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Perturbation of wavelet frames on non-Archimedean fields

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Abstract

The paper deals with two different aspects of wavelet frames. First, we obtain a necessary condition on irregular wavelet frames on local fields of positive characteristic and in the second aspect, we present some results on the perturbation of wavelet frames, when we disturb the mother function of a wavelet frame or dilation parameter. All the results have been carried without the compactness of support neither on generating function nor on its Fourier transform.

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1. Introduction

In a Hilbert space $\mathcal{H}$, a sequence $\{f_i\}_{i\in\mathbb{Z}}$ is called a frame, if there exists positive constants A and B such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} \left| \langle f, f_n\rangle \right|^2 \leq B\|f\|^2, \quad for \;\; all \;\; f \in \mathcal{H}.$$

The constants A and B are called the frame bounds. If the sequence $\{f_i\}_{i\in\mathbb{Z}}$ is a frame in $\mathcal{H}$, then the operator S such that $Sf = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle f_n$ is called the frame operator and can be used to provide the representation of every $f \in \mathcal{H}$. Various remarkable properties of this tool can be seen when when the Hilbert space $\mathcal{H}$ is the set of all measurable and square integrable function space $L^2(\mathbb{R})$. As such two very well known examples of frames can be considered. For $\psi \in L^2(\mathbb{R})$, a, $b > 0$, the sequence

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$\{\psi_{a,b,j,k}\}_{j,k\in\mathbb{Z}}$ defined by $\psi_{a,b,j,k} = a^{-j/2}\psi\big(a^{-j}x - \lambda b\big)$. It is called a wavelet frame if it forms a frame for $L^2(\mathbb{K})$.
In applications, numerical errors occur by modifying the function ψ in frame wavelet case or g in Weyl-Heisenberg case. Similar problems may arise if the perturbation on parameters a,b are done.

There is a substantial body of work that has been concerned with the construction of wavelets on local fields or more generally on local fields of positive characteristic. For example, R. L. Benedetto and J. J. Benedetto [1] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. As far as the characterization of wavelet frames on local fields is concerned, Shah and Abdullah [7] have established a complete characterization of tight wavelet frames on local fields by virtue of two basic equations in the frequency domain and show how to construct an orthonormal wavelet basis for $L^2(K)$. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine and so on. An introduction to the frame theory and its applications can be found in [2].
The paper is organized is follows. Section 2 disuses preliminary Fourier analysis on Non-Archimedean Fields and also some results which are prerequisite in the subsequent results. In Section 3, necessary condition on irregular wavelet frames on local fields of positive characteristic is established. Furthermore, a stability of wavelet frames is discussed under the disturbance of mother function of a wavelet frame or elsewhere the dilation parameter. All the results have been carried out under the relaxation of the compact support of generating function and on its Fourier transform. So the results are more general in this case.

2. Fourier Analysis on Non-Archimedean Fields

Any field $\mathbb{K}$ equipped with the discrete topology is called a Local field. If K is connected, then it is $\mathbb{R}$ or $\mathbb{C}$. However, if K is not connected, then it is totally disconnected. Thus a locally compact, indiscrete and totally disconnected field K is called a Local field. The additive and multiplicative groups of K are denoted by K^+ and K^* respectively. Other than this, example of a Local field of characteristic zero is p-adic field $\mathbb{Q}_p$, fields of positive characteristic are Cantor Dyadic group and Vilenkin p-groups. For further details we referr to [8, 6].
Let dx be the Haar measure on K^+. If $\lambda \in K\backslash 0$, the $d(\lambda x)$ is also a Haar measure on K^+. If we let $d(\lambda x) = |\lambda|dx$, then we call $|\alpha|$ as the absolute or valuation of λ, which is non-Archimedean on K. The valuation $x \to |x|$ with $|0| = 0$ has the following properties

(i) $|x| = 0$ if and only if $x = 0$,

(ii)$|xy = |x||y|$ for all $x,\ y \in K$,

(iii)$|x + y| \leq max\{|x|, |y|\}$ for all $x,\ y \in K$.

Property (iii) is called the ultra-metric inequality. Moreover, $|x + y| = max\{|x|, |y|\}$, if $|x| \neq |y|$. Define $\mathcal{B} = \{x \in \mathbb{K} : |x| < 1\}$. Then the set $\mathcal{B}$ is called the prime ideal in $\mathbb{K}$ which is maximal ideal in $\mathcal{D} = \{x \in \mathbb{K} : |x| \leq 1\}$. Thus $\mathcal{B}$ is both principal and prime. Therefore for such an ideal $\mathcal{B}$ in $\mathcal{D}$, we have $\mathcal{B} = \langle p\rangle = p\mathcal{D}$. Let $\mathcal{D}^* = \mathcal{D}\backslash\mathcal{B} = \{x \in \mathbb{K} : |x| = 1\}$. Then $\mathcal{D}^*$ is a group of units in

K^* and if $x \neq 0$, then we write $x = p^k z,\ z \in \mathcal{D}^*$. Moreover, $\mathcal{B}^k = p^k\mathcal{D} = \left\{x \in \mathbb{K} : |x| < q^{-k}\right\}$ are compact subgroups of K^+ and are known as the fractional ideals of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathcal{B}$ in $\mathcal{D}$, then any element $x \in \mathbb{K}$ can be uniquely written as $x = \sum_{r=k}^{\infty} c_r p^r,\ c_r \in \mathcal{U}$. Let χ is constant on cosets of $\mathcal{D}$ so if $y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(yx),\ x \in \mathbb{K}$. Suppose that $\chi_\sqcap$ is any character on K^+, then clearly the restriction $\chi_\sqcap|\mathcal{D}$ is also a character on $\mathcal{D}$ in K^+, then, as then, as it was proved in [8], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$. The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is

defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx.$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$. For any prime p and $a, b \in \mathbb{K}$, let D_p, $T_{u(n)a}$ and $E_{u(m)b}$ be the operators acting on $L^2(\mathbb{K})$ given by dilations, translations and modulations, respectively:

$D_p f(x) = \sqrt{q} f(p^{-1}x)$,

$T_{u(n)a} f(x) = f(x - u(n)a)$,

$E_{u(m)b} f(x) = \chi(u(m)bx) f(x)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where

$GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \le a_k < p, \quad \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Here after we use the notation $\chi_n = \chi_{u(n)},\ n \ge 0$.

Let the local field K be of characteristic $p>0$ and $\zeta_0,\zeta_1,\zeta_2,\ldots,\zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu\mathfrak{p}^{-j})=\begin{cases}\exp(2\pi i/p), & \mu=0 \text{ and } j=1,\\ 1, & \mu=1,\ldots,c-1 \text{ or } j\neq 1.\end{cases}$$

We also denote the test function space on K by $\mathcal{S}$, i.e., each function f in $\mathcal{S}$ is a finite linear combination of functions of the form $\mathbf{1}_k(x-h), h\in K, k\in\mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that $\mathcal{S}$ is dense in $L^p(K), 1\le p<\infty$, and each function in $\mathcal{S}$ is of compact support and so is its Fourier transform.

3. Perturbation of wavelet frames

Definition 3.1. *Suppose a function $\psi\in L^2(\mathbb{K})$, define a sequence $\{\psi_{j,k,\mathfrak{p}}\}_{j,k\in\mathbb{N}_0}$ by $\psi_{j,k,\mathfrak{p}}(x)=q^{-j/2}\psi(\mathfrak{p}^{-j}x-u(k))$. If this sequence is a frame in $L^2(\mathbb{K})$, then we call it to be a wavelet frame. Moreover for any sequence $\{\lambda_j\}_{j\in\mathbb{N}_0}$ in $\mathbb{K}^+$, the corresponding sequence $\{\psi_{j,k,\lambda_j}\}_{j,k\in\mathbb{N}_0}$ is called an irregular wavelet frame, where $\psi_{j,k,\lambda_j}(x)=\lambda^{j/2}\psi(\mathfrak{p}^{-j}x-u(k))$.*

Theorem 3.2. *Let $\psi\in L^2(\mathbb{R})$ and $\{\lambda_j\}_{j\in\mathbb{N}_0}$ be a sequence of scalers in $\mathbb{K}^+$. Then if $\{\psi_{\lambda_j,k}\}_{j,k\in\mathbb{N}_0}$ is an irregular wavelet frame with bounds A and B, we have*

$$A\le\sum_{j\in\mathbb{N}_0}\left|\hat{\psi}(\lambda_j^{-1}x)\right|^2\le B,\quad a.e\ x\in\mathbb{K}.$$

Proof. For any $\phi\in L^2(\mathbb{K})$, we have by hypothesis

$$A\|\phi\|^2\le\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\left|\left\langle\phi,\psi_{j,k}\right\rangle\right|^2\le B\|\phi\|^2. \tag{3.1}$$

For $\epsilon>0$ and $\hat{\phi}(x):=\dfrac{1}{\sqrt{\epsilon}}\mathbf{1}_{\mathcal{B}}$, we observe $\|\phi\|=1$, by making use of Plancheral theorem in (3.1), one has

$$A\|\hat{\phi}\|^2\le\sum_{j\in\mathbb{N}_0}q^{-1}\lambda_j\int_{\mathcal{B}}\left|\sum_{m\in\mathbb{N}_0}\hat{\phi}\left(\lambda_j(x+q^{-1}u(m))\right)\overline{\psi(x+q^{-1}u(m))}\right|^2dx\le B\|\hat{\phi}\|^2. \tag{3.2}$$

Taking the summation over j of the right hand side of (3.2), we obtain

$$\sum_{j\in\mathbb{N}_0}q^{-1}\lambda_j\int_{\mathcal{B}}\left|\sum_{m\in\mathbb{N}_0}\frac{1}{\sqrt{\epsilon}}\mathbf{1}_{\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\overline{\psi(x+q^{-1}u(m))}\right)\right|^2dx\le B. \tag{3.3}$$

For $\zeta\in\mathbb{K}$, such that $\mathcal{B}\subset\zeta+\mathfrak{p}^j\mathcal{B}$, then from (3.3), we have

$$\sum_{j\in\mathbb{N}_0}q^{-1}\lambda_j\int_{\mathcal{B}}\left|\hat{\psi}(x)\right|^2dx=\sum_{j\in\mathbb{N}_0}q^{-1}\int_{a+p^j\mathcal{B}}\left|\hat{\psi}(\lambda_j^{-1}x)\right|^2dx\le B.$$

By making use of Lebesgue's dominated convergence theorem, one has

$$\sum_{j\in\mathbb{N}_0} |\hat{\psi}(\lambda_j^{-1}x)|^2 \leq qB. \tag{3.4}$$

From the left hand inequality of (3.2), we have

$$A \leq \sum_{j\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left| \sum_{m\in\mathbb{N}_0} \mathbf{1}_{\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\right) \overline{\psi(x+q^{-1}u(m))} \right|^2 dx$$

$$\leq \sum_{j\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left| \sum_{m\in\mathbb{N}_0} \mathbf{1}_{\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\right) \overline{\psi(x+q^{-1}u(m))} \right|^2 dx$$

$$+ \sum_{j\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left| \sum_{m\in\mathbb{N}_0} \mathbf{1}_{a+p^j\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\right) \overline{\psi(x+q^{-1}u(m))} \right|^2 dx$$

$$= (\blacktriangle) + (\blacksquare),$$

where

$$(\blacktriangle) = \sum_{j\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left| \sum_{m\in\mathbb{N}_0} \mathbf{1}_{\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\right) \overline{\psi(x+q^{-1}u(m))} \right|^2 dx,$$

$$(\blacksquare) = \sum_{j\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left| \sum_{m\in\mathbb{N}_0} \mathbf{1}_{a+p^j\mathcal{B}}\left(\lambda_j(x+q^{-1}u(m))\right) \overline{\psi(x+q^{-1}u(m))} \right|^2 dx.$$

Choose $j > j_0$ and $\epsilon > 0$ such that $a + \mathfrak{p}^j\mathcal{B} \subset a + \mathfrak{p}^{j_0}q^{-1}\mathcal{B}$, it follows that

$$(\blacksquare) \leq \sum_{j>j_0, j_0\in\mathbb{N}_0} q^{-1}\lambda_j \int\limits_{a+\mathfrak{p}^{j_0}q^{-1}\mathcal{B}} \left|\hat{\psi}(x)\right|^2 dx.$$

Again by use of Lebesgue's dominated convergence theorem, we have for all $j_0 \in \mathbb{N}_0$

$$(\blacksquare) \to \sum_{j>j_0} \left|\hat{\psi}\left(\lambda_j^{-1}x\right)\right|^2$$

and for some $\eta > 0$, we have

$$A \leq (\blacktriangle) + \sum_{j\in\mathbb{N}_0} \left|\hat{\psi}\left(\lambda_j^{-1}x\right)\right|^2 + \eta.$$

For some $a \in \mathbb{K}$, $\mathcal{B} \subset a + \mathfrak{p}^j\mathcal{B}$, and $T_j = q\lambda_j$ we have $\mathbf{1}_{a+\mathfrak{p}^j\mathcal{B}} = 1$, thus we have

$$(\blacktriangle) = \sum_{j<m} q^{-1}\lambda_j \int\limits_{\mathcal{B}} \left|\mathbf{1}_{\mathcal{B}}\big(\lambda_j\big(x + q^{-1}u(m)\big)\big)\overline{\psi(x + q^{-1}u(m))}\right|^2$$

$$\leq \sum_{j<m} q^{-1}\lambda_j \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \left|\mathbf{1}_{\mathcal{B}}(\lambda_j(x + q^{-1}u(m)))\overline{\psi(x + q^{-1}u(m))}\right|dx$$

$$\leq \sum_{j<m} q^{-1}\lambda_j T_j \int_{\mathcal{B}} \mathbf{1}_{\mathcal{B}}(\lambda_j x)|\hat{\psi}|^2 dx$$

$$\leq \sum_{j<j_0} \int_{\mathcal{B}} |\hat{\psi}(x)|^2 dx.$$

and consequently

$$(\blacktriangle) \leq (\mathfrak{p} + 1) \int_{a+p^j\mathcal{B}} |\hat{\psi}(x)|^2 dx$$

and so

$$A \leq \frac{\eta}{2} + b^{-1} \sum_{j\in\mathbb{N}_0} |\hat{\psi}(\lambda_j^{-1}x)|^2 + \frac{\eta}{2}$$

Thus

$$qA \leq \sum_{j\in\mathbb{N}_0} \left|\hat{\psi}\big(\lambda_j^{-1}x\big)\right|^2 \tag{3.5}$$

(3.4) and (3.5) together yield

$$qA \leq \sum_{j\in\mathbb{N}_0} |\hat{\psi}(\lambda_j^{-1}x)|^2 \leq qB.$$

Proposition 3.3. *Let $\{f_n\}_{n\in\mathbb{N}_0}$ be a frame in $L^2(\mathbb{K})$ with frame bounds A, B. Then for $\{g_n\}_{n\in\mathbb{N}_0} \subset L^2(\mathbb{K})$ and $A' < A$ such that*

$$\sum_{n\in\mathbb{N}_0} \left| \langle f, f_n - g_n \rangle \right|^2 \leq A'\|f\|^2_{L^2(\mathbb{K})} \quad \text{for all } f \in L^2(\mathbb{K}).$$

Then $\{g_n\}_{n\in\mathbb{N}_0}$ is a frame for $L^2(\mathbb{K})$ with frame bounds $A\left\{1 - \sqrt{\frac{A'}{A}}\right\}^2$ and $B\left\{1 - \sqrt{\frac{A'}{B}}\right\}^2$.

Theorem 3.4. *Let $\psi \in L^2(\mathbb{R})$ such that $\{q^{j/2}\psi(\mathfrak{p}^j x - u(k))\}_{k,j\in\mathbb{N}_0}$ is a wavelet frame with bounds A and B. For some $\phi \in L^2(\mathbb{K})$ and $\lambda \in \mathbb{K}$ such that*

$$|\hat{\psi}(x) - \hat{\phi}(x)| \leq \lambda|\hat{r}(x)|, \quad a.e \quad x \in \mathbb{K} \tag{3.6}$$

where $\hat{r}(x) \in L^2(\mathbb{K})$ such that

$$|\hat{r}(x)| \leq \frac{k|x|^a}{(1 + |x|^2)^{\gamma/2}}, \quad a > 0,\ \gamma > a + 1,\ c > 0.$$

If $\lambda < \left(\dfrac{A}{K}\right)^{1/2}$, *where* $K = k^2 \dfrac{\mathfrak{p}}{(\mathfrak{p}-1)} \left\{1 + \dfrac{1}{(c-1)}\right\}$ *with* $c \in G_a =$ $\{x : |x| \le |a|, a \in \mathbb{K}\}$, *then* $\left\{q^{j/2}\phi(\mathfrak{p}^j x - u(k))\right\}_{j,k\in\mathbb{N}_0}$ *is a wavelet frame with bounds* $A\left\{1 - \left(\dfrac{\mu}{A}\right)^{1/2}\right\}^2, B\left\{1 - \left(\dfrac{\mu}{B}\right)^{1/2}\right\}^2$, *where* $\mu = K\lambda^2$.

Proof. Choose $h = \psi - \phi,\ \psi \in L^2(\mathbb{R})$, suppose that f is taken from the class of functions in $L^2(\mathbb{K})$ which have compact support. Therefore

$$\sum_{n\in\mathbb{N}_0} \hat{f}\big(\mathfrak{p}^j(x+u(n))\big)\overline{\hat{h}\big(x+u(n)\big)} \in L^2(\mathbb{K}),$$

by Parseval's identity theorem, it follows that

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \left|\left\langle f, h_{h,k}\right\rangle\right|^2 = \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \left| \int \hat{f}(x)\mathfrak{p}^{-j/2}\overline{\hat{h}(p^{-j}x)}\ \overline{\chi_{p^{-j}}(xu(k))}dx\right|^2$$

$$= \sum_{j\in\mathbb{N}_0} \mathfrak{p}^j \int \left| \sum_{j\in\mathbb{N}_0} \hat{f}\big(\mathfrak{p}^j(x+u(n))\overline{\hat{h}\big(x+u(n)\big)}\big)\right|^2 dx$$

$$= \sum_{j\in\mathbb{N}_0} \int \left|\hat{f}(x)\right| \left|\hat{\phi}\big(\mathfrak{p}^{-j}x\big)\right|^2 dx + \triangle(f),$$

where

$$\triangle(f) := \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \int_{\mathbb{K}} \hat{f}(x)\overline{\hat{f}\big(x+\mathfrak{p}^j u(k)\big)}\ \hat{h}\big(\mathfrak{p}^{-j}x + u(k)\big)\ \overline{\hat{h}\big(\mathfrak{p}^{-j}x\big)}dx.$$

By (3.6), it follows that

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \left|\left\langle f, h_{j,k}\right\rangle\right| \le \sum_{j\in\mathbb{N}_0} \int \lambda^2\left|\hat{r}(\mathfrak{p}^j x)\right|^2 |\hat{f}|^2 dx + \triangle(f). \tag{3.7}$$

To get a bound for $\sum_{j\in\mathbb{N}_0} \left|\hat{r}(\mathfrak{p}^j x)\right|^2$, we have for every $x \in L^2(\mathcal{B})$ there exists $j_0 \in \mathbb{N}_0$ such that $x = \mathfrak{p}^{j_0}x'$. Thus

$$\sum_{j\in\mathbb{N}_0} \left|\hat{r}(\mathfrak{p}^j x)\right|^2 = \sup_{x\in\mathcal{B}} \sum_{j_0\in\mathbb{N}_0} \left|\hat{r}(\mathfrak{p}^{j_0}x)\right|^2.$$

For $\alpha - \gamma < 0$ and $a_0 \in \mathbb{K}$, we have

$$\sup_{x\in\mathcal{B}} \left|\hat{r}(\mathfrak{p}^j x)\right|^2 \le \sup_{x\in\mathcal{B}} \sum_{j\in\mathbb{N}_0} k^2 \frac{\left|\mathfrak{p}^j x\right|^\alpha}{(1+|\mathfrak{p}^j x|^2)^\gamma}$$

$$\le k^2 q^{2\alpha}\left[\sum_{j\le j_0} \mathfrak{p}^{2pj} + \sum_{j>j_0} \mathfrak{p}^{2j(\alpha-\gamma)}\right]$$

$$\le 2k^2 \frac{q}{q-1}$$

Thus

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\left|\left\langle f, h_{j,k}\right\rangle\right|^2 \le k^2\frac{q}{q-1}\|f\|^2+\triangle(f). \tag{3.8}$$

By Cauchy-Schwarz inequality, we have

$$|\triangle(f)| = \left|\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\int \hat{f}(x)\ \overline{\hat{f}(x+\mathfrak{p}u(k))}\ \hat{h}\big(\mathfrak{p}^{-1}x+u(k)\big)\ \overline{\hat{h}\big(\mathfrak{p}^{-1}x\big)}dx\right|$$

$$\le \sum_{k\in\mathbb{N}_0}\left\{\int |\hat{f}(x)|^2\sum_{j\in\mathbb{N}_0}|\hat{h}(p_0^{-1}x)|\ \left|\hat{\phi}(\mathfrak{p}^{-1}x+u(k))\right|dx\right\}^{1/2}$$

$$\times\left\{\int |\hat{f}(x)|^2\sum_{j\in\mathbb{N}_0}|\hat{h}(\mathfrak{p}^{-1}x)|\ \left|\hat{h}\big(\mathfrak{p}x-u(k)\big)\right|\ dx\right\}^{1/2} \tag{3.9}$$

Let $\Psi(s) := \sup_{x\in\mathcal{B}}\sum_{j\in\mathbb{N}_0}|\hat{h}\big(\mathfrak{p}x\big)|\ \left|\hat{h}(\mathfrak{p}^j x+u(s))\right|$, we have

$$|\triangle(f)| \le \int |\hat{f}(x)|^2\sum_{k\in\mathbb{N}_0}\Big\{\Psi(u(k))\Psi(u(-k))\Big\}^{1/2}dx.$$

By (3.6) and since $\alpha-\gamma<0$, therefore

$$\Psi(s) \le \sup_{x\in\mathcal{B}}\sum_{j\in\mathbb{N}_0}\left|\hat{r}(\mathfrak{p}^j x)\right|\ \left|\hat{r}\big(\mathfrak{p}^j x+u(s)\big)\right|$$

$$\le \sup_{x\in\mathcal{B}}\sum_{j\in\mathbb{N}_0}k^2\frac{|\mathfrak{p}^j x|^\alpha}{(1+|\mathfrak{p}^j x|^2)^{\gamma/2}}\ \frac{|\mathfrak{p}x+u(s)|^\alpha}{(1+|\mathfrak{p}^j x+u(s)|^2)^{\gamma/2}}$$

$$\le k^2\sup_{x\in\mathcal{B}}\sum_{j<j_0}|\mathfrak{p}^j x|^\alpha\left(1+|\mathfrak{p}^j x+u(s)|^2\right)^{(\alpha-\gamma)/2}$$

$$+\,k^2\sup_{x\in\mathcal{B}}\sum_{j>j_0}\left[\left(1+|\mathfrak{p}^j x|^2\right)\left(1+|\mathfrak{p}^j+u(s)|^2\right)\right]^{(\alpha-\gamma)/2} \tag{3.10}$$

For $j<j_{_0}, j_{_0}\in\mathbb{N}_0$,$|s|\ge 2$ and $\dfrac{|s|}{2}\le|s|-1$, we have

$$|\mathfrak{p}^j x+u(s)| \ge |s|-|\mathfrak{p}^j x| \ge |s|-\mathfrak{p}^{j+1} \ge |s|-1 \ge \frac{|s|}{2}.$$

Then

$$\left(1+|\mathfrak{p}x+u(s)|^2\right)^{-1} \le \frac{4}{1+|s|^2}$$

Now, for $|s| \le 2$, we have

$$\left(1+|\mathfrak{p}^j x + u(s)|\right)^{-1} \le \frac{5}{1+|s|^2}$$

Thus

$$\Psi(s) \le k^2 \left\{\frac{5}{1+|s|^2}\right\}^{(\gamma-\alpha)/2} \sum_{j\in\mathbb{N}_0} \mathfrak{p}^{\alpha_j} + k^2 \sup_{x\in\mathcal{B}} \sum_{j\in\mathbb{N}_0} \left\{(1+|\mathfrak{p}^j x|^2)\left(1+|\mathfrak{p}^j x+u(s)|^2\right)\right\}^{(\alpha-\gamma)/2}$$

$$\le k^2 \frac{q}{q-1}\left\{\frac{5}{1+|s|^2}\right\}^{(\gamma-\alpha)/2} + k^2 \sum_{x\in\mathcal{B}} \sum_{j\in\mathbb{N}_0} \{(1+|\mathfrak{p}^j x|^2)(1+|\mathfrak{p}^j x+u(s)|^2)\}^{(\alpha-\gamma)/2}.$$

Also

$$k^2 \sup_{x\in\mathcal{B}} \sum_{x\in\mathbb{N}_0} \left\{(1+|\mathfrak{p}^j x|^2)\left(1+|\mathfrak{p}^j x+u(s)\right)\right\}^{-(\gamma-\alpha)(1-\epsilon)/2-\epsilon(\gamma-\alpha)/2}$$

$$\le k^2 \left\{\frac{4}{4+|s|^2}\right\}^{(\gamma-\alpha)(1-\epsilon)/2} \sup_{x\in\mathcal{B}} \sum_{j\in\mathbb{N}_0} \left(1+|\mathfrak{p}^j x|^2\right)^{-\epsilon(\gamma-\alpha)/2}$$

$$\le k^2 \left\{\frac{4}{4+|s|^2}\right\}^{(\gamma-\alpha)(1-\epsilon)/2} \sum_{j\in\mathbb{N}_0} \mathfrak{p}^{-j\epsilon(\gamma-\alpha)}$$

$$\le k^2 \frac{p_0}{p_0-1}\left\{\frac{4}{1+|s|^2}\right\}^{(\gamma-\alpha)(1-\epsilon)/2}.$$

Thus, we have

$$\Psi(s) \le k^2 \frac{q}{q-1}\left\{\left\{\frac{1}{|s|^2}\right\}^{(\gamma-\alpha)/2} + \left\{\frac{1}{1+|s|^2}\right\}^{(\gamma-\alpha)(1-\epsilon)/2}\right\},$$

For $0<\mu<\gamma-\alpha$, it follows that

$$\Psi(s) \le k^2 \frac{q}{q-1}\left(1+|s|^2\right)^{-\mu/2}.$$

Therefore

$$\sum_{k\in\mathbb{N}_0} \{\Psi(k)\Psi(-k)]^{1/2} \le \sum_{k\in\mathbb{N}_0} k^2 \frac{q}{q-1}\left[1+|u(k)|^2\right]^{-\mu/4}\left[1+|u(-k)|^2\right\}^{-\mu/2}$$

$$= k^2 \frac{q}{q-1} \sum_{k\in\mathbb{N}_0} \left\{1+|u(k)|^2\right\}^{-\mu/2}$$

$$\le k^2 \frac{q}{(-1)(\mu-1)}.$$

If $1<\mu<\gamma-\alpha$. Finally

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \left| \langle f, \psi - h\rangle_{j,k} \right|^2 \leq k^2 \frac{q}{q-1}\|\hat{f}\|^2 + k^2\frac{q}{(q-1)(\mu-1)}\|\hat{f}\|^2$$

$$\leq k^2\frac{q}{q-1}\left(1+\frac{1}{\mu-1}\right)\left(1+\frac{1}{\mu-1}\right)\|f\|^2$$

$$= M\|f\|^2,$$

with $M = k^2\frac{q}{q-1}\left(1+\frac{1}{\mu-1}\right)\left(1+\frac{1}{\mu-1}\right)$ and if $\lambda < \left(\frac{A}{K}\right)^{1/2}$, then $M < A$. Thus $\{(\psi-\phi)_{j,k}\}_{j,k\in\mathbb{N}_0}$ is a Bessel sequence with bound $M < A$ and hence by Proposition 3.3, $\{q^{j/2}\psi(p^jx-u(k))\}_{j,k\in\mathbb{N}_0}$ is a wavelet frame in $L^2(\mathbb{K})$ with bounds $A\left\{1-\left(\frac{M}{A}\right)^{1/2}\right\}^2$ and $B\left\{1-\left(\frac{M}{B}\right)^{1/2}\right\}^2$. This completes the proof of the theorem.

4. Pertubation about dilation

In this part section of the paper, we give a perturbation theorem dealing with the dilation parameter, which does not require of compactness on the support of Fourier transform of ψ.

Theorem 4.1. *Let $\{q^{j/2}\psi(\mathfrak{p}^jx-u(k))\}_{j,k\in\mathbb{N}_0}$ be a wavelet frame for $L^2(\mathbb{K})$ with bounds A and B. Suppose there exists $k>0$ such that*

$$|\hat{\psi}(x)| \leq k\frac{|x|^\alpha}{(1+|x|^2)^{\gamma/2}}, \quad \gamma > \alpha+1,\ \alpha>0.$$

For $\mathfrak{p}\in\mathbb{K}$, suppose

$$K := ess\sup_{x\in\mathbb{K}}\left|\sum_{j\in\mathbb{N}_0}|\hat{\psi}(\mathfrak{p}x)|^2 - |\hat{\psi}(\mathfrak{p}x)|^2\right|^2.$$

If $\mu := M + k^2\frac{1}{\theta-1}\left(\frac{q_0}{q_0-1}+\frac{q}{q-1}\right) < A$ with $1<\theta<\gamma-\alpha$, then $\{q^{j/2}\psi(\mathfrak{p}^jx-u(k))\}_{j,k\in\mathbb{N}_0}$ is a wavelet frame on $L^2(\mathbb{K})$ with frame bounds $A' = A-\mu$ and $B' = B+\mu$.

Proof. Let $f\in L^2(\mathbb{K})$, we have by in the proof of Theorem 3.4, that

$$\left|\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi_{j,k}\rangle\right|^2 - \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi_{j,k}\rangle\right|^2\right|$$

$$= \left|\int|\hat{f}(x)|^2\sum_{j\in\mathbb{N}_0}|\hat{\psi}(\mathfrak{p}^jx)|^2dx + \triangle_1(f) - \int|\hat{f}(x)|^2\sum_{j\in\mathbb{N}_0}|\hat{\psi}(\mathfrak{p}^jx)|^2dx - \triangle_2(f)\right|,$$

where

$$\triangle_1(f) = \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\int\hat{f}(x)\overline{\hat{f}(x+qu(k))}\hat{\psi}(\mathfrak{p}^jx+u(k))\overline{\psi(\mathfrak{p}^jx)}dx,$$

and

$$\triangle_2(f) = \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\int \hat{f}(x)\overline{\hat{f}(x+q^2u(k))}\hat{\psi}\big(\mathfrak{p}^j x+u(k)\big)\overline{\psi(\mathfrak{p}^j x)}dx.$$

If we let

$$\Psi_1(s) := \sup_{x\in\in\mathbb{K}}\sum_{j\in\mathbb{N}_0}|\hat{\psi}(\mathfrak{p}^j x)|\;|\hat{\psi}(\mathfrak{p}x+u(s))|,$$

and

$$\Psi_2(s) := \sup_{x\in\mathbb{K}}\sum_{j\in\mathbb{N}_0}|\hat{\psi}(\mathfrak{p}x)|\;|\hat{\psi}(\mathfrak{p}^j x+u(s))|,$$

then we can bound $\triangle_1(f)$ and $\triangle_2(f)$ as

$$|\triangle_1(f)| \le \|f\|^2\sum_{k\in\mathbb{N}_0}\left[\Psi_1(u(k))\Psi_1(u(-k))\right]^{1/2},$$

$$|\triangle_2(f)| \le |f\|^2\sum_{k\in\mathbb{N}_0}\left[\Psi_2(u(k))\Psi_2(u(-k))\right]^{1/2}.$$

Proceeding as in Theorem 3.4, we have

$$\sum_{k\in\mathbb{N}_0}\left[\Psi_1\big(u(k)\big)\Psi_1\big(u(-k)\big)\right]^{1/2} \le k^2\frac{q_0}{(q_0-1)(\mu-1)},\quad 1<\mu<\gamma-\alpha,$$

$$\sum_{k\in\mathbb{N}_0}\left[\Psi_2\big(u(k)\big)\Psi_2\big(u(-k)\big)\right]^{1/2} \le k^2\frac{q}{(q-1)(\mu-1)},\quad 1<\mu<\gamma-\alpha,$$

Therefore, we have

$$|\triangle_1(f)| \le \|f\|^2k^2\frac{q_0}{(q_0-1)(\mu-1)},\qquad 1<\mu<\gamma-\alpha,$$
$$|\triangle_2(f)| \le \|f\|^2k^2\frac{q}{(q-1)(\mu-1)},\qquad 1<\mu<\gamma-\alpha.$$

Thus

$$\left|\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{j,k,\mathfrak{p}_0}\rangle|^2 - \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}|\,\langle f,\psi_{j,k,\mathfrak{p}}\rangle\,|^2\right|$$

$$\le \int|\sum_{j\in\mathbb{N}_0}\hat{\psi}(\mathfrak{p}_0^j x)\big|^2 - |\hat{\psi}(\mathfrak{p}^j x)|^2\;|\hat{f}(x)|^2dx + k^2\frac{1}{\theta-1}\left(\frac{q_0}{q_0-1}+\frac{q}{q-1}\right)\|f\|^2$$

$$\le \left[K + k^2\frac{1}{\theta-1}\left(\frac{q_0}{q_0-1}+\frac{q}{q-1}\right)\right]\|f\|^2$$

$$\le A\|f\|^2.$$

Thus by Preposition 3.3, our result follows.

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