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Growth estimate for rational functions with prescribed poles and restricted zeros

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Abstract

Let $r(z) = f(z)/w(z)$ where $f(z)$ be a polynomial of degree at most n and $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$ for $1 \leq j \leq n$. If the rational function $r(z) \neq 0$ in $|z| < k$, then for $k = 1$, it is known that [A.Aziz and N.A.Rather, *Journal Mathematical Inequalities and Applications*, Vol.2, No.2(1999), 165 - 173]

$$|r(Rz)| \leq \left(\frac{|B(Rz)| + 1}{2}\right) \sup_{|z|=1} |r(z)| \quad for \quad |z| = 1$$

where $B(z) = \prod_{j=1}^{n} \{(1 - \bar{a}_j z)/(z - a_j)\}$. In this paper, we consider the case $k \geq 1$ and obtain certain results concerning the growth of the maximum modulus of the rational functions with prescribed poles and restricted zeros in the Chebyshev norm on the unit circle in the complex plane.

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1. Introduction and Statements of the Results

Let $\mathcal{P}_n$ be the set of all complex polynomials $f(z) = \sum_{j=1}^{n} a_j z^j$ of degree at most n and let $D_{k-} = \{z : |z| < k\}$, $D_{k+} = \{z : |z| > k\}$ and $T_k = \{z : |z| = k\}$. For f defined on the circle T_k in the complex domain, we write

$$M(f, k) = \sup_{z \in T_k} |f(z)| \;\; and \;\; m(f, k) = \inf_{z \in T_k} |f(z)|.$$

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For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, we set

$$w(z) = \prod_{j=1}^{n}(z - a_j) \quad , \quad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{f(z)}{w(z)} : f \in \mathcal{P}_n\right\}.$$

Then clearly $\mathcal{R}_n$ is the space of all rational functions with at most n poles $a_1, a_2, \ldots, a_n$ with finite limit at infinity. We note that $B(z) \in \mathcal{R}_n$. Throughout this paper, we shall assume that all the poles $a_1, a_2, \ldots, a_n$ lie in D_{1+}.

For $f \in \mathcal{P}_n$, we have

$$M(f, R \geq 1) \leq R^n M(f, 1). \tag{1.1}$$

The result is sharp and equality in (1.1) holds for polynomials having all their zeros at origin. Inequality (1.1) is a simple deduction from the maximum modulus principle (see [4], [5], [6]). For the class of polynomials $P \in \mathcal{P}_n$ having no zero in D_{-1} the inequality (1.1) can be sharpened. In fact, if $f \in \mathcal{P}_n$ does not vanish in D_{1-}, then

$$M(f, R \geq 1) \leq \frac{R^n + 1}{2} M(f, 1). \tag{1.2}$$

Inequality (1.2) is due to Ankeny and Rivlin [1]. Equality in (1.2) holds for $f(z) = az^n + b$, $|a| = |b| = 1$.

As a refinement of the inequality (1.2), Aziz and Dawood [2] proved that if $f \in \mathcal{P}_n$ does not vanish in D_{1-}, then

$$M(f, R \geq 1) \leq \left(\frac{R^n + 1}{2}\right) M(f, 1) - \left(\frac{R^n - 1}{2}\right) m(f, 1). \tag{1.3}$$

The equality in (1.3) holds for $f(z) = az^n + b$, $|a| = |b| = 1$.

Walsh [7, Lemma II] extended the inequality (1.1) to the rational functions $r(z) \in \mathcal{R}_n$ and proved that if $r(z) \in \mathcal{R}_n$, then for $z \in T_1$ and $R \geq 1$,

$$|r(Rz)| \leq |B(Rz)|\, M(r, 1). \tag{1.4}$$

Equality in (1.4) holds for $r(z) = \alpha B(z)$ where $\alpha \in T_1$.

Aziz and Rather [3] considered the class of rational functions $r(z) \in \mathcal{R}_n$ having no zero in D_{1-} and as an extension of (1.2), proved that if $r(z) \in R_n$ does not vanish in D_{1-}, then for $z \in T_1$ and $R \geq 1$,

$$|r(Rz)| \leq \left(\frac{|B(Rz)| + 1}{2}\right) M(r, 1). \tag{1.5}$$

The result is sharp and equality in (1.5) holds for $r(z) = B(z) + \beta$ where $\beta \in T_1$. As an extension of (1.3) to the rational functions $r(z) \in \mathcal{R}_n$ and a refinement of inequality (1.5), Aziz and Rather [3] also showed that if $r(z) \in \mathcal{R}_n$ and $r(z) \neq 0$ in D_{1-}, then for $z \in T_1$ and $R \geq 1$,

$$|r(Rz)| \leq \left(\frac{|B(Rz)|+1}{2}\right) M(r,1) - \left(\frac{|B(Rz)|-1}{2}\right) m(r,1). \tag{1.6}$$

The result is sharp and equality in (1.6) holds for $r(z) = B(z) + \beta$ where $\beta \in T_1$.

The main aim of this paper is to obtain certain growth estimates for rational functions $r(z) \in \mathcal{R}_n$ having no zero in D_{k-} where $k \geq 1$. In this direction, we first present a sharp extension of inequality (1.5) for class of rational functions $r \in \mathcal{R}_n$ having no zero in D_{k-}. More precisely, we prove:

Theorem 1.1. *If $r(z) \in \mathcal{R}_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$ and $R \geq 1$,*

$$|r(Rz)| \leq \frac{(R+k)^n(|B(Rz)|+1)}{(R+k)^n + (1+Rk)^n} M(r,1). \tag{1.7}$$

Remark 1.2. *For $k = 1$, inequality (1.7) reduces to inequality (1.5).*

Next, by involving $m(r,k)$, we establish the following refinement of the inequality (1.7).

Theorem 1.3. *If $r(z) \in \mathcal{R}_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$ and $R \geq 1$,*

$$|r(Rz)| \leq \frac{(R+k)^n(|B(Rz)|+1)}{(R+k)^n + (1+Rk)^n} M(r,1) - \frac{((1+Rk)^n |B(Rz)| - (R+k)^n)}{(R+k)^n + (1+Rk)^n} m(r,k). \tag{1.8}$$

Remark 1.4. *Clearly for $k = 1$, inequality (1.8) reduces to inequality (1.6).*

2. Preliminaries

For the proofs of our theorems, we need the following lemma which is due to Aziz and Rather [3].

Lemma 2.1. *If $r(z) \in R_n$, then for $z \in T_1$ and $R \geq 0$,*

$$|r(Rz)| + |r^*(Rz)| \leq (|B(Rz)|+1)\, M(r,1) \tag{2.1}$$

where $r^*(z) = B(z)\overline{r(1/\bar{z})}$.

3. Proofs of the Theorems

Proof .[Proof of Theorem 1.1] Let $f^*(z) = z^n\overline{f(1/\bar{z})}$ be the conjugate polynomial of $f(z)$. By hypothesis, $r(z) = f(z)/w(z)$, therefore,

$$r^*(z) = B(z)\overline{r(1/\bar{z})} = f^*(z)/w(z)$$

and we have

$$|r(z)/r^*(z)| = |f(z)/f^*(z)|$$

that is, for $R \geq 1$ and $z \in T_1$

$$|r(Rz)/r^*(Rz)| = |f(Rz)/f^*(Rz)|\,. \tag{3.1}$$

Further, since all the zeros of $f(z)$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, we write

$$f(z) = c\prod_{j=1}^{n}(z - r_j e^{i\theta_j})$$

where $r_j \geq k,\ j = 1, 2, \ldots, n$. Therefore, for $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$\begin{aligned}\left|\frac{f(Re^{i\theta})}{R^n f(e^{i\theta}/R)}\right|^2 &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - Rr_j e^{i\theta_j}}\right|^2 \\ &= \prod_{j=1}^{n}\left(\frac{R^2 + r_j^2 - 2Rr_j\cos(\theta - \theta_j)}{1 + R^2 r_j^2 - 2Rr_j\cos(\theta - \theta_j)}\right) \\ &\leq \prod_{j=1}^{n}\left(\frac{R + r_j}{1 + Rr_j}\right)^2 \\ &\leq \prod_{j=1}^{n}\left(\frac{R + k}{1 + Rk}\right)^2,\end{aligned}$$

This gives for $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$\left|\frac{f(Re^{i\theta})}{f^*(Re^{i\theta})}\right| = \left|\frac{f(Re^{i\theta})}{R^n f(e^{i\theta}/R)}\right| \leq \left(\frac{R + k}{1 + Rk}\right)^n$$

so that for $z \in T_1$ and $R \geq 1$, we have

$$\left|\frac{f(Rz)}{f^*(Rz)}\right| \leq \left(\frac{R + k}{1 + Rk}\right)^n.$$

Combining this inequality with (3.1), we get

$$|r(Rz)/r^*(Rz)| = |f(Rz)/f^*(Rz)| \leq \left(\frac{R + k}{1 + Rk}\right)^n \qquad \text{for } z \in T_1 \text{ and } R \geq 1.$$

Equivalently,

$$\left(\frac{1 + Rk}{R + k}\right)^n |r(Rz)| \leq |r^*(Rz)| \qquad \text{for } z \in T_1\,,\ R \geq 1, \tag{3.2}$$

This in conjunction with Lemma 2.1 yields for $z \in T_1$ and $R \geq 1$,

$$\begin{aligned}\left\{1 + \left(\frac{1 + Rk}{R + k}\right)^n\right\}|r(Rz)| &\leq |r(Rz)| + |r^*(Rz)| \\ &\leq (|B(Rz)| +)\,M(r, 1),\end{aligned}$$

which is equivalent to the desired result, thus completes the proof of Theorem 1.1. □

Proof .[Proof of Theorem 1.3] Since $m(r,k) = Inf_{z\in T_k}|r(z)|$, therefore, we have $m(r,k) \leq |r(z)|$ for $z \in T_k$. If $r(z)$ has a zero on T_k, then $m(r,k) = 0$ and the result follows from Theorem 1.1 in this case. So assume all the zeros of $r(z)$ lie in D_{k+} so that $m(r,k) > 0$ and it follows by the minimum modulus theorem that we have

$$m(r,k) < |r(z)| \;\; for \;\; z \in D_{k-}. \tag{3.3}$$

We show all the zeros of $g(z) = r(z) + \lambda m(r,k)$ lie in $T_k \cup D_{k+}$ for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. For, if there is a point $z = z_0$ in D_{k-} such that $r(z_0) + \lambda m(r,k) = g(z_0) = 0$, then

$$|r(z_0)| = |\lambda| m(r,k) \leq m(r,k),$$

which is a contradiction to (3.3). Hence all the zeros of rational function $g(z) = r(z) + \lambda m(r,k)$ lie in $T_k \cup D_{k+}$. Now proceeding similarly as in the proof of Theorem 1.1, from inequality (3.3) with rational function $r(z)$ replaced by $g(z)$, we get

$$\begin{aligned}\left(\frac{1+Rk}{R+k}\right)^n |g(Rz)| &\leq |g^*(Rz)| \\ &= |B(Rz)g(z/R)| \qquad \text{for } z \in T_1, \;\; R \geq 1.\end{aligned}$$

Equivalently,

$$\begin{aligned}\left(\frac{1+Rk}{R+k}\right)^n |r(Rz) + \lambda m(r,k)| \; &\leq \left| B(Rz)\overline{r(z/R)} + \bar{\lambda} B(Rz) m(r,k) \right| \\ &= |B(Rz)| \, |r(z/R) + \lambda m(r,k)|\end{aligned} \tag{3.4}$$

for $z \in T_1$ and $R \geq 1$. Choosing the argument of λ with $|\lambda| = 1$ such that

$$|r(z/R) + \lambda m(r,k)| = |r(z/R)| - m(r,k),$$

which is possible by (3.3), we obtain from inequality (3.4) for $z \in T_1$ and $R \geq 1$,

$$\begin{aligned}\left(\frac{1+Rk}{R+k}\right)^n \{|r(Rz)| - m(r,k)\} &\leq \left| B(Rz)\overline{r(z/R)} \right| - |B(Rz)| m(r,k) \\ &= |r^*(Rz)| - |B(Rz)| m(r,k).\end{aligned}$$

This gives for $z \in T_1$ and $R \geq 1$,

$$\left(\frac{1+Rk}{R+k}\right)^n |r(Rz)| + \left\{ |B(Rz)| - \left(\frac{1+Rk}{R+k}\right)^n \right\} m(r,k) \leq |r^*(Rz)| \,.$$

Combining this with Lemma 2.1, we obtain we get for $z \in T_1$ and $R \geq 1$,

$$\begin{aligned}\left[1 + \left(\frac{1+Rk}{R+k}\right)^n\right] |r(Rz)| + \left\{ |B(Rz)| - \left(\frac{1+Rk}{R+k}\right)^n \right\} m(r,k) & \\ \leq |r(Rz)| + |r^*(Rz)| \,. & \\ \leq \left(|B(Rz)| + 1\right) M(r,1), &\end{aligned}$$

which gives for $z \in T_1$ and $R \geq 1$,

$$\begin{aligned}|r(Rz)| \leq \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n} \left(|B(Rz)| + 1\right) M(r,1) & \\ - \left(\frac{(R+k)^n \, |B(Rz)| - (1+Rk)^n}{(R+k)^n + (1+Rk)^n}\right) m(r,k). &\end{aligned}$$

This proves Theorem 1.3. $\square$

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