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Action of topological groupoid on topological space

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Abstract

The main goal of this paper is investigating some types of topological groupoid and their action which denoted by LM- groupoid and M- groupoid. Some properties of these groupoid are written as proposition. We concentrated to research the relation of M- groupoid and LM- groupoid.

Keywords: groupoid, topological groupoid, principal groupoid, topological group, action of topological groupoid.
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1. Introduction

A cartan principal bundle is quadruple $\Gamma(F, \pi, N)$ where F and N are topological space,Γ is a group acting freely on the right of F, $F \times \Gamma \to F(z, r \to z.r)$ and $\pi : F \to N$ is surjective map subject to the following conditions:

i) The fibers of $\acute{\pi}$ equal to the orbits of Γ o equivalently, $\pi(z) = \pi(\acute{z}) \Leftrightarrow \exists r \in \Gamma$ such that $\acute{z} = z.r$.

ii) The map $T : F \times F \to \Gamma, (f, f.g) \to g$ is continuous.

iii) $\pi : F \to N$ is an identification map [1],[5]. A principal fiber bundle is quadruple $\Gamma(F, \pi, N)$ where F and N are topological space,Γ is a

iv) topological group acting freely on the right of $F, (F \times \Gamma \to F, (z, r) \to z.r)$ and $\pi : F \to N$ is continuous surjective map subject to the following conditions:

(i) The fibers of π equal to the orbits of Γ or equivalently $\pi : (z) = \pi(\acute{z}) \Leftrightarrow \exists r \in \Gamma$ such that $\acute{z} = z.r$.

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(ii) There is an open cover $\{\coprod_i \to N$ and continuous maps $\upsilon_i : \coprod_i \to N$ such that $\pi \circ \upsilon_i = I_{\coprod_i}$. A morphism of cartan principal bundles from $\Gamma(F, \pi, N)$ to $\acute{\Gamma}(\acute{F}, \acute{\pi}, \acute{N})$ is a trio of $f : F \to \acute{F}, g : N \to \acute{N}$ and $h : \Gamma \to \acute{\Gamma}$ where f, g are continuous map and h is a homomorphism of topological groups such that $\acute{\pi} \circ f = g \circ \pi$ and $f(z, r) = f(Z).h(r)$, for all $Z \in F, r \in \Gamma$ [6, 7]. An isomorphism of topological groups [4, 8].

A topological groupoid (M, N) is said to be LM- groupoid if for all $x \in N$, the map $\beta_x : M_x \to [x]$ is an identification map. A topological groupoid (M, N) is said to be M- groupoid if for all $x \in N$, the map $\delta_x : M_x \times M_x \to M; \delta \times (m_1, m_2) = m_1 m_2^{-1}$ is an identification map [2, 3].

2. The Results of M- groupoid and LM- groupoid

Proposition 2.1. *Let the map $\beta_x : M_x \to [x]$ is an open map then ${}_xM_x(M_x, \beta_x, [x])$ is bundle for every $x \in N$.*

Proof . The restriction of the law of composition β on $M_x \times_x M_x \subset M * M$ defines a law of continuous action of ${}_xM_x$ on M_x, $(m, r) \to mr$ which is free since if $mr = m$ then e is unity for each $m \in M$, m has unique right $w(\alpha(m))$ and unique left unity $w(\beta(m))$ where $(m, r) \in M_x \times_x M_x$. Now:

(i) If $m_1, m_2 \in \beta_x^{-1}(y)$, for all $y \in [x]$ then $\sum(m_1) = \sum(m_2) = (y, x)$ and $m_1^{-1}m_2 \in_x M_x$ but $m_1(m_1^{-1}m_2) = (m_1 m_1^{-1})_{m_2} = m_2$ therefore the fibers of β_x equal to the orbits of ${}_xM_x$.

(ii) The map $L_x : M_x \times_{[x]} M_x \to_x M_x, L_x(m, mr) = r$ is continuous since L_x is defined by the composition of continuous map: $M_x \times_{[x]} M_x \xrightarrow{inc} M_x \times_x M_x \times M_x \xrightarrow{\delta \times I_{m_x}} M \times M_x \xrightarrow{inc} M * M \xrightarrow{\gamma} M(m, mr) \to (m, mr) \to (m^{-1}, mr) \to r$ which takes values in M_x where $\acute{\delta}6/M_x$.

(iii) $\beta_x : M_x \to [x]$ is open map for all $x \in N$. Hence ${}_xM_x(M_x, \beta_x[x])$ is cartan principle bundle, for all $x \in N$.

□

Proposition 2.2. *Let ${}_xM_x(M_x, \beta_x[x])$ is cartan principle bundle for every $x \in N$ then for any $m \in {}_yM_{x,x}M_x(M_x, \beta_x[x])$ and ${}_yM_y(M_y, \beta_y[y])$ are isomorphic cartan principle bundles.*

Proof . M_x homeomorphic to M_y by Rm^{-1} and ${}_xM_x$ isomorphic to ${}_yM_y$ isomorphic to ${}_yM_y$ by $int(m)(h) = mhm^{-1}$ m for all $h \in_x M_x$ and since $m : x \to y$ then $[x] = [y]$. Now the following diagrams are commutative

Where $\acute{\gamma} = \gamma \backslash M_x \times_x M_x$ and $\breve{\acute{\gamma}} = \gamma \backslash M_y \times_y M_y$. Hence the maps Rm^{-1}, $I_{[x]}$ and $Int(m)$ represent an isomorphism of cartan principle bundles. □

Proposition 2.3. *Every transitive groupoid M is transitive LM- groupoid.*

Proof . Let M is transitive groupoid and $x \in N$, then consider the following commutative diagram:

In which δ_x, β and P_{r_1} are identification maps.

Hence $\beta_x : M_x \to N$ is an identification map, and then (M, N) is transitive LM- groupoid. □

Proposition 2.4. *Let (M, N) be transitive LM-groupoid then Ehresmannn groupoid $(M_x \times M_x/{}_xM_x, M_x/{}_xM_x)$ is isomorphic to (M, N), for all $x \in N$.*

$$
\begin{tikzcd}
M_x \arrow[r, "R_{m^{-1}}"] \arrow[d, "\beta_x"'] & M_y \arrow[d, "\beta_y"] \\
{[x]} \arrow[r, "I"'] & {[x]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M_x \times \arrow[d, "R_{m^{-1}}"', "\times"] & {}_xM_x \arrow[r, "\acute{\gamma}"] \arrow[d, "Int_{(m)}"] & M_x \arrow[d, "R_{m^{-1}}"] \\
M_y \times & {}_yM_y \arrow[r, "\acute{\acute{\gamma}}"'] & M_y
\end{tikzcd}
$$

$$
\begin{tikzcd}
M_x \times {}_xM_x \arrow[r, "\delta_x"] \arrow[d, "P_{r_1}"'] & M \arrow[d, "\beta"] \\
M_x \arrow[r, "\beta_x"'] & N
\end{tikzcd}
$$

$$
\begin{tikzcd}
M_x \times M_x \arrow[r, "\eta"] \arrow[d, "\delta_x"'] & M_x \times M_x / {}_xM_x \arrow[dl, shift right, "f"'] \\
M \arrow[ur, shift right, "f^{-1}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
M_x \arrow[r, "\pi"] \arrow[d, "\beta_x"'] & M_x/\,_xM_x \arrow[dl, shift right, "\eta_x"'] \\
{[x]} \arrow[ur, shift right, "\eta_x^{-1}"'] &
\end{tikzcd}$$

Proof . The map $\delta_x : M_x \times M_x \to M$ and $\eta : M_x \times M_x \to M_x \times M_x/_xM_x$ and both identification maps and content on the fibers of each other. Hence the dotted arrows in the following diagram

are exist and unique by the universal properly of identification map, and the map f is given by $f([(m_1, m_2)]) = m_1m_2^{-1}$ becomes homeomorphism.
Clearly $f, \eta_x)$ be an isomorphic of topological groupoids where $\eta_x : M_x/_xM_x \to [x], \pi : M_x \to M_x/_xM_x, \eta_x(\pi(m)) = \beta_x(m)$ for all $x \in N$.
Hence $M_x \times M_x/_xM_x$ is isomorphic to M in topological groupoid. □

Proposition 2.5. *Every transitive LM-groupoid is principal groupoid.*

Proof . Let (M, N) be transitive LG-groupoid then M is transitive and for all $x \in N$ the map $\beta_x : M_x \to N$ is identification since every G- groupoid is transitive LM-groupoid. Hence M is principal groupoid. □

Proposition 2.6. *Let Ehresmannn groupoid* $(M_x \times M_x/_xM_x, M_x/_xM_x)$ *is isomorphic to* (M, N)*, for all* $x \in N$ *then the map* $\beta_x : M_x \to [x]$ *is an open map.*

Proof . The restriction of the law of composition γ on $M_x \times_x M_x$ defines a law of continuous free action of $_xM_x$ on M_x. Hence we have an open continuous surjective map (identification map) $\pi : M_x \to M_x/_xM_x$ where $M_x/_xM_x$ has the identification topological over π. Now the maps $\pi : M_x \to M_x/_xM_x$ and $\beta_x : M_x \to [x]$ are both identification maps and constant on the fibers of each other. hence the dotted arrows in the following diagram are exist and unique by the universal property of identification map

and the map η_x is given by $\eta_x(\pi(m)) = \beta_x(m)$ hence we have $M_x/_xM_x$ homeomorphism to $[x]$ and consequently we have $\beta_x : M_x \to [x]$ is an open map for all $x \in N$ □

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4. Conclusions

We have studied the topological groupoid and their action which denoted by LM-groupoid and M-groupoid. We discussed properties of these groupoid written as proposition. In addition, we focused to search the relation between two types M-groupoid and LM-groupoid.

5. open problems

Some suggestions for future works are listed as follow:

1. Studying the theory of topological groups and their action on topological space.
2. Studying the theory of topological groupoid and their action on topological space.
3. The relation between topological groups and topological groupoids.
4. The equivalence between topological groups and topological groupoids.

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