according to the cases $p > 2$ and $1 < p < 2$.

In literature, there have been intensive research activities for equations, or systems, of p-Laplacian singular or degenerate type. Consider first the case when $\Theta \equiv 0$. Several types of degenerate elliptic equations were studied by different methods in [22]. Problem (1.1), with $f = \mu$ an $\mathbb{R}^m$-valued Radon measure, is studied in [14] where the existence of a distributional solution is attained. Breit et al. [10]

present global estimates under minimal boundary regularity to the p-Laplacian system with the right-hand side in divergence form. Regularity results is achieved in [23] for the second order derivatives of the solution of nonlinear N-systems of p-Laplacian type in n space variables. In [12], the authors proved the existence and uniqueness of solutions that are high regular for the p-Laplacian system $-\nabla.(|\nabla u|^{p-2}\nabla u) = f$, with $p \in (1,2)$. Chabrowski [11] investigates the existence of a nontrivial solution of the degenerate equtaion $-D_i(a(x)D_iu) + \lambda u = K(x)|u^{p-2}u$. Nonlinear elliptic system of PDE's of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(Du) = 0, \quad \alpha = 1, 2, ..., m,$$

is treated in [19]. The authors showed local Lipschitz-continuity and regularity of weak solutions. Pucci and Servadei [21] considered p-Laplacian equations in the entire $\mathbb{R}^n$ of the form $\Delta_p u = g(x,u)$, $1 < p < n$, and established regularity and qualitative properties of the solutions. The main purpose of [20] is to analyse the interaction between the gradient term and the function f to obtain existence results for the quasilinear elliptic problem

$$\begin{aligned} -\Delta_p u &= \pm|\nabla u|^{\nu} + f(x,u) \quad \text{in } \Omega, \\ u \geq 0 \quad &\text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $p > 1$ and $0 < \nu \leq p$. Li and Haiyun [24] discussed the existence of solution for the equation

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + g(x,u) = f \quad \text{in } \Omega, \\ -\langle \nu, |\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x)) \quad \text{on } \partial\Omega, \end{cases}$$

with $f \in L^p(\Omega)$ and $\frac{2n}{n+2} < p < \infty$. Authors in [25] have extended the above problem to the following

$$\begin{cases} -\text{div}\big((c(x) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\big) + \epsilon|u|^{p-2}u + g(x,u) = f \quad \text{in } \Omega, \\ -\langle \nu, (c(x) + |\nabla u|^2)^{\frac{p-2}{p}}\nabla u\rangle \in \beta_x(u(x)) \quad \text{on } \partial\Omega. \end{cases}$$

Now, consider the case when the Lipschitz continuous term Θ is present. Problem (1.1) was discussed in [2] as mentioned before. Existence and uniqueness of entropy solutions ($f \in L^1(\Omega)$) to the nonlinear elliptic problems

$$\begin{cases} -\text{div}\,\Phi(Du - \Theta(u)) + \alpha(u) &= f \quad \text{in } \Omega, \\ \quad \Phi(Du - \Theta(u)) \cdot \eta + \gamma(u) &= g \quad \text{on } \partial\Omega, \end{cases}$$

are established in [1]. We refer the reader to [4] for an extension result of (1.1) to a general quasilinear elliptic operator in divergence form.

Problem (1.4) is motivated in the particular case where $p \equiv 2$. Some special parabolic cases, which are raised in many different physical contexts such as infiltration phenomena of a fluid in a partially saturated porous media, have been stated in [13].

It is our purpose in this paper to extend the problem (1.1) to a more general form that contains a nonlinearity $g(.)$ having the same sign of $s \in \mathbb{R}^m$. Furthermore, we use the theory of Young measures to achieve our result. To be more flexible with the utilization of the theory of Young measures, we refer the reader to see [3, 4, 5, 16, 7, 6].

As usual, we define weak solutions as follows.

Definition 1.1. *A weak solution of* (1.4) *is a function* $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

$$\int_{\Omega} \big(\Phi(Du - \Theta(u)) : D\varphi + |u|^{p-2}u \cdot \varphi + g(x,u,Du) \cdot \varphi\big)dx = \langle f, \varphi\rangle$$

holds for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Throughout this paper, $\langle .,. \rangle$ denotes the duality pairing of $W^{-1,p'}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p}(\Omega;\mathbb{R}^m)$ for some $p \in (1,\infty)$.

We shall prove the following existence theorem.

Theorem 1.2. *Suppose that Θ, g verify Eq.* (1.2) *and (H) respectively. If g satisfies either (H)(a) or (H)(b) then there exists in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ a weak solution to the problem* (1.4).

2. Notations and properties

Let Ω be a bounded open domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary condition $\partial\Omega$. Throughout this paper, $1 < p < \infty$. First we recall that, by the Poincaré and the Sobolev inequality, there exists a constant $\beta > 0$ such that

$$\max\left(\|u\|_p, \|u\|_{p^*}\right) \leq \beta \|Du\|_p \qquad \forall u \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{2.1}$$

Note that we write β, in general without further comment, to point to the use of (2.1) contrary to [2]. This relation and the Hölder inequality are central in this paper to establish the required estimates to prove the desired results.

Lemma 2.1. *Let $\xi, \eta \in \mathbb{R}^m$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi \cdot (\xi - \eta).$$

Proof . We consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $x \mapsto x^p - px + (p-1)$. We have

$$f(x) \geq \min_{y \in \mathbb{R}^+} f(y) = f(1) = 0 \qquad \text{for all } x \in \mathbb{R}^+.$$

Therefore, we take $x = \frac{|\eta|}{|\xi|}$ (if $|\xi| = 0$, the result is obvious) in the inequality above to get the result of the lemma by using the Cauchy-Schwartz inequality. □ By $C_0(\mathbb{R}^m)$ we denote the space of continuous functions $\varphi \in C(\mathbb{R}^m)$ satisfying $\lim_{|\lambda|\to\infty} \varphi(\lambda) = 0$. Its dual can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$, by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

Lemma 2.2 ([15]). *Let $\{z_j\}_{j\geq 1}$ be a measurable sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in C_0(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ for a.e. $x \in \Omega$.

Definition 2.3. *We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{z_k\}_k$.*

Remark 2.4. *(1) In [8], it is shown that for any Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ and $\{z_k\}_k$ generates the Young measure ν_x, we have*

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x,.) \rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda) d\nu_x(\lambda)$$

weakly in $L^1(\Omega')$ for all measurable $\Omega' \subset \Omega$, provided that the negative part $\varphi^-(x, z_k)$ is equiintegrable.

(2) The above results remains true if $z_k = Du_k$ for $u_k : \Omega \to \mathbb{R}^m$.

To conclude this section, we recall the following useful lemma:

Lemma 2.5 ([2]). *Let (u_k) be a bounded sequence in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Then the Young measure ν_x generated by Du_k has the following properties:*

(1) ν_x is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $x \in \Omega$.

(2) The weak L^1-limit of Du_k is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(3) ν_x satisfies $\langle \nu_x, id\rangle = Du(x)$ for almost every $x \in \Omega$.

3. Existence of weak solutions

3.1. Galerkin approximations and a priori estimates

To construct the approximating solutions, we will use the Galerkin method. To this purpose, we consider the operator $T : W^{-1,p'}(\Omega;\mathbb{R}^m) \to W_0^{1,p}(\Omega;\mathbb{R}^m)$ defined by

$$\langle T(u), \varphi\rangle = \int_\Omega \Big(\Phi(Du - \Theta(u)) : D\varphi + |u|^{p-2}u\cdot\varphi + g(x,u,Du)\cdot\varphi\Big)dx - \langle f, \varphi\rangle$$

for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Assertion 1: We claim that $T(u)$ is linear, well defined and bounded.
For arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear. By Hölder's inequality and (2.1) it follows that

$$\begin{aligned}|\langle T(u), \varphi\rangle| &\le \int_\Omega \big(|Du - \Theta(u)|^{p-1}|D\varphi| + |u|^{p-1}|\varphi| + |g(x,u,Du)||\varphi|\big)dx \\ &\qquad\qquad\qquad\qquad + \|f\|_{-1,p'}\|\varphi\|_{1,p} \\ &\le \Big(\int_\Omega |Du-\Theta(u)|^p dx\Big)^{\frac{1}{p'}}\|D\varphi\|_p + \beta\|u\|_p^{p-1}\|D\varphi\|_p \\ &\qquad\qquad + \beta\big\||g(x,u,Du)|\big\|_{p'}\|D\varphi\|_p + \|f\|_{-1,p'}\|\varphi\|_{1,p}.\end{aligned}$$

Since

$$\int_\Omega |g(x,u,Du)|^{p'}dx \le \int_\Omega \big(|d(x)|^{p'} + |u|^p + |Du|^p\big)dx < \infty$$

and

$$|a+b|^p \le 2^{p-1}(|a|^p + |b|^p) \qquad (p > 1), \tag{3.1}$$

it follows that

$$|\langle T(u), \varphi\rangle| \le C\|\varphi\|_{1,p} \qquad \forall \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$$

for some constant positive C. Hence T is well defined and bounded.

Assertion 2: We show that the restriction of T to a finite linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.
Let W be a finite subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that $\dim W = r$ and $(w_i)_{i=1}^r$ a basis of W. Let $(u_k = a_k^i w_i)$ be a sequence in W such that $u_k \to u = a^i w_i$ in W (with conventional summation). On

the one hand, $u_k \to u$ and $Du_k \to Du$ almost everywhere. By virtue of the continuity properties of Θ and g, it follows that

$$\begin{aligned}\Phi(Du_k - \Theta(u_k)) : D\varphi + |u_k|^{p-2}u_k \cdot \varphi + g(x, u_k, Du_k) \cdot \varphi \\ \longrightarrow \Phi(Du - \Theta(u)) : D\varphi + |u|^{p-2}u \cdot \varphi + g(x, u, Du) \cdot \varphi\end{aligned}$$

almost everywhere as $k \to \infty$. On the other hand, since $u_k \to u$ strongly in W,

$$\int_\Omega |u_k - u|^p dx \to 0 \quad \text{and} \quad \int_\Omega |Du_k - Du|^p dx \to 0.$$

Then according to [9] (Chapter IV, Section 3, Theorem 3) there exists a subsequence of (u_k) (still denoted by (u_k)) and $l_1, l_2 \in L^1(\Omega)$ such that

$$|u_k - u|^p \le l_1 \quad \text{and} \quad |Du_k - Du|^p \le l_2.$$

By using (3.1), we have

$$\begin{aligned}|u_k|^p = |u_k - u + u|^p &\le 2^{p-1}(|u_k - u|^p + |u|^p) \\ &\le 2^{p-1}(l_1 + |u|^p).\end{aligned}$$

Similarly,

$$|Du_k|^p \le 2^{p-1}(l_2 + |Du|^p).$$

Hence $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded by a constant denoted by C. Let Ω' be a measurable subset of Ω and let $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$. By (3.1) and the Hölder inequality, we get

$$\begin{aligned}\int_{\Omega'} |\Phi(Du_k - \Theta(u_k)) : D\varphi| dx &\le \int_{\Omega'} |Du_k - \Theta(u_k)|^{p-1} |D\varphi| dx \\ &\le \Big(\int_{\Omega'} |Du_k - \Theta(u_k)|^p dx \Big)^{\frac{1}{p'}} \Big(\int_{\Omega'} |D\varphi|^p dx \Big)^{\frac{1}{p}} \\ &\le 2^{\frac{(p-1)^2}{2}} \big(\underbrace{\|Du_k\|_p^p}_{\le C} + c^p \underbrace{\|u_k\|_p^p}_{\le C} \big)^{\frac{1}{p'}} \Big(\int_{\Omega'} |D\varphi|^p dx \Big)^{\frac{1}{p}}.\end{aligned}$$

By the growth condition in (H)(a) and Eq. (2.1),

$$\begin{aligned}\int_{\Omega'} |g(x, u, Du) \cdot \varphi| dx &\le \Big(\int_{\Omega'} \big(|d(x)|^{p'} + |u_k|^p + |Du_k|^p \big) dx \Big)^{\frac{1}{p'}} \Big(\int_{\Omega'} |\varphi|^p dx \Big)^{\frac{1}{p}} \\ &\le \beta \big(\|d\|_{p'} + \underbrace{\|u_k\|_p^p}_{\le C} + \underbrace{\|Du_k\|_p^p}_{\le C} \big) \Big(\int_{\Omega'} |D\varphi|^p dx \Big)^{\frac{1}{p}}.\end{aligned}$$

In the above two estimations, we have used Hölder's inequality. Since $\int_{\Omega'} |D\varphi|^p dx$ is arbitrary small if the measure of Ω' is chosen small enough, then the sequences $\big(\Phi(Du_k - \Theta(u_k)) : D\varphi\big)$ and $\big(g(x, u_k, Du_k) \cdot \varphi\big)$ are equiintegrable. Hence, the Vitali Convergence Theorem (see e.g. [18]) implies

$$\lim_{k\to\infty} \langle T(u_k), \varphi \rangle = \langle T(u), \varphi \rangle.$$

Assertion 3: We claim that T is coercive.
Indeed, let $\varphi = u$ in the definition of T, then

$$\begin{aligned}\langle T(u), u\rangle &= \int_\Omega \big(\Phi(Du-\Theta(u)) : Du + |u|^p + g(x,u,Du)\cdot u\big)dx - \langle f, u\rangle \\ &\geq \int_\Omega \big(\Phi(Du-\Theta(u)) : Du + |u|^p\big)dx - \|f\|_{-1,p'}\|u\|_{1,p}\end{aligned}$$

by the sign condition in (H)(a) and Hölder's inequality. On the other hand, by using Lemma 2.1,

$$\begin{aligned}\Phi(Du-\Theta(u)) : Du &= |Du-\Theta(u)|^{p-2}(Du-\Theta(u)) : Du \\ &= |Du-\Theta(u)|^{p-2}(Du-\Theta(u)) : (Du-\Theta(u)+\Theta(u)) \\ &\geq \frac{1}{p}|Du-\Theta(u)|^p - \frac{1}{p}|\Theta(u)|^p.\end{aligned}$$

As

$$\begin{aligned}\frac{1}{2^{p-1}}|Du|^p &= \frac{1}{2^{p-1}}|Du-\Theta(u)+\Theta(u)|^p \\ &\leq \frac{1}{2^{p-1}}\big[2^{p-1}\big(|Du-\Theta(u)|^p + |\Theta(u)|^p\big)\big] \qquad \text{(by (3.1))} \\ &= |Du-\Theta(u)|^p + |\Theta(u)|^p,\end{aligned}$$

thus

$$\Phi(Du-\Theta(u)) : Du \geq \frac{1}{p2^{p-1}}|Du|^p - \frac{2}{p}|\Theta(u)|^p.$$

Consequently, by the choice of the constant c, we deduce that

$$\begin{aligned}\langle T(u), u\rangle &\geq \frac{1}{p2^{p-1}}\int_\Omega |Du|^p dx - \frac{2}{p}\int_\Omega |\Theta(u)|^p dx + \int_\Omega |u|^p dx - \|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{1}{p2^{p-1}}\int_\Omega |Du|^p dx - \frac{2}{p}c^p\int_\Omega |u|^p dx + \int_\Omega |u|^p dx - \|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{1}{p2^{p-1}}\int_\Omega |Du|^p dx - \|f\|_{-1,p'}\|u\|_{1,p} \to \infty \quad \text{as } \|u\|_{1,p}\to\infty.\end{aligned}$$

Hence T is coercive.

Now, the problem (1.4) is equivalent to find $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\langle T(u), \varphi\rangle = 0 \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{3.2}$$

In order to find such a solution we apply a Galerkin scheme. Let $W_1 \subset W_2 \subset ... \subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional with the property that $\underset{k\geq 1}{\cup} W_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Let us fix k and assume that $\dim W_k = r$ and $w_1, ..., w_r$ is a basis of W_k. Then we define the map

$$S : \mathbb{R}^r \to \mathbb{R}^r, \quad a \mapsto \Big(\langle T(a^i w_i), w_j\rangle\Big)_{j=1,..,r}$$

for $a = (a^1, ..., a^r) \in \mathbb{R}^r$.

Lemma 3.1. *S is continuous and*

$$S(a)\cdot a \longrightarrow \infty \quad as \quad \|a\|_{\mathbb{R}^r} \to \infty.$$

Proof . Since T restricted to W_k is continuous by Assertion 2, S is continuous. Let be $a \in \mathbb{R}^r$ and $u = a^i w_i \in W_k$ (with the conventional summation). Then

$$S(a)\cdot a = \langle T(u), u\rangle$$

and $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p} \to \infty$. By Assertion 3, it follows that $S(a)\cdot a \to \infty$ as $\|a\|_{\mathbb{R}^m} \to \infty$. □ The properties of S allow us to construct our Galerkin approximations.

Lemma 3.2. (1) *For all $k \in \mathbb{N}$ there exists $u_k \in W_k$ such that*

$$\langle T(u_k), \varphi\rangle = 0 \quad \textit{for all}\, \varphi \in W_k. \tag{3.3}$$

(2) *The sequence (u_k) constructed in (1) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that*

$$\|u_k\|_{1,p} \le R \quad \textit{for all}\, k \in \mathbb{N}. \tag{3.4}$$

Proof . (1) Since by Lemma 3.1, $S(a)\cdot a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$, it follows the existence of $R > 0$ such that $S(a)\cdot a > 0$ for all $a \in \partial B_R(0) \subset \mathbb{R}^r$. According to the usual topological arguments [26, Proposition 2.8], we have that $S(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k \in \mathbb{N}$, there exists $u_k \in W_k$ such that

$$\langle T(u_k), \varphi\rangle = 0 \quad \text{for all}\ \varphi \in W_k.$$

(2) We have $\langle T(u), u\rangle \to \infty$ as $\|u\|_{1,p} \to \infty$ by Lemma 3.1, we deduce that there exists $R > 0$ with the property, that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,p} > R$. This gives a contradiction with the Galerkin approximations u_k which satisfy (3.3). Hence (u_k) is uniformly bounded. □

3.2. Passage to the limit

As stated in the introduction, we use the theory of Young measures to pass to the limit in the Galerkin approximating equations and to identify weak limits. Remark that, since (u_k) is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ by (3.4), it follows by Lemma 2.2 the existence of a Young measure ν_x generated by Du_k in $L^p(\Omega;\mathbb{M}^{m\times n})$ satisfying the properties of Lemma 2.5.

Lemma 3.3. *If $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, Θ and g satisfy the conditions* (1.2), *(H)(a) and (b), then for every $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$*

$$\int_\Omega \big(\Phi(Du_k - \Theta(u_k)) : D\varphi + |u_k|^{p-2}u_k\cdot\varphi + g(x,u_k,Du_k)\cdot\varphi\big)dx$$
$$\longrightarrow \int_\Omega \big(\Phi(Du-\Theta(u)) : D\varphi + |u|^{p-2}u\cdot\varphi + g(x,u,Du)\cdot\varphi\big)dx$$

as $k \to \infty$.

Proof . The proof will be devided into two cases which correspond to the cases of the condition (H). Let (u_k) be a sequence such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. By the compact embedding of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ into $L^p(\Omega;\mathbb{R}^m)$, we have $u_k \to u$ in $L^p(\Omega;\mathbb{R}^m)$ (for a subsequence). Let $E_{k,\epsilon} = \{x \in \Omega;\ |u_k - u| \ge \epsilon\}$. Hence

$$\int_\Omega |u_k - u|^p dx \ge \int_{E_{k,\epsilon}} |u_k - u|^p dx \ge \epsilon^p |E_{k,\epsilon}|.$$

Thus

$$|E_{k,\epsilon}| \le \frac{1}{\epsilon^p}\int_\Omega |u_k - u|^p dx \to 0 \quad \text{as } k \to \infty,$$

which implies that $u_k \to u$ in measure on Ω and almost everywhere. Take the continuity of Θ and the weak limit defined in Lemma 2.5 together with the equiintegrability of $(Du_k - \Theta(u_k))$ into consideration, we deduce

$$\begin{aligned} Du_k - \Theta(u_k) \rightharpoonup & \int_{\mathbb{M}^{m\times n}} (\lambda - \Theta(u)) d\nu_x(\lambda) \\ & = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - \Theta(u)\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) \\ & = Du - \Theta(u) \end{aligned}$$

weakly in $L^1(\Omega)$. Hence

$$\Phi(Du_k - \Theta(u_k)) \rightharpoonup \Phi(Du - \Theta(u)) \quad \text{weakly in } L^1(\Omega).$$

Now, let start with the case (H)(a). By similar argument in $E_{k,\epsilon}$, we can deduce that $u_k \to u$ and $Du_k \to Du$ almost everywhere. It follows by the continuity condition in (H) that

$$g(x, u_k, Du_k)\cdot\varphi \to g(x, u, Du)\cdot\varphi \quad \text{a.e.}$$

for arbitrary $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$. On the other hand, by the growth condition in (H)(a), $(g(x, u_k, Du_k)\cdot \varphi)$ is equiinetgrable (see Assertion 3 if necessary). Hence

$$g(x, u_k, Du_k)\cdot\varphi \to g(x, u, Du)\cdot\varphi$$

by the Vitali Convergence Theorem.
For the case (H)(b), if g is independent of the third variable, it is sufficient to use the same arguments as discussed above. Now, let $A \to g(x, u, A)$ be linear. Since $(g(x, u_k, Du_k))$ is equiintegrable, we deduce by Remark 2.4 that

$$\begin{aligned} g(x, u_k, Du_k) \rightharpoonup \langle \nu_x, g(x, u, .)\rangle & = \int_{\mathbb{M}^{m\times n}} g(x, u, \lambda) d\nu_x(\lambda) \\ & = g(x, u, .) o \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} \\ & = g(x, u, Du), \end{aligned}$$

by linearity of g. Hence the convergence in Lemma 3.3 follows. □

To complete the proof of Theorem 1.2, it is then sufficient to show that for any $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, $\langle T(u), \varphi\rangle = 0$ holds.

Let $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, since $\underset{k\ge 1}{\cup} W_k$ is dense in $W_0^{1,p}(\Omega; \mathbb{R}^m)$, there exists $(\varphi_k) \subset \underset{k\ge 1}{\cup} W_k$ such that

$\varphi_k \to \varphi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ as $k \to \infty$. Therefore

$$\begin{aligned}
&\langle T(u_k), \varphi_k\rangle - \langle T(u), \varphi\rangle \\
&= \int_\Omega \Big[\Phi(Du_k - \Theta(u_k)) : D\varphi_k - \Phi(Du - \Theta(u)) : D\varphi \\
&\qquad + |u_k|^{p-2}u_k \cdot \varphi_k - |u|^{p-2}u \cdot \varphi \\
&\qquad\qquad + g(x, u_k, Du_k) \cdot \varphi_k - g(x, u, Du) \cdot \varphi\Big] dx + \langle f, \varphi_k - \varphi\rangle \\
&= \int_\Omega \Big[\Phi(Du_k - \Theta(u_k)) : (D\varphi_k - D\varphi) + \big(\Phi(Du_k - \Theta(u_k)) - \Phi(Du - \Theta(u))\big) : D\varphi \\
&\qquad + |u_k|^{p-2}u_k \cdot (\varphi_k - \varphi) + \big(|u_k|^{p-2}u_k - |u|^{p-2}u\big) \cdot \varphi \\
&\qquad\quad + g(x, u_k, Du_k) \cdot (\varphi_k - \varphi) + \big(g(x, u_k, Du_k) - g(x, u, Du)\big) \cdot \varphi\Big] dx \\
&\qquad\qquad + \langle f, \varphi_k - \varphi\rangle \longrightarrow 0
\end{aligned}$$

as $k \to \infty$ by Lemma 3.3, i.e.

$$\lim_{k\to\infty} \langle T(u_k), u_k\rangle = \langle T(u), \varphi\rangle.$$

By virtue of Eq. (3.3), it follows that $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

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