Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 115-125
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.19675.2094

Bifurcation analysis and chaos control of the population model with harvest

Özlem Ak Gümüş[a,*], A.George Maria Selvam[b], R.Dhineshbabu[b]

[a]Faculty of Arts and Sciences, Department of Mathematics, Adıyaman University, Adıyaman-02040, Turkey
[b]Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this article, we investigated the dynamic behavior of a discrete-time population model with the harvest. We give numerical simulation and chaos control by using the linear feedback control method.

Keywords: Discrete-time model, stability, bifurcation, chaos control, harvesting.
2010 MSC: 39A10, 39A28, 39A30, 39A33.

1. Introduction

The population models in ecology and biology can be expressed via continuous or discrete-time equations. These models are dynamic models with inputs, outputs and positive feedback's. Discrete-time models present more realistic approach if births and deaths occur within certain intervals of time in small population size [10]. It's an important topic to rigorously analyze of population dynamics with some effects [2, 8, 9, 17, 18]. One of the important factors affecting population dynamics is harvest. In recent years, the harvest effect is called as hydra effect which is an unexpected outcome [1, 11]. In studies population dynamics with harvest, it is possible to encounter unexpected results that affect population changes. In such cases, it becomes important to keep under control the population by feedback control technique [7, 15].

Density-dependent processes such as growth, survival, reproduction and movement are compensatory if their rates change in response to variation in population density (or numbers) such

*Corresponding author
Email addresses: akgumus@adiyaman.edu.tr (Özlem Ak Gümüş), agmshc@gmail.com (A.George Maria Selvam), vrdhineshbabu10@gmail.com (R.Dhineshbabu)

Received: January 2020 *Accepted:* June 2021

that they result in a slowed population growth rate at high densities and promote a numerical increase of the population at low densities. Compensatory density dependence is important to fisheries management because it operates to offset the losses of individuals [20].

Population dynamics describes the ways in which a given population grows and shrinks over time, as controlled by birth, death, and emigration or immigration. It is the basis for understanding changing fishery patterns and issues such as habitat destruction, predation and optimal harvesting rates. The population dynamics of fisheries is used by fisheries scientists to determine sustainable yields [12, 21, 24].

A fishery population is affected by three dynamic rate functions. These are birth rate (or recruitment), growth rate and mortality. Recruitment means reaching a certain size or reproductive stage. With fisheries, recruitment usually refers to the age a fish can be caught and counted in nets. Growth rate measures the growth of individuals in size and length. This is important in fisheries where the population is often measured in terms of biomass. Mortality includes harvest mortality and natural mortality. Natural mortality includes non-human predation, disease and old age. If these rates are measured over different time intervals, the harvestable surplus of a fishery can be determined. The harvestable surplus is the number of individuals that can be harvested from the population without affecting long term stability (average population size). The harvest within the harvestable surplus is called compensatory mortality, where the harvest deaths are substituting for the deaths that would otherwise occur naturally. Harvest beyond that is additive mortality, harvest in addition to all the animals that would have died naturally [13, 23]. The Beverton-Holt and the Logistic (or Ricker) ecological models are respectively suitable to define compensatory and over compensatory dynamics in the population [3, 4, 5, 16, 22, 25, 26].

The purpose of this paper is to investigate the stability and bifurcation analysis of the positive equilibrium point of the following model by adding harvest to the model without delay in [6]

$$N_{t+1} = rN_t + N_t f_\alpha(N_t) - eN_t = F(r, \alpha, e, N_t), \tag{1.1}$$

where, $f_\alpha(N_t)$ represents interactions (competitions) among mature individuals, e is the population surviving constant effort harvesting, and r is the growth rate such that $r \in (0, 1)$. The following assumptions are imposed on the function f:

(i) $f_\alpha(N_t)$ is sufficiently smooth on $[0, \infty)$ and f'_α continuous such that $f'_\alpha(N_t) < 0$ for $N_t \in [0, \infty)$; i.e., as the density increases, f decreases continuously.

(ii) $f(0)$ has a finite positive value.

$N_t f_\alpha(N_t)$ is one of the usual overcompensatory population maps like the scaled $N_t(\alpha - N_t)$. So, the remainder of the article is discussed by taking $f_\alpha(N_t) = (\alpha - N_t)$.

2. Stability Analysis of Equilibrium Points of Eq (1.1) and Period-Doubling Bifurcation.

We will begin by reviewing some definitions and theorems (see, for instance [14, 19]). For the first order difference equation as follows:

$$x(t + 1) = f(x(t)) \tag{2.1}$$

Definition 2.1. *x^* is defined as equilibrium point of Eq (2.1) if the following equality holds:*

$$x^* = f(x^*).$$

Theorem 2.2. *Let x^* be a positive equilibrium point of Eq (2.1). Assume that f' is a continuous and smooth on an open interval I containing equilibrium point. Then x^* is locally asymptotically stable if:*

$$|f'(x^*)| < 1,$$

and unstable if:

$$|f'(x^*)| > 1.$$

Definition 2.3. *$m > 1$ is defined as period for Eq (2.1) if:*

$$f^m(x^*) = x^* \text{ and } f^i(x^*) \neq x^*, \text{for } i = 1, 2, ..., m-1.$$

Theorem 2.4. *Let $O^+(b) = \{b = (x(0), x(1), ..., x(k-1))\}$ be a k cycle of a continuously differentiable function f. Then the following statements hold.*

(i) The k cycle $O^+(b)$ is attracting if

$$|f'(x(0)), f'(x(1)), ..., f'(x(k-1))| < 1.$$

(ii) The k cycle $O^+(b)$ is repelling if

$$|f'(x(0)), f'(x(1)), ..., f'(x(k-1))| > 1.$$

Theorem 2.5. *Suppose that for an equilibrium point x^* of Eq (2.1), $f'(x^*) = 1$. Then the following statements then hold:*

(i) If $f''(x^*) \neq 0$, then x^* is unstable.
(ii) If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.
(iii) If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.

Theorem 2.6. *Suppose that for an equilibrium point x^* of Eq (2.1), $f'(x^*) = -1$.Then the following statements then hold:*

(i) If $-2f'''(x^*) - 3[f''(x^*)]^2 < 0$, then x^* is asymptotically stable.
(ii) If $-2f'''(x^*) - 3[f''(x^*)]^2 > 0$, then x^* is unstable.

We then obtain the following theorem.

Theorem 2.7. *Let N^* be a positive equilibrium point of Eq (1.1). Then, N^* is a local asymptotic stable equilibrium point if*

$$0 < N^* \leq 2. \tag{2.2}$$

and N^* is an unstable equilibrium point if $N^* > 2$.

Proof . Let's consider (i) - (ii). We can say that F' is a continuous function. The linearized form of Eq (1.1) in a neighbourhood of N^* is given by:

$$u_{t+1} = F'(N^*)u_t, \quad F'(N^*) = 1 - N^*,$$

such that $u_t = N_t - N^*$. By applying Theorem 2, we find that:

$$-2 < -N^* < 0.$$

Also, from Theorem 6, we can easily seen $N^* = 2$ is asymptotically stable. Thus the inequality (2.2) is proved. □

Remark 2.8. *The equilibrium points of the model (1.1) with $f_\alpha(N_t) = (\alpha - N_t)$ are 0 and $N^* = r + \alpha - e - 1$.*

Now, let's give some definitions and theorems (see, for instance [14, 19]) for stability analysis of the 2 cycle solution and period doubling bifurcation.

Let consider functions of two variables of the form

$$G(x, \lambda) = f_\lambda(x) \tag{2.3}$$

where, $f_\lambda(x)$ is a C^∞ function of the variable x, for fixed λ. Also, G depends smoothly on λ. We will begin by reviewing some definitions and theorems.

Definition 2.9. *The Schwarzian derivative of a function f at x is*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right). \tag{2.4}$$

Theorem 2.10. *Let f_λ be one parameter family of functions and suppose that $f_\lambda(x_0) = x_0$ and $f'_\lambda(x_0) \neq 1$. Then there are intervals I about x_0 and N about λ_0 and a smooth functions $p : N \to I$ such that $p(\lambda_0) = x_0$ and $f_\lambda(p(\lambda)) = p(\lambda)$, Moreover, f_λ has no other fixed points in I.*

Lemma 2.11. *Suppose $f(x)$ has finitely many critical points and $Sf < 0$. Then f has only finitely many periodic points of period m for any integer m.*

Theorem 2.12. *(Period doubling bifurcations) Suppose*

(i) $f_\lambda(0) = 0$ *for all λ in an interval about λ_0.*
(ii) $f'_{\lambda_0}(0) = -1$.
(iii) $\frac{\partial (f_\lambda^2)'}{\partial \lambda}|_{\lambda=\lambda_0}(0) \neq 0$.

Then there is an interval I about 0 and a function $p : I \to \mathbb{R}$ such that

$$f_{p(x)}(x) \neq x \tag{2.5}$$

and

$$f^2_{p(x)}(x) = x. \tag{2.6}$$

Remark 2.13. *The following expressions confirm in addition to the hypothesis of Theorem 12.*

(i) $Sf_{\lambda_0}(0) \neq 0$. *Then the curve $\lambda = p(x)$ satisfies $p''(0) \neq 0$.*
(ii) This means that either $\lambda = p(x)$ is concave in the direction shown in to the right concave or in the opposite direction.
(iii) If we assume that $Sf_\lambda < 0$ for all λ near λ_0, then the family f_λ cannot have a "reverse" period doubling bifurcation at λ_0.

We then obtain the following theorems.

Theorem 2.14. *2 cycles solutions $N^*_{1,2}$ of Eq (1.1) are the asymptotic stability if*

$$2 < N^* < \sqrt{5}. \tag{2.7}$$

Proof . As N^* passes through 2, the equilibrium point becomes unstable and iterative paths are attracted to the fixed point at $N^* = r+\alpha-e-1$. We get the 2 cycles, if the equation $F^2_{r,\alpha,e}(N_t) = N_t$ is solved, then we find the equilibrium points of period two (2 cycles) that

$$N^*_{1,2} = \frac{r+\alpha-e+1 \pm \sqrt{(r+\alpha-e+1)^2 - 4(r+\alpha-e+1)}}{2}. \tag{2.8}$$

The following inequalities are defined for equilibrium point of period two founded.

(i) If $N^* > 2$, then we have two real roots.
(ii) If $N^* = 2$, then we have two equal real roots.

From Theorem 4, we get that

$$\left|(r+\alpha-e+1)^2 - 4(r+\alpha-e+1)\right| < 1.$$

By considering (i) and $N^* > 0$, we reach the required inequality. □

Theorem 2.15. *Suppose that the conditions (i)-(iii) in Theorem 12 are provided. There is an interval I about 0 and a function $p : I \to \mathbb{R}$ such that $f_{p(x)}(x) \neq x$ and $f^2_{p(x)}(x) = x$. We have the polynomial as follows:*

$$p(x) = \frac{x^2-x+1}{x-1}$$

Proof . Since $f_{p(x)}(x) \neq x$, we can see that $p(x) \neq x+1$. If $f^2_{p(x)}(x) = x$ is solved, we get the function $p(x)$. □

Remark 2.16. *When $N^* = \sqrt{5}$, in this case $\left[F^2_{r,\alpha,e}(N^*_1)\right]' = \left[F_{r,\alpha,e}(N^*_1)\right]' \left[F_{r,\alpha,e}(N^*_2)\right]' = -1$. Similarly, we get 2^2 cycles by solving $F^4_{r,\alpha,e}(N_t) = N_t$. It appears that there is a 2^2 cycles when $N^* > \sqrt{5}$ which is attracting for $\sqrt{5} < N^* < \lambda_1$. This process of double bifurcation continues with indefinite step. We can show the dynamical behavior by numerical simulations.*

3. Bifurcations Analysis and Chaos Control

In this section, to show the bifurcations analysis, chaos control, maximum lyapunov exponents and complex dynamics behavior of the population model, numerical simulations are carried out. We consider the dynamic behavior of a discrete-time one dimensional population model with harvest as follows:

$$N_{t+1} = rN_t + N_t f_\alpha(N_t) - eN_t \text{ with } f_\alpha(N_t) := \alpha - N_t, \tag{3.1}$$

where the parameters α, r and e are positive. We can see easily that for the positive equilibrium point N^* if r varies in the small neighborhood of PD_{N^*}, then period doubling bifurcation will appear in the population model (3.1) where

$$PD_{N^*} = \{(r,\alpha,e) : r = r_0 = 3-\alpha+e, r > 1, \alpha, e > 0\}.$$

Now we introduce the study of control strategy in order to move the unstable periodic orbits or the chaotic orbits towards the stable one. First, we apply the linear feedback control method [7] to the discrete one dimensional population model with harvest (3.1). For this, we assume that the discrete - time controller of (3.1) is defined by

$$N_{t+1} = rN_t + N_t(\alpha - N_t) - eN_t + S_t, \tag{3.2}$$

where $S_t = -p(N_t - N^*)$ is feedback controlling force, p stands for the feedback gains, and N^* is an unique positive equilibrium point of (3.1).

Example 3.1. *Let us consider* $\alpha = 1.3$, $e = 0.5$ *and* $r \in [2, 3.2]$ *with initial condition* $N_0 = 0.2$. *Then Equation (3.1) undergoes period doubling bifurcation which emerges from the equilibrium point* $N^* = 2$ *at the intrinsic growth bifurcation parameter* r. *The value of* r *varies in a small neighborhood of* $r_0 = 2.2$ *and the equation (3.1) dynamics converges to a period - 2 orbit. From Figure 1, we observe that the interior equilibrium point* N^* *of map (3.1) is stable for* $r < 2.2$ *and loses its stability through a period doubling bifurcation for* $r = 2.2$ *and for* $r > 2.2$. *There is a period doubling cascade in orbits of periods - 2, 4, 8, 16, 32 and non-periodic oscillations, that is usually referred to a chaos illustrated in Figures 2(c) and 2(d).*

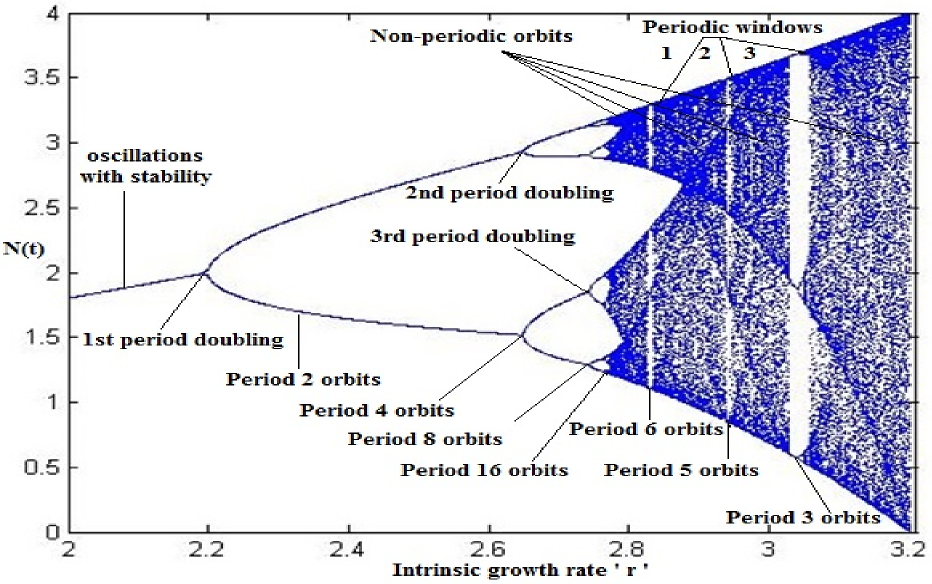

Figure 1: Period doubling bifurcation diagram for (3.1) around N^* in $(r - N)$ plane.

The maximum Lyapunov exponent (LE) related to Figure 1 are computed and plotted in Figure 2(a) which confirms the existence of the chaotic behavior and aperiodic orbits in the parametric space. It is observed that some LE values are positive and some are negative so there exist stable equilibrium points or stable period windows in the chaotic region. In general the positive Lyapunov exponent is considered to be one of the characteristics implying the existence of chaos. Figures 2(d) - 2(f) are local amplifications of the bifurcation diagram with three periodic windows that occurs in the chaotic region. In these periodic windows, periods - 6, 5 and 3 orbits appear (see Figures 3, 4 and 5). Furthermore, a first sub period-doubling (periodic window-1) cascade in orbits of periods - 6, 12 and 24 as in Figure 3, a second sub period doubling (periodic window-2) cascade in orbits of periods - 5, 10, and 20 are presented in Figure 4 and a third sub period doubling (periodic window-3) appears in orbits of periods - 3 , 6, 12 shown in Figure 5. Finally a sub period doubling cascade occurs in each periodic window leading to boundary chaotic and dynamic transition from periodic behaviors to chaos.

Example 3.2. *In this example, we take* $\alpha = 0.63$, $r = 0.75$ *and* $e \in [2.2, 3.4]$ *with initial condition* $N_0 = 0.1$. *It can be observed that the first bifurcation value of (3.1) is between the numbers 2.2 and 2.89, and the second one is between 2.81 and 2.85, and so on. Finally, Figure 7 presents the time series corresponding to Figure 6(a) for varying values of* e. *We also observe that in Figures 7(a) - 7(f) , the growth pattern becomes periodic after an initial aperiodicity. In Figure 7(b), the period is 2-value cycle at* $e = 2.6$ *whereas Figure 7(c) portrays 4-value cycle at* $e = 2.88$. *The 8-value cycle at*

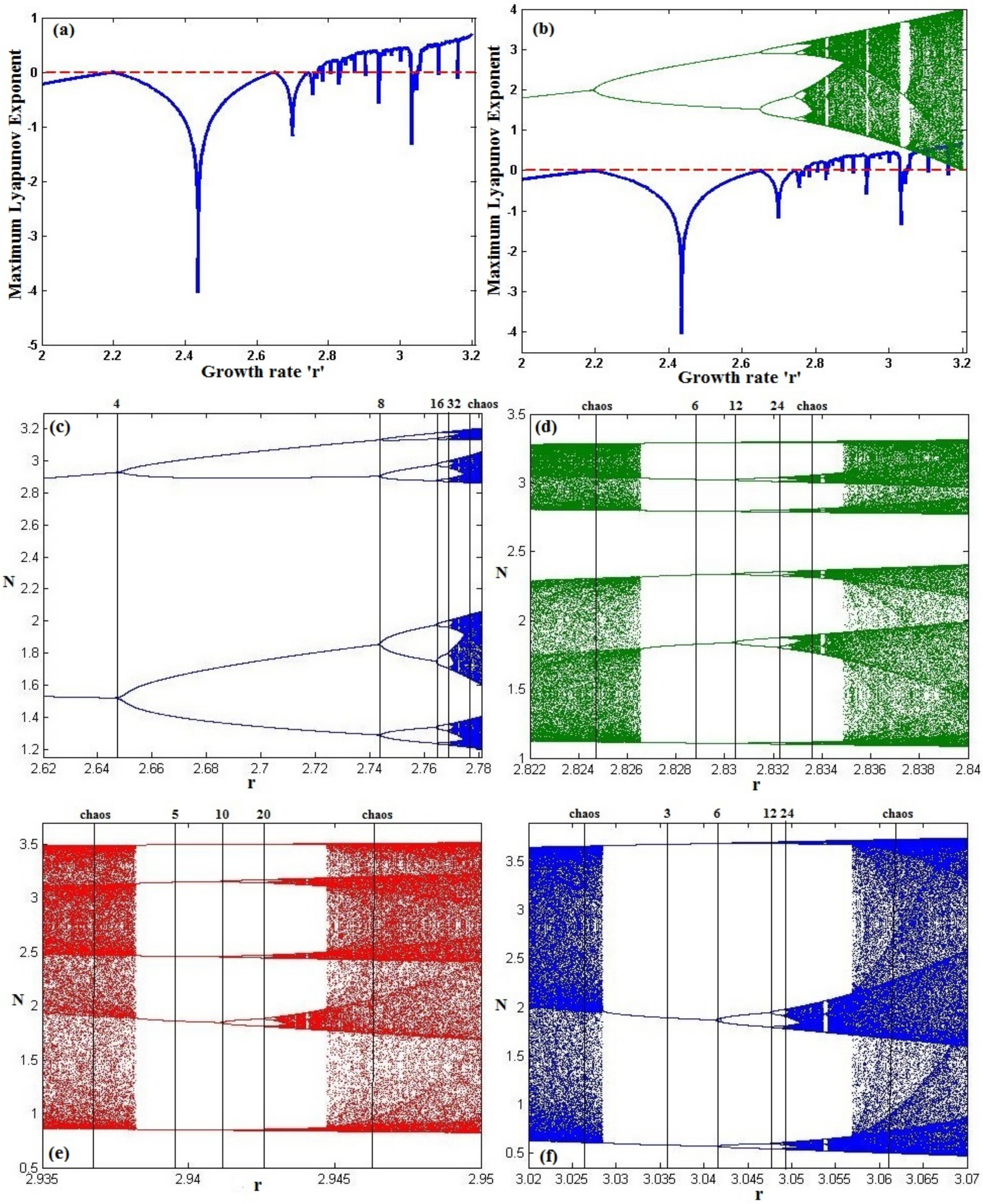

Figure 2: (a) Maximum Lyapunov exponents corresponding to Figure 1, (b) Maximum Lyapunov exponents are superimposed on bifurcation diagrams, (c) - (f) local amplifications of Figure 1 showing periodic windows.

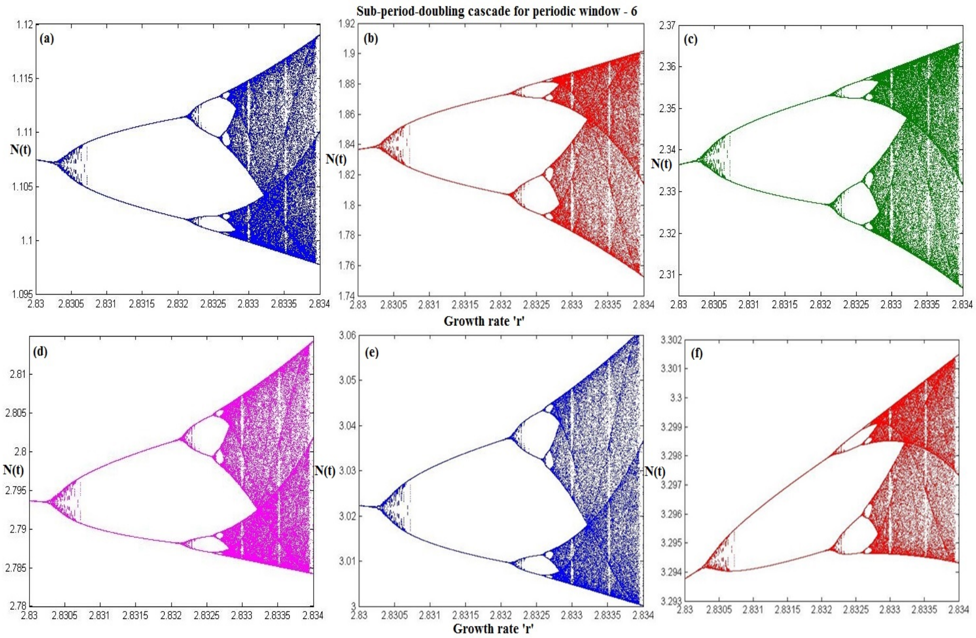

Figure 3: Sub-period-doubling cascade for periodic window - 6.

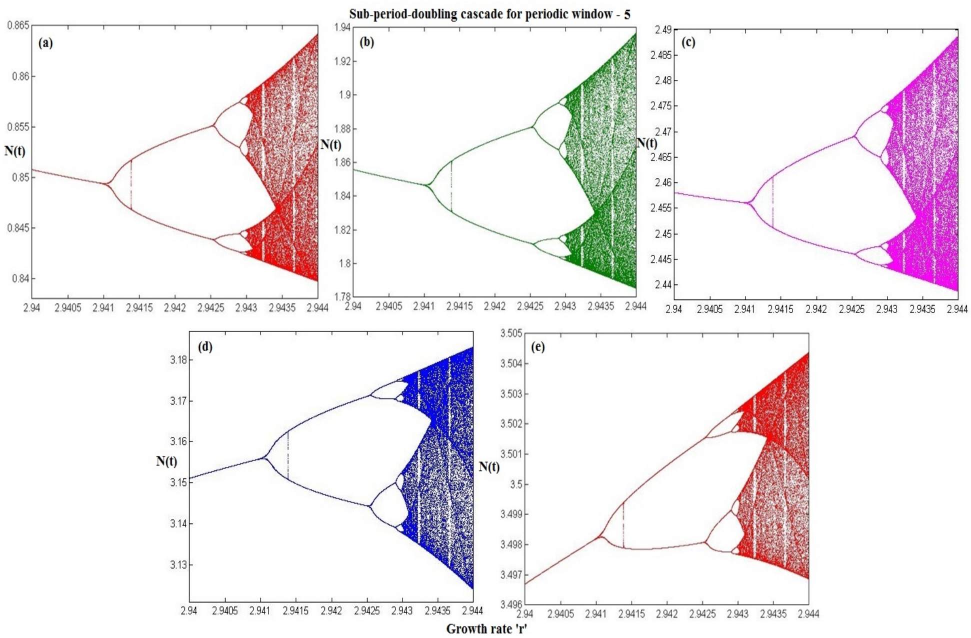

Figure 4: Sub-period-doubling cascade for periodic window - 5.

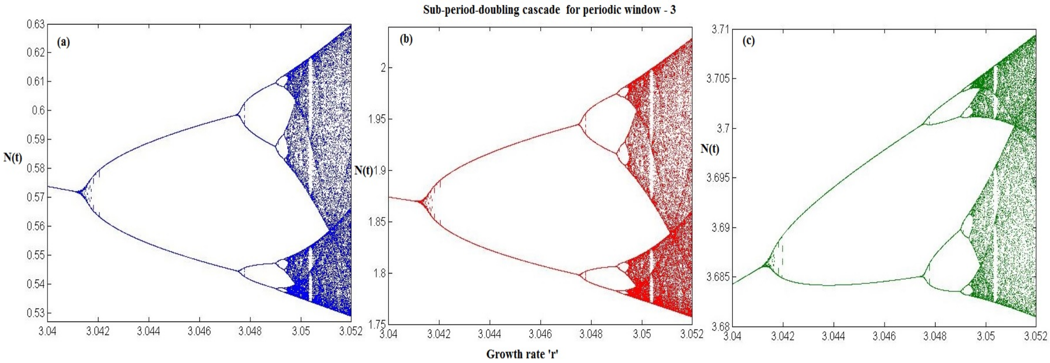

Figure 5: Sub-period-doubling cascade for periodic window - 3.

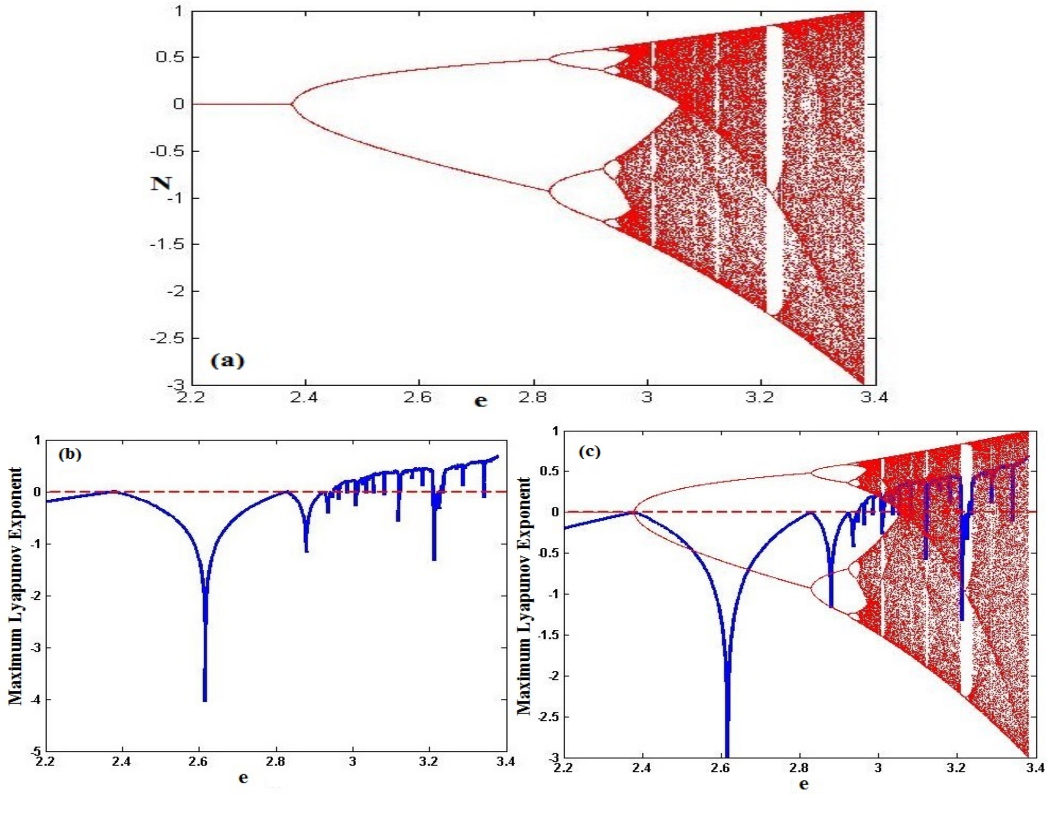

Figure 6: (a) Period doubling bifurcation diagram for (3.1) in $(e - N)$ plane, (b) Maximum Lyapunov exponents for corresponding to Figure 6(a), (c) Maximum Lyapunov exponents are superimposed on bifurcation diagrams.

$e = 2.94$, 16 value cycle at $e = 2.947$ and non periodic oscillations at $e = 2.99878$ are displayed in Figures 7(d), 7(e), 7(f) respectively.

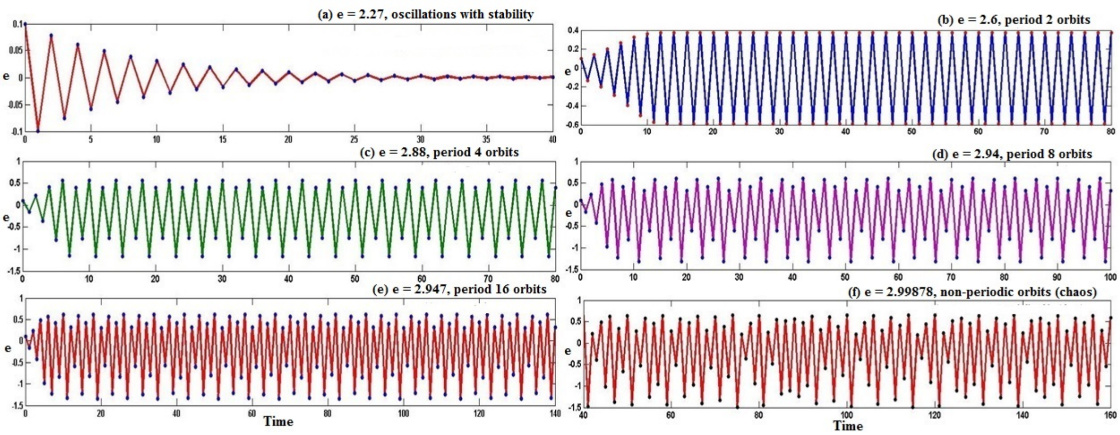

Figure 7: Time series for various values of e corresponding to Figure 6(a).

Example 3.3. *Next, we take $\alpha = 1.3$, $e = 0.5$ and $r = 2.2$ with initial value $N_0 = 0.2$. In this case, the unique positive equilibrium point $N^* = 2$ of (3.1) is unstable. The plot of N_t is shown in Figure 8(a) for the model (3.1). Then, the stable controlled model (3.2) is presented in Figure 8(b). In order to make the equilibrium point N^* locally asymptotically stable, linear feedback control strategy is used. For this, we consider the corresponding controlled system (3.2) in which the feedback controlling force is taken as $S_t = -p\,(N_t - 2)$ with feedback gain $p = -0.11$. The plot of N_t is shown in Figure 8(b) for the population model (3.2). The outcomes of theoretical analysis are confirmed.*

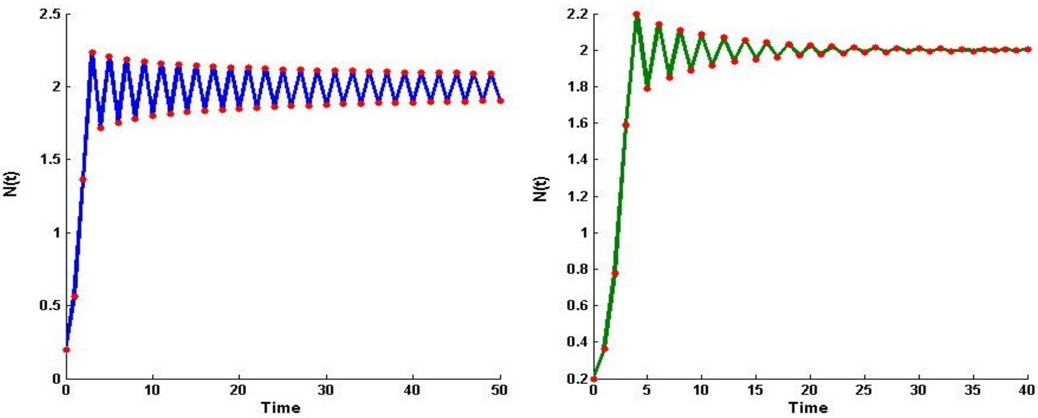

Figure 8: (a) Plot of N_t for model (3.1), (b) Plot of N_t for model (3.2).

4. Conclusion

This paper focused on complex dynamic behavior of the population model with harvesting (1.1). Numerical simulations which show the bifurcations analysis, chaos control, maximum Lyapunov exponents are performed to demonstrate this dynamics behavior.

References

[1] P.A. Abrams, *When does greater mortality increase population size? The long history and diverse mechanisms underlying the hydra effect*, Eco. Lett. 12 (2009) 462–474.

[2] O. Ak Gumus, *Global and local stability analysis in a nonlinear discrete time population model*, Adv. Diff. Equ. 299 (2014) 1687–1847.

[3] M. Begon, J.L. Harper and C.R. Townsend, *Ecology: Individuals, Populations and Communities*, Blackwell Science Ltd, Osney Mead, Oxford, 1996.

[4] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Texts in Applied Mathematics 40, Springer Verlag, New York, 2001.

[5] C. Castillo–Chavez and A. Yakubu, *Intraspecific competition, dispersal and disease dynamics in discrete-time patchy environments, In Mathematical approaches for emerging and reemerging infectious diseases: an introduction (Minneapolis, MN, 1999)*, IMA Math. Appl. 125 (2002) 165–181.

[6] C.W. Clark, *A delayed-recruitment model of population dynamics, with an application tobaleen whale populations*, J. Math. Biol. 3 (1976) 381–391.

[7] Q. Din, *Complexity and chaos control in a discrete-time prey predator model*, Commun. Nonlinear Sci. Numerical Simul. 49 (2017) 113–134.

[8] A. George Maria Selvam, R.Dhineshbabu and S. Britto Jacob, *Quadratic harvesting in a fractional order scavenger model*, J. Phys. Conf. Series 1139 (2018).

[9] A. George Maria Selvam, R.Janagaraj and D. Vignesh, *Allee effect and Holling type II response in a discrete fractional order prey-predator model*, J. Phys. Conf. Series 1139 (2018) 1–8.

[10] K. Hinggins, A. Hastings and L. Botsford, *Density dependence and age structure: nonlinear dynamics and population behavior*, Amer. Nat. 149 (1997) 247–269.

[11] F. M. Hilker and E. Liz, *Harvesting, census timing and hidden hydra effects*, Ecol. Complex. 14 (2013) 95–107.

[12] V. Kapur, D. Troy and J. Oris, *A Sustainable Fishing Simulation Using Mathematical Modeling* Archived, 2009-05-05 at the Wayback Machine Crossroads, 1997.

[13] P.A. Larkin, *An epitaph for the concept of maximum sustained yield*, [permanent dead link] Trans. Amer. Fish. Soc. 106 (1977) 1–11.

[14] J.S. Allen Linda, *An Introduction to Mathematical Biology*, Pearson, New Jersey, 2007.

[15] E, Liz, *How to control chaotic behavior and population size with proportional feedback*, Phys. Lett. A 374 (2010) 725–728.

[16] RM. May, *Simple mathematical models with very complicated dynamics*, Nat. 261 (1976) 459–467.

[17] H. Merdan, O. Ak Gumus, *Stability analysis of a general discrete-time population model involving delay and Allee effects*, Appl. Math. Comput. 219 (2012) 1821–1832.

[18] H. Merdan, O. Ak Gumus and G. Karahisarli, *Global stability analysis of a general scalar difference equation*, Discon. Nonlin. Complex. 7(3)(2018) 225–232.

[19] Robert L. Devaney, *A First Course in Chaotic Dynamical Systems*, Second Edition (Addison-Wesley Studies in Nonlinearity), California, 1948.

[20] K.A. Rose, J.H. Cowan, K.O. Winemiller, J.M. Myers and R. Hilborn, *Compensatory density dependence in fish populations: importance, controversy, understanding and prognosis.* Fish Fish. 2(2001) 293–327.

[21] W.Z. Richard, Ch. J. Harvey, S.L. Katz, Th. P. Good, Ph.S. Levin, *Ecologically Sustainable Yield*, Archived 2011-06-11 at the Wayback Machine American Scientist, 2003.

[22] W.E. Ricker, *Stock and recruitment.* J. Fish. Res. Bd. Can. 11(1954) 559–623.

[23] C. Walters and J. Maguire, *Lessons for stock assessment from the northern cod collapse*, Rev. Fish Bio. Fish. 6(1996) 125–137.

[24] T.K. Wilderbuera, K. Thomas and Ch. Ik. Zhang, *Evaluation of the population dynamics and yield characteristics of Alaska plaice, Pleuronectes quadrituberculatus, in the eastern Bering Sea*, Fish. Res. 41(2) 1999.

[25] A.A. Yakubu, *Multiple attractors in juvenile-adult single species models*, J. Diff. Eqns Appl. 9(2003) 1083–1098.

[26] A.A. Yakubu, *Discrete-time meta population dynamics and unidirectional dispersal*, J. Diff. Eqns Appl. 9(7) (2003) 633–653.