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Legendre Kantorovich methods for Uryshon integral equations

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Abstract

In this paper, the *Kantorovich* method for the numerical solution of nonlinear *Uryshon* equations with a smooth kernel is considered. The approximating operator is chosen to be either the orthogonal projection or an interpolatory projection using *Legendre* polynomial basis. The order of convergence of the proposed method and those of superconvergence of the iterated versions are established. We show that these orders of convergence are valid in the corresponding discrete methods obtained by replacing the integration by a quadrature rule. Numerical examples are given to illustrate the theoretical estimates.

Keywords: *Uryshon* equation, *Kantorovich* method, Projection operator, *Legendre* polynomial, Discrete methods , Superconvergence.
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1. Introduction

We consider the following *Urysohn* integral equation defined on $\mathscr{X} = \mathcal{C}[-1,1]$ by

$$x(s) - \int_{-1}^{1} \kappa(s,t,x(t))dt = f(t), \quad s \in [-1,1] \tag{1.1}$$

where the kernel $\kappa(.,.,.)$ is a real smooth function and u is the unknown function to be determined. Classical methods for solving (1.1) are the *Galerkin* method based on the orthogonal projection

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onto a finite dimensional subspace of $\mathscr{X}$ and the collocation method based on an interpolatory projection. The iterated *Galerkin*/iterated collocation solutions are obtained by one step of iteration and were studied for *Urysohn* integral equations in [6]. The discrete version of collocation and iterated collocation methods was considered in *Atkinson-Flores* [3]. The obtained solution is shown to converge faster than the iterated *Galerkin* solution. Recently a modified projection method was introduced in [13]. More recently, a superconvergent *Nyström* method which converges as rapid as the modified projection method was proposed in [2].

The purpose of this paper is to investigate the *Kantorovich* method for solving (1.1), which is based on "Kantorovich regularization" (*Kantorovich*, 1948) using piecewise polynomial basis functions. This method is discussed in *Schock* [15] and *Sloan* [16] for linear *Fredholm* integral equations. Among these polynomials, *Legendre* or *Chebyshev* polynomial can be used as bases functions which possess nice property of orthogonality and low computational cost.

Various polynomially based projection methods for nonlinear equations were studied. The *Kumar* and *Sloan's* method using *Legendre* polynomials was introduced in [7] and its discrete version was proposed in [8]. The same method using *Chebyshev* polynomials basis functions was early considered in *Kumar* [10]. A superconvergent version of the *Kumar* and *Sloan* method for solving *Hammerstein* equations with smooth kernels was analysed in [1]. Other important results on the numerical solutions of nonlinear integral equations using *Legendre* polynomials can be found in [5, 9, 16].

Now for a summary of the paper. In Section 2, notation is set, the numerical methods are described, and some relevant results are recalled. In Section 3, the orders of convergence of the proposed method and its iterated version for both the orthogonal projection and the interpolatory projection are obtained. In Section 4, we show that these orders of convergence are preserved after taking into account the errors introduced by the numerical quadrature rule. Numerical results are given in Section 5.

2. Preliminaries and method

Let $\mathbb{X}_n$ denote the space of all polynomials of degree $\leq n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the *Legendre* polynomials $\{L_0, L_1, \ldots, L_n\}$ defined by

$$\begin{gathered} L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1,1] \\ (i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1, 2, \ldots, n-1 \end{gathered} \tag{2.1}$$

form an orthogonal basis for $\mathbb{X}_n$. Since

$$\langle L_i, L_j \rangle = \begin{cases} \dfrac{2}{2i+1}, & i = j \\ 0, & i \neq j, \end{cases}$$

then, an orthonormal basis for $\mathbb{X}_n$ is given by

$$\left\{ \varphi_i(s) = \sqrt{\frac{2i+1}{2}} L_i(s) : i = 0, 1, \ldots, n \right\}.$$

We consider two types of projections from $\mathcal{C}[-1,1]$ to $\mathbb{X}_n$.

Orthogonal projection. For $u, v \in \mathcal{C}[-1,1]$, the inner product is given by

$$\langle u, v \rangle = \int_{-1}^{1} u(t)v(t)dt \quad \text{and the associated norm is} \quad \|u\|_{\mathscr{L}^2} = \left(\int_{-1}^{1} u(t)^2 dt \right)^{\frac{1}{2}}.$$

Let $\pi_n^G x$ be the orthogonal projection operator defined from $\mathcal{C}[-1,1]$ to $\mathbb{X}_n$. Then for all $x \in \mathcal{C}[-1,1]$, we have

$$\begin{aligned}(\pi_n^G x)(s) &= \sum_{i=0}^{n} \langle x, \varphi_i \rangle \varphi_i(s), \\ \left\langle \pi_n^G x, \varphi_i \right\rangle &= \langle x, \varphi_i \rangle, \quad i = 0, 1, \ldots, n.\end{aligned} \tag{2.2}$$

Interpolatory projection. For $x \in \mathcal{C}[-1,1]$, let $\pi_n^C x$ denote the unique polynomial of degree n satisfying

$$(\pi_n^C x)(\tau_i) = x(\tau_i), \quad i = 0, 1, \ldots, n, \tag{2.3}$$

where $\{\tau_0, \tau_1, \ldots, \tau_n\}$ are the zeros of the *Legendre* polynomial L_{n+1}. In the *Lagrange* form, $\pi_n^C x$ is

$$(\pi_n^C x)(s) = \sum_{j=0}^{n} x(\tau_j) \ell_j(s), \quad s \in [-1,1],$$

where ℓ_j is the unique polynomial of degree n that satisfies $\ell_j(\tau_i) = \delta_{ij}$. Clearly, π_n^C is a linear operator on $\mathcal{C}[-1,1]$, with the property $\pi_n^C = (\pi_n^C)^2$. It is therefore a projection, having as range the set $\mathbb{X}_n$. Henceforth, we write π_n^C or π_n^G as π_n. The crucial properties of π_n are given in the following lemma.

Lemma 2.1. *(Golberg and Chen [5]) Let $\pi_n : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be the orthogonal projection or the interpolatory projection operator defined by (2.2) and (2.3) respectively. There exists a constant $p > 0$ independent of n such that for $x \in \mathcal{C}[-1,1]$,*

$$\|\pi_n x\|_{\mathscr{L}^2} \leq p \|x\|_{\mathscr{L}^2}, \tag{2.4}$$

$$\|x - \pi_n x\|_{\mathscr{L}^2} \leq (1+p) \inf_{\phi \in \mathbb{X}_n} \|x - \phi\|_{\mathscr{L}^2}. \tag{2.5}$$

Moreover, for any $x \in \mathcal{C}^r[-1,1]$,

$$\|x - \pi_n x\|_{\mathscr{L}^2} \leq c_1 n^{-r} \|x^{(r)}\|_{\mathscr{L}^2}, \tag{2.6}$$

$$\|x - \pi_n x\|_\infty \leq c_1 n^{\beta - r} \|x^{(r)}\|_\infty, \tag{2.7}$$

where c_1 is a constant independent of $n, \beta = \frac{3}{4}$ for the orthogonal projection and $\beta = \frac{1}{2}$ for the interpolatory projection.

Remark 2.2. *The estimate (2.7) shows that $\|x - \pi_n x\|_\infty \not\to 0$ as $n \to \infty$ for any $x \in \mathcal{C}^r[-1,1]$, whereas the estimate (2.5) imply that $\|x - \pi_n x\|_{\mathscr{L}^2} \to 0$ as $n \to \infty$ for all $x \in \mathcal{C}[-1,1]$.*

Let $\mathscr{K}$ be the *Urysohn* integral operator defined by

$$(\mathscr{K} x)(s) = \int_{-1}^{1} \kappa(s, t, x(t)) dt, \quad s \in [-1,1]. \tag{2.8}$$

Thus, equation (1.1) can be writing in operator form as

$$x - \mathscr{K}(x) = f. \tag{2.9}$$

For our convenience we let

$$z = \mathscr{K}(x). \tag{2.10}$$

Thus, writing the solution of (2.9) as $x = z + f$, we have

$$z = \mathscr{K}(z + f). \tag{2.11}$$

The *Kantorovich* method, is obtained by applying the classical projection method to the equation (2.11). Thus, the approximate solution is given by

$$x_n = z_n + f, \tag{2.12}$$

where z_n satisfies

$$z_n - \pi_n \mathscr{K}(z_n + f) = 0. \tag{2.13}$$

The theoretical advantage of the proposed method is that the inhomogeneous term is now 0 rather than $\pi_n f$ in projection methods which may be smoother than f.
Note that the above equations are equivalent to a single equation for x_n

$$x_n - \pi_n \mathscr{K}(x_n) = f. \tag{2.14}$$

Throughout this paper, this method will be called respectively *Kantorovich-Galerkin* method or *Kantorovich*-collocation method when the orthogonal projection or the interpolatory projection is used.
Finally, the iterated *Kantorovich* approximation is defined by

$$\begin{aligned} \widetilde{x}_n &= \mathscr{K}(x_n) + f, \\ &= \widetilde{z}_n + f, \end{aligned} \tag{2.15}$$

where

$$\widetilde{z}_n = \mathscr{K}(z_n + f). \tag{2.16}$$

From (2.13) and (2.15) we observe that $z_n = \pi_n \widetilde{z}_n$, and hence

$$\widetilde{z}_n - \mathscr{K}(\pi_n \widetilde{z}_n + f) = 0. \tag{2.17}$$

For the implementation of the method, we define

$$F_n(y) = y - \pi_n \mathscr{K}(y + f).$$

Then, equation (2.13) becomes

$$F_n(z_n) = 0.$$

This last equation is solved iteratively by using the *Newton-Kantorovich* method. For an initial approximation $z_n^{(0)}$, define

$$z_n^{(k+1)} = z_n^{(k)} - [F_n'(z_n^{(k)})]^{-1} F_n(z_n^{(k)}),$$

where $F_n'(z_n^{(k)})$ is the *Fréchet* derivative of F_n given by

$$F_n'(z_n^{(k)})h = h - \pi_n \mathscr{K}'(z_n^{(k)} + f)h.$$

By a simple calculus, we get

$$z_n^{(k+1)} - \pi_n \mathscr{K}'(z_n^{(k)}) z_n^{(k+1)} = \pi_n \mathscr{K}(z_n^{(k)} + f) - \pi_n \mathscr{K}'(z_n^{(k)}) z_n^{(k)}. \tag{2.18}$$

Since $z_n^{(k)} \in \mathbb{X}_n$, we can write in the case of orthogonal projection

$$z_n^{(k)} = \sum_{j=0}^{n} \langle z_n^{(k)}, \varphi_j \rangle \varphi_j = \sum_{j=0}^{n} y_n^{(k)}(j) \varphi_j.$$

Then, (2.18) is equivalent to the following linear system of size $n+1$

$$(I - A_n^{(k)}) y_n^{(k+1)} = r_n^{(k)},$$

where for $i, j = 0, \ldots, n$,

$$\begin{aligned} A_n^{(k)}(i,j) &= \langle \mathscr{K}'(z_n^{(k)}) \varphi_j, \varphi_i \rangle, \\ r_n^{(k)}(i) &= \langle \mathscr{K}(z_n^{(k)} + f), \varphi_i \rangle - (C_n^{(k)} y_n^{(k)})(i). \end{aligned}$$

For the interpolatory projection, we can write

$$z_n^{(k)} = \sum_{j=0}^{n} z_n^{(k)}(\tau_j) \ell_j = \sum_{j=0}^{n} y_n^{(k)}(j) \ell_j.$$

Then, we obtain the system of linear equations

$$(I - B_n^{(k)}) y_n^{(k+1)} = q_n^{(k)},$$

where for $i, j = 0, \ldots, n$,

$$\begin{aligned} B_n^{(k)}(i,j) &= \mathscr{K}'(z_n^{(k)})(t_j), \\ q_n^{(k)} &= \mathscr{K}(z_n^{(k)} + f)(t_i) - (B_n^{(k)} y_n^{(k)})(i). \end{aligned}$$

3. Convergence rates

For the rest of the paper we assume that $r \geq 1$. Let x_0 be an isolated solution of (1.1), and let a, b be real numbers such that

$$\left[\min_{s \in [-1,1]} x_0(s), \max_{s \in [-1,1]} x_0(s) \right] \subset [a, b].$$

Define

$$\Omega = [-1, 1] \times [-1, 1] \times [a, b].$$

Assume that $\kappa, \frac{\partial \kappa}{\partial u} \in \mathcal{C}^r(\Omega)$. Then, $\mathscr{K}$ is a compact operator from $\mathscr{L}^\infty[-1,1]$ to $\mathcal{C}^r[-1,1]$. If $f \in \mathcal{C}[-1,1]$, then, since

$$x_0 - \mathscr{K}(x_0) = f, \tag{3.1}$$

the solution x_0 belongs to $\mathcal{C}[-1,1]$. Moreover, the operator $\mathscr{K}$ is *Fréchet* differentiable and the *Fréchet* derivative is given by

$$(\mathscr{K}'(x) g)(s) = \int_{-1}^{1} \frac{\partial \kappa}{\partial u}(s, t, x(t)) g(t) dt.$$

For $\delta_0 > 0$, let $\mathcal{B}(x, \delta_0) = \{ y \in \mathbb{X} : \|x - y\|_\infty < \delta_0 \}$. Since $\frac{\partial \kappa}{\partial u} \in \mathcal{C}^r(\Omega)$, it follows that $\mathscr{K}'$ is *Lipschitz* continuous in a neighborhood $\mathcal{B}(x_0, \delta_0)$ of x_0, that is, there exists a constant γ such that

$$\|\mathscr{K}'(x_0) - \mathscr{K}'(x)\| \leq \gamma \|x_0 - x\|, \quad x \in \mathcal{B}(x, \delta_0). \tag{3.2}$$

For $j = 0, 1, \ldots, r$ we have

$$\begin{aligned}\| \, [\mathscr{K}'(x_0)g]^{(j)} \, \|_\infty &= \sup_{s\in[-1,1]} \left| \int_{-1}^{1} \frac{\partial^{j+1}\kappa}{\partial s^j \partial u}(s,t,x_0(t))g(t)dt \right| \\ &\leq \sup_{s,t\in[-1,1]} \left| \frac{\partial^{j+1}\kappa}{\partial s^j \partial u}(s,t,x_0(t)) \right| \int_{-1}^{1} |g(t)| \, dt \\ &\leq 2\|\kappa\|_{r,\infty}\|g\|_\infty,\end{aligned} \tag{3.3}$$

where

$$\|\kappa\|_{r,\infty} = \max_{s,t\in[-1,1]} \sum_{j=0}^{r} \left\{ \left| \frac{\partial^j \kappa}{\partial s^j}(s,t,x_0(t)) \right| + \left| \frac{\partial^{j+1}\kappa}{\partial s^j \partial u}(s,t,x_0(t)) \right| \right\}.$$

The operator $\mathscr{K}'(x_0)$ is compact. Assume that $(I - \mathscr{K}'(x_0))^{-1} : \mathcal{C}[-1,1] \to \mathcal{C}[-1,1]$ is a bounded linear operator and that 1 is not an eigenvalue of $\mathscr{K}'(x_0)$. Then, it can be shown that

$$M = (I - \mathscr{K}'(x_0))^{-1}\mathscr{K}'(x_0)$$

is the compact linear integral operator (See *Riesz-Nagy* [17])

$$(Mg)(s) = \int_{-1}^{1} m(s,t)g(t)dt, \quad s \in [-1,1], \quad g \in \mathscr{X}, \tag{3.4}$$

where the smoothness of kernel m is the same as that of kernel κ, that is,

$$m \in \mathcal{C}^r([-1,1] \times [-1,1]).$$

The following lemma, which can be shown easily, will be used to prove the main results of this section.

Lemma 3.1. *Let $x_0 \in \mathcal{C}[-1,1]$ be an isolated solution of (1.1). Assume that $\kappa \in \mathcal{C}^r(\Omega)$ and that 1 is not an eigenvalue of $\mathscr{K}'(x_0)$. Then for n large enough, the operators $I - \pi_n\mathscr{K}'(x_0)$ are invertible i.e. there exists a constant $A_1 > 0$ such that $\|(I - \pi_n\mathscr{K}'(x_0))^{-1}\|_\infty \leq A_1 < \infty$.*

Proof . Using estimates (2.7) and (3.3), we have

$$\begin{aligned}\|(\pi_n\mathscr{K}'(x_0) - \mathscr{K}'(x_0))g\|_\infty &= \|(I - \pi_n)\mathscr{K}'(x_0)g\|_\infty, \\ &\leq 2c_1 n^{\beta - r}\|\kappa\|_{r,\infty}\|g\|_\infty.\end{aligned}$$

Since $0 < \beta < 1$, for $\beta < r = 1, 2 \ldots$, it follows that

$$\|\pi_n\mathscr{K}'(x_0) - \mathscr{K}'(x_0)\|_\infty = \mathcal{O}(n^{\beta - r}) \to 0, \quad \text{as} \quad n \to \infty.$$

Hence by Lemma 2.6 in [4], the operators $(I - \pi_n\mathscr{K}'(x_0))^{-1}$ exists and are uniformly bounded, for some sufficiently large n. This completes the proof. □ The following theorem can be proved by using *Theorem 2* of *Vainikko* [18].

Theorem 3.2. *Let $x_0 \in \mathcal{C}[-1,1]$ be an isolated solution of (1.1). Assume that $\kappa \in \mathcal{C}^r(\Omega)$ and that 1 is not an eigenvalue of $\mathscr{K}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation (2.9) has a unique solution x_n in $\mathcal{B}(x_0, \delta_0)$ for a sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\alpha_n}{1+q} \leq \|x_n - x_0\|_\infty \leq \frac{\alpha_n}{1-q}, \tag{3.5}$$

where $\alpha_n = \left\|(I - \pi_n\mathscr{K}'(x_0))^{-1}(\mathscr{K}(x_0) - \pi_n\mathscr{K}(x_0))\right\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

The next theorem establish the rate of convergence of the approximation x_n to the exact solution x_0.

Theorem 3.3. *Assume that $\kappa \in \mathcal{C}^r(\Omega)$. Let x_0, x_n be the solutions of (2.9) and (2.14) respectively. Then, under the hypothesis of Theorem 3.2, for n large enough, we have*

$$\|x_n - x_0\|_\infty = \mathcal{O}(n^{\beta-r}). \tag{3.6}$$

Proof . Using estimates (2.7) , (3.5) and Lemma 3.1, we have

$$\begin{aligned}\|x_n - x_0\|_\infty &\leq A_1\|(I-\pi_n)\mathscr{K}(x_0)\|_\infty,\\ &\leq 2A_1c_1n^{\beta-r}\|\kappa\|_{r,\infty}.\end{aligned}$$

This completes the proof. □ For the rest of the paper, we set

$$z_0 = \mathscr{K}(x_0) \quad \text{and} \quad m_s(t) = m(s,t), \quad s,t \in [-1,1].$$

Theorem 3.4. *Assume that $\kappa, \frac{\partial\kappa}{\partial u} \in \mathcal{C}^r(\Omega)$ and that $f \in \mathcal{C}[-1,1]$. Let $\widetilde{x}_n^G$ be the iterated Kantorovich-Galerkin approximation of x_0 given by (2.15). Then, for a sufficiently large n, we have*

$$\|\widetilde{x}_n^G - x_0\|_\infty = \mathcal{O}(n^{-2r}). \tag{3.7}$$

Proof . Note that from equations (2.15) and (3.1) we have

$$\begin{aligned}\widetilde{x}_n - x_0 &= \mathscr{K}(x_n) - \mathscr{K}(x_0)\\ &= \mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) + \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0).\end{aligned} \tag{3.8}$$

Noting that

$$x_n - x_0 = \pi_n(\widetilde{x}_n - x_0) - (I - \pi_n)\mathscr{K}(x_0), \tag{3.9}$$

yields

$$\widetilde{x}_n - x_0 = [\mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0)] + \mathscr{K}'(x_0)\pi_n(\widetilde{x}_n - x_0) - \mathscr{K}'(x_0)(I - \pi_n)\mathscr{K}(x_0). \tag{3.10}$$

Hence, using again (3.8), we get

$$\begin{aligned}\mathscr{K}'(x_0)\pi_n(\widetilde{x}_n - x_0) &= \mathscr{K}'(x_0)(\pi_n - I)[\mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) + \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0)]\\ &\quad + \mathscr{K}'(x_0)(\widetilde{x}_n - x_0)\end{aligned}$$

and replacing in (3.10), we obtain the formula

$$\begin{aligned}\widetilde{x}_n - x_0 &= \{[I - \mathscr{K}'(x_0)]^{-1}[\mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0)]\}\\ &\quad - M(I - \pi_n)[\mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0)]\\ &\quad - M(I - \pi_n)\mathscr{K}'(x_0)(x_n - x_0) - M(I - \pi_n)\mathscr{K}(x_0).\end{aligned} \tag{3.11}$$

By the mean value theorem, the *Lipschitz* continuity of $\mathscr{K}'$ and estimate (3.6) we obtain

$$\begin{aligned}\|\mathscr{K}(x_n) - \mathscr{K}'(x_0)(x_n - x_0) - \mathscr{K}(x_0)\| &= \|[\mathscr{K}'(x_n + \theta(x_0 - x_n)) - \mathscr{K}'(x_0)](x_n - x_0)\|\\ &\leq \gamma(1-\theta)\|x_n - x_0\|_\infty^2\\ &= \mathcal{O}(n^{2(\beta-r)}).\end{aligned} \tag{3.12}$$

where $0 < \theta < 1$. For each $s \in [-1,1]$, we have

$$M(I-\pi_n^G)\mathscr{K}(x_0)(s) = \int_{-1}^{1} m(s,t)(I-\pi_n^G)z_0(t)dt$$
$$= \left\langle (I-\pi_n^G)m_s, (I-\pi_n^G)z_0 \right\rangle.$$

Hence, using the *Cauchy-Schwarz* inequality and estimate (2.6), we can show that

$$\begin{aligned}\|M(I-\pi_n^G)\mathscr{K}(x_0)\|_\infty &\leq \max_{s\in[-1,1]} \|(I-\pi_n^G)m_s\|_{\mathscr{L}^2}\|(I-\pi_n^G)z_0\|_{\mathscr{L}^2},\\ &\leq c_1^2 n^{-2r} \max_{s\in[-1,1]} \|m_s^{(r)}\|_{\mathscr{L}^2}\|z_0^{(r)}\|_{\mathscr{L}^2},\\ &\leq 2c_1^2 n^{-2r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}.\end{aligned} \tag{3.13}$$

where

$$\|m\|_{r,\infty} = \max_{s,t\in[-1,1]} \left\{ \left| \frac{\partial^j m}{\partial t^j}(s,t) \right| : j = 0,1,\dots,r \right\}.$$

By (3.4) we get

$$\begin{aligned}M(I-\pi_n^G)\mathscr{K}'(x_0)g(s) &= \int_{-1}^{1} m(s,t)(I-\pi_n^G)\mathscr{K}'(x_0)g(t)dt,\\ &= \left\langle m_s, (I-\pi_n^G)\mathscr{K}'(x_0)g \right\rangle,\\ &= \left\langle (I-\pi_n^G)m_s, (I-\pi_n^G)\mathscr{K}'(x_0)g \right\rangle.\end{aligned}$$

Then, using the *Cauchy-Schwarz* inequality and estimates (2.6),(3.3), we obtain

$$\begin{aligned}\|M(I-\pi_n^G)\mathscr{K}'(x_0)g\|_\infty &\leq \max_{s\in[-1,1]} \|(I-\pi_n^G)m_s\|_{\mathscr{L}^2}\|(I-\pi_n^G)\mathscr{K}'(x_0)g\|_{\mathscr{L}^2},\\ &\leq c_1^2 n^{-2r} \max_{s\in[-1,1]} \|m_s^{(r)}\|_{\mathscr{L}^2}\|[\mathscr{K}'(x_0)g]^{(r)}\|_{\mathscr{L}^2},\\ &\leq 2\sqrt{2}c_1^2 n^{-2r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}\|g\|_\infty.\end{aligned}$$

This implies that

$$\|M(I-\pi_n^G)\mathscr{K}'(x_0)\| \leq 2\sqrt{2}c_1^2 n^{-2r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}. \tag{3.14}$$

Now combining (3.11), (3.12), (3.13) and (3.14), the estimate (3.7) holds. □

Theorem 3.5. *Assume that $\kappa, \frac{\partial\kappa}{\partial u} \in \mathcal{C}^r(\Omega)$ and that $f \in \mathcal{C}[-1,1]$. Let $\widetilde{x}_n^C$ be the iterated Kantorovich-Collocation approximation of x_0 given by (2.15). Then, for a sufficiently large n, we have*

$$\|\widetilde{x}_n^C - x_0\|_\infty = \mathcal{O}(n^{-r}). \tag{3.15}$$

Moreover, we have the following superconvergence estimate for x_n^C at the collocation points

$$\max_{0\leq i\leq n} |x_n^C(\tau_i) - x(\tau_i)| = \mathcal{O}(n^{-r}).$$

Proof . From (3.11) we have

$$M(I-\pi_n^C)\mathscr{K}(x_0)(s) = \int_{-1}^{1} m(s,t)(I-\pi_n^C)z_0(t)dt, \quad s \in [-1,1]$$

Then, taking supremum and using the *Cauchy-Schwarz* inequality, we get

$$\begin{aligned}\|M(I-\pi_n^C)\mathscr{K}(x_0)\|_\infty &\leq \max_{s\in[-1,1]}\|m_s\|_{\mathscr{L}^2}\|(I-\pi_n^C)z_0\|_{\mathscr{L}^2},\\ &\leq \sqrt{2}c_1n^{-r}\|m\|_{r,\infty}\|z_0^{(r)}\|_{\mathscr{L}^2},\\ &\leq 2c_1n^{-r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}.\end{aligned} \tag{3.16}$$

For the third term in (3.11) , using the estimates (2.6),(3.3) we obtain,

$$\begin{aligned}\|M(I-\pi_n^C)\mathscr{K}'(x_0)g\|_\infty &\leq \max_{s\in[-1,1]}\|m_s\|_{\mathscr{L}^2}\|(I-\pi_n^C)\mathscr{K}'(x_0)g\|_{\mathscr{L}^2},\\ &\leq \sqrt{2}c_1n^{-r}\|m\|_{r,\infty}\|[\mathscr{K}'(x_0)g]^{(r)}\|_{\mathscr{L}^2},\\ &\leq 2\sqrt{2}c_1n^{-r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}\|g\|_\infty,\end{aligned}$$

which means that

$$\|M(I-\pi_n^C)\mathscr{K}'(x_0)\|_\infty \leq 2\sqrt{2}c_1n^{-r}\|m\|_{r,\infty}\|\kappa\|_{r,\infty}. \tag{3.17}$$

Combining estimates (3.11), (3.12), (3.16) and (3.17), we obtain (3.15).
Now, applying π_n^C to both sides of equation (2.14), we have that

$$\begin{aligned}\pi_n^C x_n &= \pi_n^C\mathscr{K}(x_n)+\pi_n^C f\\ &= \pi_n^C\widetilde{x}_n,\end{aligned}$$

and therefore

$$x_n^C(\tau_i)=\widetilde{x}_n^C(\tau_i), \quad i=0,1,\ldots,n.$$

Hence, the required result follows from (3.15). □

4. Discrete methods

In practice, the integrals in the definitions of the orthogonal projection π_n^G and the operator $\mathscr{K}$ involved in equations (2.2) and (2.8) are not computed exactly. It is necessary to replace them by a numerical quadrature formula giving rise to discrete and iterated discrete *Legendre-Kantorovich* methods, respectively. To introduce these discrete methods, we consider a quadrature formula defined by

$$\int_{-1}^{1} f(t)dt \simeq \sum_{i=1}^{M} w_i f(t_i), \tag{4.1}$$

where the weights are such that

$$w_i>0, \quad i=1,\ldots,M$$

and the number of nodes is written simply M, with the dependence on n understood implicitly. We suppose that this formula has degree of precision $d\geq 2n$, that is

$$\int_{-1}^{1} P(t)dt = \sum_{i=1}^{M} w_i P(t_i), \tag{4.2}$$

for all polynomials P of degree $\leq d$. Following *Golberg* [11] and *Sloan* [12] we define the discrete inner product as

$$\langle f,g\rangle_M=\sum_{i=1}^{M} w_i f(t_i)g(t_i), \quad f,g\in\mathcal{C}[-1,1]. \tag{4.3}$$

Let $\mathscr{Q}_n^G : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be the hyperinterpolation operator defined by *Sloan* [12] as

$$(\mathscr{Q}_n^G x)(s) = \sum_{i=0}^{n} \langle x, \varphi_i \rangle_M \varphi_i(s), \tag{4.4}$$

and satisfying

$$\left\langle \mathscr{Q}_n^G x, \varphi_i \right\rangle_M = \langle x, \varphi_i \rangle_M, \quad i = 0, 1, \ldots, n.$$

For the discrete *Legendre* collocation method we will use the interpolatory projection operator π_n^C defined by (2.3). For notational convenience from now on we write $\pi_n^C \equiv \mathscr{Q}_n^C$ and $\mathscr{Q}_n \equiv \mathscr{Q}_n^G$ or $\mathscr{Q}_n^C$. The following crucial properties of $\mathscr{Q}_n$ are quoted from *Sloan* [12].

Lemma 4.1. *Let $\mathscr{Q}_n : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be the the hyperinterpolation or interpolatory projection operator defined by (4.4) and (2.3). Then we have*

$$\|\mathscr{Q}_n x\|_{\mathscr{L}^2} \le \sqrt{2}\|x\|_\infty, \tag{4.5}$$

$$\|x - \mathscr{Q}_n x\|_{\mathscr{L}^2} \le 2\sqrt{2} \inf_{\phi \in \mathbb{X}_n} \|x - \phi\|_\infty. \tag{4.6}$$

Moreover, for any $x \in \mathcal{C}^r[-1,1]$,

$$\|x - \mathscr{Q}_n x\|_\infty \le c_2 n^{\gamma - r} \|x^{(r)}\|_\infty, \tag{4.7}$$

where c_1 is a constant independent of n, $\gamma = 1$ for the hyperinterpolation operator and $\gamma = \frac{1}{2}$ for the interpolatory projection. Note that for any $x \in \mathcal{C}^r[-1,1]$, we have also (see [8])

$$\langle x - \mathscr{Q}_n x, x - \mathscr{Q}_n x \rangle_M^{\frac{1}{2}} \le c_2 \sqrt{2} n^{-r} \|x^{(r)}\|_\infty, \tag{4.8}$$

where c_1 is a constant independent of n and $n \ge r$.

Using the numerical integration method (4.1), the *Nyström* approximation of the integral operator $\mathscr{K}$ is defined as

$$(\mathscr{K}_n x)(s) = \sum_{i=1}^{M} w_i \kappa(s, t_i, x(t_i)), \quad s \in [-1,1]. \tag{4.9}$$

The *Fréchet* derivative of $\mathscr{K}_n$ is given by

$$(\mathscr{K}_n'(x) g)(s) = \sum_{i=1}^{M} w_i \frac{\partial \kappa}{\partial u}(s, t_i, x(t_i)) g(t_i).$$

Since $w_j > 0$ and $2 = \int_{-1}^{1} dt = \sum_{i=1}^{M} w_i$, we have for $j = 0, 1 \ldots d$,

$$\begin{aligned} \| \left[\mathscr{K}_n'(x_0) g\right]^{(j)} \|_\infty &\le \sum_{i=1}^{M} w_i \sup_{s \in [-1,1]} \left| \frac{\partial^{j+1} \kappa}{\partial s^j \partial u}(s, t_i, x_0(t_i)) \right| |g(t_i)| \\ &\le 2\|\kappa\|_{j,\infty} \|g\|_\infty. \end{aligned} \tag{4.10}$$

Theorem 4.2. *Assume that $\kappa, \frac{\partial \kappa}{\partial u} \in \mathcal{C}^d(\Omega)$, then we have*

$$\|\mathscr{K}(x_0) - \mathscr{K}_n(x_0)\|_\infty \le c_3 n^{-d} \|\kappa\|_{d,\infty}, \tag{4.11}$$

$$\|\mathscr{K}'(x_0) g - \mathscr{K}_n'(x_0) g\|_\infty \le c_3 n^{-d} \|\kappa\|_{d,\infty} \|g\|_{d,\infty}, \tag{4.12}$$

where c_3 is a constant independent of n.

Proof . For any $p \in \mathbb{X}_d$, we have

$$|(\mathscr{K}(x_0) - \mathscr{K}_n(x_0))(s)| = \left| \int_{-1}^{1} [\kappa(s,t,x_0(t)) - p(t)]\, dt - \sum_{i=1}^{M} w_i\, [\kappa(s,t_i,x_0(t_i)) - p(t_i)] \right|,$$
$$\leq \|\kappa - p\|_\infty \left[\int_{-1}^{1} dt + \sum_{i=1}^{M} w_i \right],$$
$$\leq 4\|\kappa - p\|_\infty.$$

According to *jackson's theorem* we have for all $x \in \mathcal{C}^r[-1,1]$

$$\inf_{\phi \in \mathbb{X}_n} \|x - \phi\|_\infty \leq c_3 n^{-r} \|x^{(r)}\|_\infty, \tag{4.13}$$

where c_3 is a constant independent of n. Thus

$$|(\mathscr{K}(x_0) - \mathscr{K}_n(x_0))(s)| \leq 4 \inf_{p \in \mathbb{X}_d} \|\kappa - p\|_\infty,$$
$$\leq 4c_3 d^{-d} \|\kappa\|_{d,\infty}.$$

Since $d \geq 2n$, we conclude that

$$|(\mathscr{K}(x_0) - \mathscr{K}_n(x_0))(s)| \leq 4c_3 (2n)^{-d} \|\kappa\|_{d,\infty} \tag{4.14}$$

and therefore

$$\|\mathscr{K}(x_0) - \mathscr{K}_n(x_0)\|_\infty \leq c_3 n^{-d} ||\kappa||_{d,\infty}. \tag{4.15}$$

Similarly, it can be shown that

$$\|(\mathscr{K}'(x_0) - \mathscr{K}_n'(x_0))g\|_\infty \leq c_3 n^{-d} \|\kappa\|_{d,\infty} \|g\|_{d,\infty}. \tag{4.16}$$

This completes the proof. □ The discrete version of the approximate equation (2.14) is given by

$$y_n - \mathscr{Q}_n \mathscr{K}_n(y_n) = f, \tag{4.17}$$

while the discrete iterated *Kantorovich* solution is defined as follows

$$\widetilde{y}_n = \mathscr{K}_n(y_n) + f. \tag{4.18}$$

Theorem 4.3. *Assume that $\kappa, \frac{\partial \kappa}{\partial u} \in \mathcal{C}^r(\Omega)$. Let x_0, y_n be the solutions of (2.9) and (4.17) respectively. Then, for a sufficiently large n, we have*

$$\|x_0 - y_n\|_\infty = \mathcal{O}(n^{\gamma - r}). \tag{4.19}$$

Proof . According to *Theorem 2* of *Vainikko* [18], we can show that

$$\|x_0 - y_n\|_\infty \leq \left\| (I - \mathscr{Q}_n) \mathscr{K}_n'(x_0)^{-1} (\mathscr{K}(x_0) - \mathscr{Q}_n \mathscr{K}_n(x_0)) \right\|_\infty. \tag{4.20}$$

As in Lemma 3.1 it can be also shown that for a sufficiently large n, there exists a constant $A_2 > 0$ such that $\left\|(I - \mathscr{Q}_n\mathscr{K}_n'(x_0))^{-1}\right\|_\infty \leq A_2$. Now, we write

$$\|(\mathscr{Q}_n\mathscr{K}_n'(x_0) - \mathscr{K}'(x_0))g\|_\infty \leq \|(I - \mathscr{Q}_n)\mathscr{K}'(x_0)g\|_\infty + \|(\mathscr{K}'(x_0) - \mathscr{K}_n'(x_0))g\|_\infty. \tag{4.21}$$

Since $0 < \gamma < r$ and by the estimates (4.7) and (4.10) we get

$$\|(I - \mathscr{Q}_n)\mathscr{K}'(x_0)g\|_\infty \leq 2c_2 n^{\gamma - r}\|\kappa\|_{r,\infty}\|g\|_\infty \to 0, \quad \text{as} \quad n \to \infty. \tag{4.22}$$

Thus, for any $g \in \mathcal{C}^r[-1,1]$ and $d > 1$, it follows from (4.12), (4.21) and (4.22) that

$$\|(\mathscr{Q}_n\mathscr{K}_n'(x_0) - \mathscr{K}'(x_0))g\|_\infty \leq 2c_2 n^{\gamma - r}\|\kappa\|_{r,\infty}\|g\|_\infty + c_3 n^{-d}\|\kappa\|_{d,\infty}\|g\|_{d,\infty}.$$

This shows that $\mathscr{Q}_n\mathscr{K}_n'(x_0)$ converges pointwise to the operator $\mathscr{K}'(x_0)$ in infinity norm. Hence from (4.7), (4.10) and (4.11) we have

$$\begin{aligned}\|\mathscr{K}(x_0) - \mathscr{Q}_n\mathscr{K}_n(x_0)\|_\infty &= \|(I - \mathscr{Q}_n)\mathscr{K}_n(x_0) - \mathscr{K}_n(x_0) + \mathscr{K}(x_0)\|_\infty \\ &\leq \|(I - \mathscr{Q}_n)\mathscr{K}_n(x_0)\|_\infty + \|\mathscr{K}(x_0) - \mathscr{K}_n(x_0)\|_\infty \\ &\leq 2c_2 n^{\gamma - r}\|\kappa\|_{r,\infty} + c_3 n^{-d}\|\kappa\|_{d,\infty}\end{aligned}$$

and therefore

$$\|x_0 - y_n\|_\infty \leq A_2(2c_2 n^{\gamma - r}\|\kappa\|_{r,\infty} + c_3 n^{-d}\|\kappa\|_{d,\infty}). \tag{4.23}$$

Since $d > r - \gamma$, the proof is reached. □ The following theorem give the order of convergence of the discrete iterated *Legendre-Kantorovich* method.

Theorem 4.4. *Assume that $\kappa, \frac{\partial \kappa}{\partial u} \in \mathcal{C}^d(\Omega)$ and that $x_0 \in \mathcal{C}[-1,1]$. Let $\widetilde{y}_n^G$ be the iterated discrete Kantorovich-Galerkin approximation of x_0 given by (4.18). Then, for a sufficiently large n, we have*

$$\|x_0 - \widetilde{y}_n^G\|_\infty = \mathcal{O}(n^{-2r}). \tag{4.24}$$

Proof . Again by [18], we can show that

$$\|x_0 - \tilde{y}_n\| \leq \|(I - \mathscr{K}_n'(x_0))^{-1}(\mathscr{K}(x_0) - \mathscr{K}_n(x_0))\|_\infty.$$

From estimate (4.12) we see that $\|\mathscr{K}_n'(x_0) - \mathscr{K}'(x_0)\|_\infty \to 0$ as $n \to \infty$ and similarly to Lemma 3.1 there exists a constant $A_2 > 0$ such that $\|(I - \mathscr{K}_n'(x_0))^{-1}\|_\infty \leq A_2$. Note that

$$\begin{aligned}\|\mathscr{K}(x_0) - \mathscr{K}_n(x_0)\|_\infty &= \|\mathscr{K}(z_0 + f) - \mathscr{K}_n(\mathscr{Q}_n z_0 + f)\|_\infty, \\ &\leq \|\mathscr{K}(z_0 + f) - \mathscr{K}_n(z_0 + f)\|_\infty + \|\mathscr{K}_n(z_0 + f) - \mathscr{K}_n(\mathscr{Q}_n z_0 + f)\|_\infty.\end{aligned} \tag{4.25}$$

First we have from (4.11)

$$\|\mathscr{K}(z_0 + f) - \mathscr{K}_n(z_0 + f)\|_\infty \leq c_3 n^{-d}\|\kappa\|_{d,\infty}. \tag{4.26}$$

Next, by *Taylor*'s theorem,

$$\mathscr{K}_n(z_0 + f) - \mathscr{K}_n(\mathscr{Q}_n z_0 + f) = \mathscr{K}_n'(x_0)(z_0 - \mathscr{Q}_n z_0) + O(\|z_0 - \mathscr{Q}_n z_0\|^2). \tag{4.27}$$

For a fixed $s \in [-1,1]$ let

$$\ell_s(t) = \frac{\partial \kappa}{\partial u}(s, t, x_0(t)), \quad t \in [-1,1].$$

Using *Cauchy-Schwarz* inequality and estimate (4.8), we have for each $s \in [-1,1]$

$$\begin{aligned}
\left|\mathscr{K}_n'(x_0)(z_0 - \mathscr{Q}_n^G z_0)(s)\right| &= \left|\sum_{i=1}^{M} w_i \frac{\partial \kappa}{\partial u}(s,t_i,x_0(t_i))[(I-\mathscr{Q}_n^G)z_0](t_i)\right| \\
&= \left\langle \ell_s, (I-\mathscr{Q}_n^G)z_0\right\rangle_M \\
&= \left\langle (I-\mathscr{Q}_n^G)\ell_s, (I-\mathscr{Q}_n^G)z_0\right\rangle_M \\
&= \left|\sum_{i=1}^{M} w_i (I-\mathscr{Q}_n^G)\frac{\partial \kappa}{\partial u}(s,t_i,x_0(t_i))(I-\mathscr{Q}_n^G)z_0(t_i)\right| \\
&\leq \left(\sum_{i=1}^{M} w_i \left[(I-\mathscr{Q}_n^G)\frac{\partial \kappa}{\partial u}(s,t_i,x_0(t_i))\right]^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{M} w_i \left[(I-\mathscr{Q}_n^G)z_0(t_i)\right]^2\right)^{\frac{1}{2}} \\
&= \left\langle (I-\mathscr{Q}_n^G)\ell_s, (I-\mathscr{Q}_n^G)\ell_s\right\rangle_M^{\frac{1}{2}} \left\langle (I-\mathscr{Q}_n^G)z_0, (I-\mathscr{Q}_n^G)z_0\right\rangle_M^{\frac{1}{2}} \\
&\leq 2c_2^2 n^{-2r}\|\kappa^{(r)}\|_\infty \|z_0^{(r)}\|_\infty.
\end{aligned}$$

This implies that

$$\|\mathscr{K}_n'(x_0)(z_0 - \mathscr{Q}_n^G z_0)\|_\infty \leq 2c_2^2 n^{-2r}\|\kappa\|_{r,\infty}^2. \tag{4.28}$$

Since $d \geq 2r$, then combining estimates (4.25), (4.26), (4.27) and (4.28), we get

$$\|x_0 - \widetilde{y}_n^G\| \leq A_2\left(c_3 n^{-d}\|\kappa\|_{d,\infty} + 2c_2^2 n^{-2r}\|\kappa\|_{r,\infty}^2\right). \tag{4.29}$$

which completes the proof of (4.24). □

Theorem 4.5. *Assume that* $\kappa, \frac{\partial \kappa}{\partial u} \in \mathcal{C}^d(\Omega)$ *and that* $x_0 \in \mathcal{C}[-1,1]$. *Let* $\widetilde{y}_n^C$ *be the iterated discrete Kantorovich-Collocation approximation of* x_0 *given by (4.18). Then, for a sufficiently large* n, *we have*

$$\|x_0 - \widetilde{y}_n^C\|_\infty = \mathcal{O}(n^{-r}). \tag{4.30}$$

Proof . Using *Cauchy-Schwarz* inequality and estimates (4.8) and (4.25), we have

$$\begin{aligned}
\left|\mathscr{K}_n'(x_0)(z_0 - \mathscr{Q}_n^C z_0)(s)\right| &= \left|\mathscr{K}_n'(x_0)(I-\mathscr{Q}_n^C)z_0(t_i)\right| \\
&= \left|\sum_{i=1}^{M} w_i \frac{\partial \kappa}{\partial u}(s,t_i,x_0(t_i))(I-\mathscr{Q}_n^C)z_0(t_i)\right| \\
&\leq \|\kappa\|_{r,\infty}\left(\sum_{i=1}^{M} w_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{M} w_i \left[(I-\mathscr{Q}_n^C)z_0(t_i)\right]^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\|\kappa\|_{r,\infty}\left\langle (I-\mathscr{Q}_n^C)z_0, (I-\mathscr{Q}_n^C)z_0\right\rangle_M^{\frac{1}{2}}
\end{aligned}$$

which means that

$$\|\mathscr{K}_n'(x_0)(z_0 - \mathscr{Q}_n^C z_0)\|_\infty \leq 2c_2 n^{-r}\|\kappa\|_{r,\infty}^2. \tag{4.31}$$

By combining estimates (4.26), (4.27) and (4.31), we obtain

$$\|x_0 - \widetilde{y}_n^C\| \leq A_2(c_3 n^{-d}\|\kappa\|_{d,\infty} + 2c_2 n^{-r}\|\kappa\|_{r,\infty}^2). \tag{4.32}$$

Since $d > r$ the proof is completed. □

5. Numerical results

In this section, numerical examples are given to illustrate the theory established in the previous sections. Note that, all required integrals were calculated by high precision with a 6-points Gauss quadrature rule. Let $\mathbb{X}_n$ denote the space of polynomials of degree $\leq n$. The computations are done for $n = 2, 3, 4, 5, 6, 7$. We give the errors obtained by the discrete version of the *Kantorovich* method and its iterated version. In the case of interpolatory projection we give also the error for y_n^C at the collocation points

$$\max_{0\leq i\leq n} |y_n^C(\tau_i) - x(\tau_i)| = \max_i |y_{n,i}^C - x_i|.$$

Exemple .1 We consider the following *Hammerstein* equation with a degenerate kernel

$$x(s) - \int_{-1}^{1} \sinh(\xi s - 1)\cosh(t-1)[x(t)]^2 dt = f(s) \quad s \in [-1, 1],$$

where $\xi = \sqrt{2}$ and $f \in \mathcal{C}[-1, 1]$ is selected so that $x_0(s) = \sqrt{s+1}$. The results are given in Tables (5.1) and (5.2).

n	$\|y_n^G - x_0\|_\infty$	$\|\tilde{y}_n^G - x_0\|_\infty$
2	9.08×10^{-1}	1.27×10^{-2}
3	1.67×10^{-1}	1.34×10^{-4}
4	3.11×10^{-2}	3.85×10^{-5}
5	3.11×10^{-3}	3.24×10^{-6}
6	4.11×10^{-4}	8.44×10^{-7}
7	3.02×10^{-5}	6.31×10^{-7}

Table 1: ***Kantorovich-Galerkin*** **method**

n	$\|y_n^C - x_0\|_\infty$	$\max_i \|y_{n,i}^C - x_i\|$	$\|\tilde{y}_n^C - x_0\|_\infty$
2	9.89×10^{-1}	1.27×10^{-1}	1.35×10^{-2}
3	1.93×10^{-1}	3.04×10^{-3}	2.86×10^{-4}
4	3.37×10^{-2}	5.39×10^{-4}	4.80×10^{-5}
5	3.46×10^{-3}	3.85×10^{-5}	3.88×10^{-6}
6	4.37×10^{-4}	3.02×10^{-6}	8.71×10^{-7}
7	3.30×10^{-5}	1.50×10^{-7}	6.32×10^{-7}

Table 2: ***Kantorovich-Collocation*** **method**

6. Conclusion

The above tables illustrate that a high precision is reached even when the polynomials are of low degree and the exact solution is only continuous. Therefore the numerical results prove that the discrete version achieves relevant results. Note that to obtain an accuracy of comparable order by piecewise polynomials a very much larger nonlinear systems are needed to be solved. It should be mentioned that the analysis given in this paper can be extended to the case of weakly singular kernels.

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