Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 97-102
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5461

Z-prime gamma submodule of gamma modules

Ali Abd Alhussein Zyarah[a,*], Ahmed Hadi Hussain[b], Hayder Kadhim Zghair[c]

[a] Iraqi Ministry of Education, General Directorate of Education for the Holy Karbala, Karbala, Iraq
[b] Department of Energy Engineering, College of Engineering Al-Musayab, University of Babylon, Babil, Iraq
[c] Department of Software, Information Technology College, University of Babylon, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

Let R be a Γ-ring and ∂ be an $R\Gamma$-module. A proper $R\Gamma$-submodule. T of an $R\Gamma$-module ∂ is called Z-prime $R\Gamma$-submodule if for each $t \in \partial, \gamma \in \Gamma$ and $f \in \partial^* = Hom_{R_\Gamma}(\partial, R), f(t)\gamma t \in T$ implies that either $t \in T$ or $f(t) \in [T :_{R_\Gamma} \partial]$. The purpose of this paper is to introduce interesting theorems and properties of Z- prime $R\Gamma$-submodule of $R\Gamma$-module and the relation of Z-prime $R\Gamma$-submodule, which represents of generalization Z-prime R-submodule of R-module.

Keywords: Γ-ring, $R\Gamma$-module, $R\Gamma$-submodule, and prime $R\Gamma$-submodule.
2010 MSC: Please write mathematics subject classification of your paper here.

1. Introduction

The topic of a Γ-ring was introduced in 1964 by Nobusawa [4]. He considered a set of homeomorphisms of a module to another module, which as closed under the addition and subtraction defined naturally but has no more a structure of a ring since he cannot have defined the product. After that, Barnes in [4, 6] weakened the generalization of Nobusawa. Then, many papers studied the Γ-ring in several algebraic structures. In [3], Ameri and Sadeghi presented the concept of a gamma modules in R investigate at some such modules. In this regard, we investigate submodules and homomorphism of a gamma modules and give the related basic results of a gamma modules. In 2005, Tekır and Sengul [7] presented the concept of prime ΓM-submodules of ΓM-modules and discussed some interesting and useful properties. Also, Zyarah and al-Mothafar provided the defining the semiprime $R\Gamma$-submodule of $R\Gamma$-module and the relation of semiprime $R\Gamma$-submodule. With multiplication $R\Gamma$-modules [11]. Also, in another work [10], they introduced some results and properties of primary

*Corresponding author
Email addresses: aliziara107@gmail.com (Ali Abd Alhussein Zyarah), met.ahmed.hadi@uobabylon.edu.iq (Ahmed Hadi Hussain), hyderkadum8@gmail.com (Hayder Kadhim Zghair)

Received: May 2021 *Accepted:* August 2021

$R\Gamma$-submodule and the definition for primary radical of $R\Gamma$-submodule of $R\Gamma$-module besides some of its basic properties. In this paper, Z-prime $R\Gamma$-submodule of $R\Gamma$-module and are investigated the basic properties, some theorems, and propositions. In addition, the relation between Z- prime $R\Gamma$-submodule with other $R\Gamma$-modules is investigated.

2. Preliminaries

Definition 2.1. *[6] Let R and Γ be an additive abelian groups, so we'll consider R is a Γ-ring R, shortly (ΓR) if there exists a mapping $\hbar : R \times \Gamma \times R \to R$ such that for every $d_1, d_2, d_3 \in R$ and $\gamma, \delta \in \Gamma$, the following conditions are hold:*

i. $(d_1 + d_2)\gamma d_3 = d_1\gamma d_3 + d_2\gamma d_3$.

ii. $d_1(\gamma + \delta)d_3 = d_1\gamma d_3 + d_1\delta d_3$.

iii. $d_1\gamma(d_2 + d_3) = d_1\gamma d_2 + d_1\gamma d_3$.

iv. $(d_1\gamma d_2)\delta d_3 = d_1\gamma(d_2\delta d_3)$.

Definition 2.2. *[3] A left $R\Gamma$-module is an additive abelian group ∂ together with a mapping $\hbar :$ $R \times \Gamma \times \partial \to \partial$ such that for all $h, h_1, h_2 \in \partial$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $r_1, r_2, r_3 \in R$ the following hold:*

i. $r_3\gamma(h_1 + h_2) = r_3\gamma h_1 + r_3\gamma h_2$.

ii. $(r_1 + r_2)\gamma h = r_1\gamma h + r_2\gamma h$.

iii. $r_3(\gamma_1 + \gamma_2)h = r_3\gamma_1 h + r_3\gamma_2 h$.

iv. $r_1\gamma_1(r_2\gamma_2)h = (r_1\gamma_1 r_2)\gamma_2 h$, *aright $R\Gamma$-module is defined in analogous manner.*

Definition 2.3. *[7] A proper $R\Gamma - S.$ T of ∂ is called prime $R\Gamma$-submodule, shortly $(P.R\Gamma - S.)$ if for any an ideal J of ΓR and for any $R\Gamma - S.$ H of ∂, $J\Gamma H \subseteq T$ implies $H \subseteq T$ or $J \subseteq [T :_{R_\Gamma} \partial]$.*

Definition 2.4. *[12] Let T be a proper $R\Gamma - S.$ of a $R\Gamma$-module ∂. The $R\Gamma - S.$ T of ∂ is called that S-prime $R\Gamma - S.$, whenever $\varphi(K) \subseteq T$, for some K be a $R\Gamma - S.$ of ∂ and $\varphi \in End_{R_\Gamma}(\partial)$, implies that $K \subseteq T$ or $\varphi(\partial) \subseteq T$.*

Definition 2.5. *[1] An $R\Gamma$-module ∂ is called Jacobson radical, denoted by $J_\Gamma(\partial)$, by $J_\Gamma(\partial) = \sum\{Y | Y \text{ is } R_\Gamma - small\ R_\Gamma - submodule\ of\ \partial\}$.*

Definition 2.6. *[5] An $R\Gamma$-module ∂ is called $R\Gamma$-faithful if it's $R\Gamma$-annihilator is the zero ideal of a ΓR.*

Definition 2.7. *[8] An ideal A of a ΓR is called prime if for any ideals I and J of R, $I\Gamma J \subseteq A$ implies, either $I \subseteq A$ or $J \subseteq A$.*

Definition 2.8. *[2] Let ∂ be an $R\Gamma$-module. We said that ∂ is a multiplication $R\Gamma$-module if any proper $R\Gamma - S.$ T of ∂, then there exist any ideal I of ΓR such that $T = I\Gamma\partial$.*

3. Z-Prime $R\Gamma$-submodule of $R\Gamma$-modules

In this section, we introduced $Z-P.R\Gamma-S.$ of $R\Gamma$-modules some propositions, and theorems.

Definition 3.1. *A proper $R\Gamma-S.$ T of an $R\Gamma$-module ∂ is called $Z-P.R\Gamma-S.$ if for each $t\in\partial$, $\gamma\in\Gamma$ and $f\in\partial^*=Hom_{R_\Gamma}(\partial,R)$, $f(t)\gamma t\in T$ implies that either $t\in T$ or $f(t)\in[T:_{R_\Gamma}\partial]$.*

Remark and Example 3.2.

1. *Every Z-prime R-submodule is $Z-P.R\Gamma-S.$ but the converse isn't true in general, as in the following example:*
   Let Z be a Z_{2Z}-module, $\Gamma=2Z$ and $6Z$ be A proper $Z_{2Z}-S.$ of Z. Then $6Z$ is $Z-P.Z_{2Z}-S.$ of Z, since $\varphi\in Z^*=Hom_{Z_{2Z}}(Z,Z)=Z$ and $\varphi:Z\to Z$; $\varphi(a)=3a$, $a\in Z$ and so $\varphi(a)\gamma(a)\in 6Z$ also $\varphi(a)\in[6Z:_{Z_{2Z}}Z]=3Z$. But $6Z$ is not Z-prime of $Z-S.$ of Z, since $\varphi\in Z^*=Hom_Z(Z,Z)=Z$ and $\varphi:\ Z\to Z$; $\varphi(a)=3a, a\in Z$ and so $\varphi(a).a\in 6Z$ also $\varphi(a)\notin 6Z=[6Z:_Z Z]$.
2. *Every $P.R\Gamma-S.$ is $Z-P.R\Gamma-S.$, but the converse is not true in general, as in the following example: Let $\partial=Z_8$ be a Z_{2Z}-module, $\Gamma=2Z$ and $T=<\bar{4}>$ be a proper Z_{2Z}-submodule of Z_8. Then $<\bar{4}>$ is Z-prime Z_{2Z}-submodule, since $f\in Z^*=Hom_{Z_{2Z}}(z_8,Z)=0$ and so $f(a)\alpha a=0\in<\bar{4}>$ for all $a\in Z_8$ and $0\in[<\bar{4}>:_{Z_{2Z}}Z_8]$. But $<\bar{4}>$ is not prime Z_{2Z}-submodule, since $2\in 2Z, 2\in Z_8, 1\in Z$ such that $(1)(2)(2)\in<\bar{4}>$ but $2\notin<\bar{4}>$ and $2\notin[<\bar{4}>:_{Z_{2Z}}Z_8]$.*
3. *Let I be an ideal of a ΓR, then I be a Z-prime ideal if for every $r\in R, f\in R^*=Hom_{R_R}(R,R)$ such that $f(r)\gamma r\in I$ implies that either $r\in I$ or $f(r)\in I$.*

Lemma 3.3. *Let D and F be any two $R\Gamma-S.s$ of an $R\Gamma$-module ∂, if $[D:_{R_\Gamma}x]$ is a Z-prime ideal of a ΓR for each $x\in F$, then $[D:_{R_\Gamma}F]$ is a Z-prime ideal of a ΓR.*

Proof . Let $f\in R^*=Hom_{R_R}(R,R), b\in R$ such that $f(b)\alpha b\in[D:_{R_\Gamma}F]$ and so, $f(b)\alpha b\alpha u\in E$ for all $\alpha\in\Gamma, u\in D$, then

$$f(b)\alpha b\in[D:_{R_\Gamma}<u>] \tag{3.1}$$

But $[D:_{R_\Gamma}<u>]$ is Z-prime ideal, so either $f(b)\in[D:_{R_\Gamma}<u>]$ or $b\in[D:_{R_\Gamma}<u>]$. Thus for any $\alpha\in\Gamma, u\in D$, either $f(b)\alpha u\in D$ or $b\alpha u\in D$. Suppose that $f(b)\notin[D:_{R_\Gamma}F]$ and $b\notin[D:_{R_\Gamma}F]$, there exists $v,w\in F$ such that $f(b)\alpha v\notin D$ and $b\alpha v\notin D$. Hence $f(b)\notin[D:_{R_\Gamma}<v>]$ and $b\notin[D:_{R_\Gamma}<w>]$. But by (3.1), $f(b)\alpha b\in[D:_{R_\Gamma}<v>]$ which is a Z-prime ideal, hence $b\in[D:_{R_\Gamma}<v>]$. Thus $b\alpha v\in D$, similarly, $f(b)\alpha b\in[D:_{R_\Gamma}<w>]$ implies that $f(b)\alpha b\alpha w\in D$. On the other hand, by (3.1) $f(b)\alpha b\in[D:_{R_\Gamma}<v+w>]$, so either $f(b)\in[D:_{R_\Gamma}<v+w>]$ or $b\in[D:_{R_\Gamma}<v+w>]$. Hence either $f(b)\alpha<v+w>\in D$ or $b\alpha<v+w>\in D$, which means either $f(b)\alpha v+f(b)\alpha w=d_1\in D$ or $b\alpha v+b\alpha w=d_2\in D$. Then either $f(b)\alpha v-d_1=f(b)\alpha w\in D$ or $b\alpha w-d_2=b\alpha w\in D$, which is contradiction. Therefore either $f(b)\in[D:_{R_\Gamma}F]$ or $b\in[D:_{R_\Gamma}F]$. □

Proposition 3.4. *Let L be a $Z-P.R\Gamma-S.$ of an $R\Gamma$-module ∂ and T be a summand of ∂, then either $T\subseteq L$ or $T\bigcap L$ is a $Z-P.R\Gamma-S.$ of ∂.*

Proof . Let $f\in T^*=Hom_{R_\Gamma}(T,R)$ and $a\in T$ such that $f(a)\gamma a\in T\bigcap L$. Suppose that $T\not\subseteq L$, then $T\bigcap L$ be a proper $R\Gamma-S.$ of T. Suppose that $a\notin T\bigcap L$, since T be a summand of ∂ then there exist a projection $\rho:\partial\to T$ and $f:T\to R$ such that $f(a)\gamma a=f\circ\rho(a)\gamma a\in L,\gamma\in\Gamma$ and $a\notin L$. Then $f\circ\rho(a)\in[L:_{R_\Gamma}\partial]\subseteq[L:_{R_\Gamma}T]$, since L be a $Z-P.R\Gamma-S.$ of ∂. Thus $f(a)\Gamma T\subseteq L$ and $f(a)\Gamma T\subseteq T$, and therefore, $f(a)\in[L\bigcap T:_{R_\Gamma}T]$. □

Remark 3.5. *Let T be a $Z-P.R\Gamma-S.$ of $R\Gamma$-module ∂, then T is called P-Z-prime $R\Gamma-S.$, where $P = rad_\Gamma([T :_{R_\Gamma} \partial])$ and hence if $< 0 >$ is a $Z - P.R\Gamma - S.$ of ∂, then $< 0 >$ is $P = rad_\Gamma([0 :_{R_\Gamma} \partial]) = rad_\Gamma(ann_\Gamma(\partial))$-$Z - P.R\Gamma - S.$ of ∂.*

Proposition 3.6. *Let P be a Z-prime ideal of a ΓR and let n be a positive integer. T_i be a $P-Z-P.R\Gamma - S.$ of an $R\Gamma$-module ∂ such that $1 \leq i \leq n$. Then $\bigcap_{i=1}^n T_i$ is also $P - Z - P.R\Gamma - S.$ of ∂.*

Proof . Let $f \in \partial^* = Hom_{R_\Gamma}(\partial, R)$ and $x \in \partial$ such that $f(x)\gamma x \in \bigcap_{i=1}^n T_i$. It's clear that $P = rad_\Gamma([\bigcap_{i=1}^n T_i :_{R_\Gamma} \partial])$. Suppose that $x \notin \bigcap_{i=1}^n T_i$, then there exist $m \in \mathbb{Z}^+$ with $1 \leq m \leq n$ such that $x \notin T_m$. But $f(x)\gamma x \in T_m$ and T_m is a $P - Z - P.R\Gamma - S.$ of ∂. It follows that $f(x) \in P$ and hence $\bigcap_{i=1}^n T_i$ is a $P - Z - P.R\Gamma - S.$ of ∂. $\square$

Proposition 3.7. *Let T be a $R\Gamma - S.$ of an $R\Gamma$-module ∂ and let P be a prime ideal of a ΓR. If $[T :_{R_\Gamma} K] \subseteq P$ for each $R\Gamma-S.K$ of ∂ containing T properly $P \subseteq [T :_{R_\Gamma} \partial]$, then T be a $Z-P.R\Gamma-S.$ of ∂.*

Proof . Let $\xi \in \partial^* = Hom_{R_\Gamma}(\partial, R)$ and $t \in \partial$ such that $\xi(t)\gamma t \in T$. Suppose that $x \notin T$ and let $K = T+ < t >$ and so K $R\Gamma - S.$ properly containing T properly, but $\xi(t)\gamma K = \xi(t)\gamma T + \xi(t)\gamma < t >\subseteq T$. And hence $\xi(t) \in [T :_{R_\Gamma} K] \subseteq P \subseteq [T :_{R_\Gamma} \partial]$. Thus T be a $Z - P.R\Gamma - S.$ of ∂. $\square$

Proposition 3.8. *Let ∂_1 and ∂_2 be two $R\Gamma$-modules and $\partial = \partial_1 \bigoplus_\Gamma \partial_2$. If $T = T_1 \bigoplus_\Gamma T_2$ is a $Z - P.R\Gamma - S.$ of ∂, then T_1 and T_2 are a Z-prime $Z - P.R\Gamma - S.s$ of ∂_1 and ∂_2 respectively.*

Proof . To show that ∂_1 is a $Z - P.R\Gamma - S.$ of ∂_1. Let $f \in \partial_1^* = Hom_{R_\Gamma}(\partial_1^*, R)$, $t \in \partial_1$ and $\gamma \in \Gamma$ such that $(t)\gamma t \in T_1$, then $(f \circ \rho)(t, 0)\gamma(t, 0) \in T_1 \bigoplus_\Gamma T_2$, where $\rho : \partial_1 \bigoplus_\Gamma \partial_2 \to \partial_1$. Since T is a $Z - P.R\Gamma - S.$ of ∂, then either $(t, 0) \in T_1 \bigoplus_\Gamma T_2$ or $f(t) \in [T_1 \bigoplus_\Gamma T_2 :_{R_\Gamma} \partial_1 \bigoplus_\Gamma \partial_2]$. Thus either $t \in T_1$ or $f(t) \in [T_1 :_{R_\Gamma} \partial_1] \bigcap [T_2 :_{R_\Gamma} \partial_2]$ and $f(t) \in [T_1 :_{R_\Gamma} \partial_1]$. Therefore,$T_1$ is a $Z - P.R\Gamma - S.$ of ∂_1 and similarly to prove T_2 is a $Z - P.R\Gamma - S.$ of ∂_2. $\square$

Proposition 3.9. *Let ∂, ∂' be an $R\Gamma$-modules and $\varphi : \partial \to \partial'$ be an $R\Gamma$-epimorphism. If T is a $Z - P.R\Gamma - S.$ of ∂ and $Ker\ \varphi \subseteq T$, then $\varphi(T)$ is a $Z - P.R\Gamma - S.$ of ∂'.*

Proof . To show that $\varphi(T)$ is a proper $R\Gamma - S.$ of ∂'. Suppose that $\varphi(T) = \partial'$, since φ is an $R\Gamma$-epimorphism, then $\varphi(T) = \varphi(\partial)$ and $\partial = T + Ker\varphi$, but $Ker\varphi \subseteq T$, hence $T = \partial$ which is contradiction, since T is $Z - P.R\Gamma - S.$ of ∂. Now, we define $\psi \in (\partial')^* = Hom_{R_\Gamma}(\partial', R)$ and $w \in \partial'$, let $\psi(w)\gamma w \in \varphi(T)$, $\gamma \in \Gamma$ and, $w \notin \varphi(T)$. Since φ is an $R\Gamma$-epimorphism, then there exist $u \in \partial$ such that $\varphi(u) = w$ and $u \notin T$. Then $\psi(w)\gamma w = \psi(w)\gamma\varphi(u) \in \varphi(T)$ and $\varphi(\psi(w)\gamma(u)) \in \varphi(T)$, since $Ker\varphi \subseteq T$, then $\psi(w)\gamma(u) \in T$. Since T is a $Z - P.R\Gamma - S.$ of ∂ and $u \notin T$, then $\psi(w) \in [T :_{R_\Gamma} \partial]$. Thus $\varphi(\psi(w)\Gamma\partial) \subseteq \varphi(T)$ and $\psi(w)\Gamma\varphi(\partial) \subseteq \varphi(T)$, then $\psi(w) \in [\varphi(T) :_{R_\Gamma} \partial']$. Therefore $\varphi(T)$ is a $Z - P.R\Gamma - S.$ of ∂'. $\square$

Proposition 3.10. *Let ∂, ∂' be an $R\Gamma$-modules and $\varphi : \partial \to \partial'$ be an $R\Gamma$-monomorphism. If T' is a $Z - P.R\Gamma - S.$ of ∂' and $\varphi(\partial) \not\subset T'$, then $\varphi^{-1}(T')$ is a $Z - P.R\Gamma - S.$ of ∂.*

Proof . To show that $\varphi^{-1}(T')$ is a proper $R\Gamma - S.$ of ∂. Suppose that $\varphi^{-1}(T') = \partial$, let $x \in \partial$ and $x \in \partial^{-1}T'$, then $\varphi(\partial) \subseteq T'$ which is contradiction. Now, we define $f \in (\partial)^* = Hom_{R_\Gamma}(\partial, R)$ and $w \in \partial$. Suppose that $w \notin \varphi^{-1}(T')$ and $\gamma \in \Gamma$, then $\varphi(w) \notin T'$. Let $f(w)\gamma w \in \varphi^{-1}(T')$, then $\varphi(f(w)\gamma w) \in T'$ and $f(w)\gamma\varphi(w) \in T'$. Since φ is an $R\Gamma$-monomorphism, we put $\varphi^{-1}\varphi(w) = w$, then $f(\varphi^{-1}(\varphi(w)))\gamma\varphi(w) \in T'$ and $f\varphi^{-1}(\varphi(w))\gamma\varphi(w) \in T'$ is a $Z - P.R\Gamma - S.$ of ∂' and $\varphi(w) \notin T'$, then $f\varphi^{-1}(\varphi(w)) \in [T' :_{R_\Gamma} \partial']$. Thus $f(w)\Gamma\varphi(\partial) \subseteq f(w)\Gamma\partial' \subseteq T'$ and $f(w)\Gamma\partial \subseteq \varphi^{-1}(T')$, hence $f(w) \in [\varphi^{-1}(T') :_{R_\Gamma} \partial]$. Therefore $\varphi^{-1}(T')$ is a $Z - P.R\Gamma - S.$ of ∂. $\square$

Corollary 3.11. *Let T and L be a two $Z - P.R\Gamma - S.s$ of $R\Gamma$-module ∂ and $L \subseteq T$, then T is a $Z - P.R\Gamma - S.$ of ∂ if and only if T/L is a $Z - P.R\Gamma - S.$ of ∂/L [9].*

4. Z-Prime $R\Gamma - S.s$ of a Faithful Multiplication $R\Gamma$-modules

We present in this section Z-prime $R\Gamma - S.s$ of multiplication $R\Gamma$-modules and also give some examples, propositions and theorems of this.

Proposition 4.1. *Let T be a proper $R\Gamma - S.$ of cyclic faithful $R\Gamma$-module ∂. If T is a $Z - P.R\Gamma - S.$ of ∂, then T is a $P.R\Gamma - S.$ of ∂.*

Proof . Let $t \in \partial, k \in R$ and $\beta \in \Gamma$ such that $k\beta t \in T$ and $t \notin T$. Suppose that $\partial =< x >, x \in \partial$, then $t = x\beta r, r \in R$. Define $\eta : \partial \to R$ by $\eta(t) = \eta(k\beta x) = k$. Since ∂ is a faithful $R\Gamma$-module, then η is well-define and which implies that $\eta(t)\beta(t) \in T$ and $x \notin T$, since T is a $Z - P.R\Gamma - S.$ of ∂, then $\eta(t) \in [T :_{R_\Gamma} \partial]$. Thus $k \in [T :_{R_\Gamma} \partial]$ and therefore T is $P.R\Gamma - S.$ of ∂. □

Corollary 4.2. *Let T be a proper $R\Gamma - S.$ of a cyclic faithful $R\Gamma$-module ∂. If T is a $Z - P.R\Gamma - S.$ of ∂, then $[T :_{R_\Gamma} \partial]$ is a Z-prime ideal of a ΓR.*

Proposition 4.3. *Let T be a proper $R\Gamma - S.$ of a multiplication $R\Gamma$-module ∂. If $[T :_{R_\Gamma} \partial]$ is a Z-prime ideal of a ΓR, then T is a $Z - P.R\Gamma - S.$ of ∂.*

Proof . Let $f \in \partial^* = Hom_{R_\Gamma}(\partial, R), t \in \partial$ and $\gamma \in \Gamma$ such that $f(t)\gamma t \in T$, then $f(t)\Gamma < t >\subseteq T$ and $< t >= I\Gamma\partial$ for some an ideal I in a ΓR. Since ∂ is a multiplication $R\Gamma$-module and so $f(t)\Gamma I\Gamma\partial \subseteq T$, then $f(t)\Gamma I \subseteq [T :_{R_\Gamma} \partial]$ and $< f(t) > \Gamma I \subseteq [T :_{R_\Gamma} \partial]$. Now, we define $g : R \to R$, it's clear that $g \in R^*$. Now, $g(< f(t) >)\Gamma I \subseteq [T :_{R_\Gamma} \partial]$. Since $[T :_{R_\Gamma} \partial]$ s a Z-prime ideal of a ΓR, then $< f(t) >\subseteq [T :_{R_\Gamma} \partial]$ or $I \subseteq [T :_{R_\Gamma} \partial]$. If $< f(t) >\subseteq [T :_{R_\Gamma} \partial]$, then $f(t) \in [T :_{R_\Gamma} \partial]$. If $I \subseteq [T :_{R_\Gamma} \partial]$, then $< t >\subseteq T$ i.e., $t \in T$. Thus T is a $Z - P.R\Gamma - S.$ of ∂. □

Corollary 4.4. *Let T be a proper $R\Gamma - S.$ of a cyclic faithful $R\Gamma$-module ∂. Then $[T :_{R_\Gamma} \partial]$ is a Z-prime ideal of a ΓR if and only if T is a $Z - P.R\Gamma - S.$ of ∂.*

Proposition 4.5. *Let ∂ be a finitely generated multiplication $R\Gamma$-module. If I is a Z-prime ideal of a Γr such that $ann_{R_\Gamma}(\partial) \subseteq I$, then $I\Gamma\partial$ is a $Z - P.R\Gamma - S.$ of ∂.*

Proof . Let $f \in \partial^* = Hom_{R_\Gamma}(\partial^*, R), t \in \partial$ and $\gamma \in \Gamma$ such that $f(t)\gamma t \in I\Gamma\partial$, then $f(t)\Gamma < t >\subseteq I\Gamma\partial$. Since ∂ is a multiplication $R\Gamma$-module, then $< t >= A\Gamma\partial$ for some A be an ideal in a ΓR, and $f(t)\Gamma A\Gamma\partial \subseteq I\Gamma\partial$. Then $f(t)\Gamma A \subseteq I + ann_{R_\Gamma}(\partial) = I$ by [?]. Now, we define $g : R \to R$, it's clear that $g \in R^*$. Now, $g(f(t))\Gamma A \subseteq I$. Since I is a Z-prime ideal of a ΓR, then $f(t) \in I$ and $f(t) \in [I\Gamma\partial :_{R_\Gamma} \partial]$ or $A \subseteq I$ and $A\Gamma\partial \subseteq I\Gamma\partial$ also $< t >\subseteq I\Gamma\partial$. Thus $f(t) \in [I\Gamma\partial :_{R_\Gamma} \partial]$ or $t \in I\Gamma\partial$ and therefore, $I\Gamma\partial$ is a $Z - P.R\Gamma - S.$ of ∂. □

Proposition 4.6. *Let ∂ be a cyclic $R\Gamma$-projective $R\Gamma$-module.If T is a $Z - P.R\Gamma - S.$ of ∂, then T is a $S - P.R\Gamma - S.$ of ∂.*

Proof . Let $f \in End_{R_\Gamma}(\partial), w \in \partial$ and $\partial = R\Gamma w, \gamma \in \Gamma$ such that $f(w) \in T$ and $w \notin T$. Since ∂ is a cyclic $R\Gamma$-module, then there exist $h : R \to \partial$ define by $h(r) = r\gamma w$, for each $r \in R$. Since ∂ is projective $R\Gamma$-modules, then there an exist $R\Gamma$-homomorphism $\theta : \partial \to R$, such that $h \circ \theta = f$. Clearly $h \circ \theta \in End_{R_\Gamma}(\partial)$, $f(w) = h(\theta(w)) = \theta(w)\gamma w \in T$ since $\theta \in \partial^* = Hom_{R_\Gamma}(\partial^*, R)$ and T is a $Z - P.R\Gamma - S.$ of ∂, $w \notin T$, then $\theta(w)\Gamma\partial \subseteq T$. Now, $f(\partial) = (h \circ \theta)(\partial) = h(\theta(\partial)) = \theta(\partial)\Gamma\partial \subseteq T$ and therefore, T is a $S - P.R\Gamma - S.$ of ∂. □

Proposition 4.7. *Let ∂ be a cyclic $Rw3b_\Gamma$-projective $R\Gamma$-module and T be a proper $R\Gamma - S.$ of ∂, then the following are equivalent:*

1. *T is a $Z - P.R\Gamma - S.$ of ∂.*
2. *T is a $S - P.R\Gamma - S.$ of ∂.*
3. *T is a $P.R\Gamma - S.$ of ∂.*

5. Conclusions

In this paper, Z-prime $R\Gamma$-submodule of $R\Gamma$-module and are investigated the basic properties, some theorems, and propositions. In addition, the relation between Z- prime $R\Gamma$-submodule with other $R\Gamma$-modules is investigated.

References

[1] M.S. Abbas, H.R. Hassan and H.A. Abbas, *On ΓR-projective gamma modules*, Int. J. Algebra, 12(2) (2018) 53–60.

[2] M.S. Abbas, H.R. Hassan and H.A. Abbas, *ΓR-multiplication and ΓR-projective gamma Modules*, Int. J. Contemporary Math. Sci. 13(2) (2018) 87–94.

[3] R. Ameri and R. Sadeghi, *Gamma modules*, Ratio Math. 20(1) (2010) 127–147.

[4] W. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. 18(3) (1966) 411–422.

[5] A. ezaei and B. Davvaz, *Tensor product of gamma modules*, Afrika Mat. 26(7) (2015) 1601–1608.

[6] N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. 1(1) (1964) 81–89.

[7] U. Tekır, U. Sengül and G. Ziverbey, *On prime ΓM-submodules of ΓM-modules*, Int. J. Pure Appl. Math. 19(1) (2005) 123–128.

[8] S. Uddin and S. Islam, *Semi-prime ideals of gamma rings*, Ann. Pure Appl. Math. 1(2) (2012) 186–191.

[9] A.A.A. Zyarah and A.K.H. Alghafil, *General formula for particular solution to the ordinary differential equation by variation of parameters of k^{th} order*, J. Interdis. Math.24(4) (2021) 1–6.

[10] A.A.A. Zyarah and N.S. Al-Mothafar, *On primary radical $R\Gamma$-submodules of $R\Gamma$-modules*, J. Discrete Math. Sci. Crypt. 23(5) (2020) 1001–1007.

[11] A.A.A. Zyarah and N.S. Al-Mothafar, *Semiprime $R\Gamma$-submodules of multiplication $R\Gamma$-modules*, Iraqi J. Sci. 61(5) (2020) 1104–1114.

[12] A.A.A. Zyarah and N.S. Al-Mothafar, *On S-prime and S-semiprime $R\Gamma$-submodules of $R\Gamma$-module*, J. Phys. Conf. Ser. IOP Pub. (2021).