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Solving binary semidefinite programming problems and binary linear programming problems via multi objective programming

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Abstract

In recent years the binary quadratic program has grown in combinatorial optimization. Quadratic programming can be formulated as a semidefinite programming problem. In this paper, we consider the general form of binary semidefinite programming problems (BSDP). We show the optimal solutions of the BSDP belong to the efficient set of a semidefinite multiobjective programming problem (SDMOP). Although finding all efficient points for multiobjective is not an easy problem, but solving a continuous problem would be easier than a discrete variable problem. In this paper, we solve an SDMOP, as an auxiliary, instead of BSDP. We show the performance of our method by generating and solving random problems.

Keywords: Semidefinite programming, Positive semidefinite matrix, Multiobjective programming, Binary programming.

1. Introduction

As is well known, NP-hard problems contain a large spectrum of applications in computer science, operations research and engineering [1, 5, 9, 12, 13]. Recently, it has emerged as a unified framework for modelling a wide variety of combinatorial optimization problems [7]. Therefore, the study of its robust and effective solvers becomes a prolonged research subject. Common continuous approaches

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are nearly all related to certain relaxation schemes of binary programming in continuous space such as linear relaxations, Lagrangian relaxations and semidefinite programming relaxations [6, 14, 15]. Semidefinite programming (SDP) is a subfield of convex optimization concerned with the optimization of a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space [8].

At first glance it might seem solving a discrete variable problem would be easier than a continuous problem. After all, for a variable within a given range, a set of discrete values within the range is finite whereas the number of continuous values is infinite. When searching for an optimum, it seems it would be easier to search in a finite set rather than in an infinite set. Moreover, continuous approaches often need to cooperate with branch and bound algorithms or some heuristic strategies so as to generate an exact solution or a desirable approximate one.

In [17], authors developed semidefinite programming relaxation techniques for some mixed binary quadratically constrained quadratic programs and analyze their approximation performance. Also in [2], authors review and compare some semidefinite relaxations for quadratically constrained quadratic programming. In this paper, we construct a semidefinite multiobjective programming as an axillary problem, and show some efficient solutions are the optimal solution of the main problem.

This paper is organized as follows. In Section 2, we recall some concepts in semidefinite programming and multiobjective programming. In Section 3, we construct a relaxation of main problem and prove two problems is equivalent, and some another results are discused. In Section 4, we report numerical results for binary programming.

2. Preliminaries

2.1. Semidefinite programming

Definition 2.1. *([16]) A symmetric matrix $S \in \mathbb{R}^{n\times n}$ denoted by $S \succeq 0$, is said to be positive semidefinite (PSD) if*

$$\forall x \in \mathbb{R}^n, \quad x^T S x \geq 0.$$

It is positive definite (PD) ($S \succ 0$) if

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \quad x^T S x > 0.$$

The set of all PSD matrices ($n\times n$) and all PD matrices ($n\times n$) is denoted by S^n_+ and S^n_{++}, respectively.

In matrix space the inner product is defined as follows:

$$\langle A, B\rangle = A \bullet B = tr(A^T B) = tr(A^T B) = \sum_{i,j} A_{ij} B_{ij}.$$

The general form of SDP problems is as follows:

$$\begin{aligned} min \quad & C \bullet X \\ s.t. \quad & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \\ & X \succeq 0, \end{aligned} \tag{2.1}$$

where C and A_i are $n \times n$ symmetric matrices, $b_i \in \mathbb{R}$ and X is an $n \times n$ symmetric matrix, with entries as the decision variables. The notation $\bullet$ shows the inner product between two matrices, finally $X \succeq 0$ indicate the matrix X should be PSD. The dual problem of (2.1) can be defined as follows:

$$\begin{aligned} max \quad & \sum_{i=1}^{m} b_i y_i \\ s.t. \quad & S = C - \sum_{i=1}^{m} y_i A_i, \\ & S \succeq 0, \end{aligned} \tag{2.2}$$

where $y_i \in \mathbb{R}$ and S is an $n \times n$ symmetric matrix, for more details see [16].
Similar to LP, we can derive weak and strong duality theorems for SDP problems. However, to construct strong duality we need some well-known conditions, such as Slater's condition.

Definition 2.2. *(Slater's condition) [16] Problems (2.1) and (2.2) have the Slater's condition if there exists feasible points X^0 and S^0 for problems (2.1) and (2.2), respectively, such that $X^0 \succ 0$ and $S^0 \succ 0$.*

Suppose that, some of the variables be binary, so consider a generall form of mixed linear and semidefinite programming problem with some binary are as follows:

$$\begin{aligned} min \quad & C \bullet X + c^T x \\ s.t. \quad & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \\ & Bx = f, \\ & X \succeq 0, \quad x \geq 0, \\ & x_q \in \{0, 1\}, \quad q \in Q, \\ & X_{ij} \in \{0, 1\}, \quad ij \in L. \end{aligned} \tag{2.3}$$

where C and A_i are $n \times n$ symmetric matrices, $b_i \in \mathbb{R}$ and X is an $n \times n$ symmetric matrix, with entries as the decision variables., and also $B \in \mathbb{R}^{k \times l}$, $x \in \mathbb{R}^l$, $f \in \mathbb{R}^k$ and $c \in \mathbb{R}^l$ is the decision vector, also $Q \subseteq \{1, 2, ..., l\}$ and $L \subseteq \{(v, w) : v = 1, 2, ..., n, \ w = 1, 2, ..., n\}$.

2.2. Multiobjective semidefinite programming

A multiobjective semidefnite programming problem is an optimization problem that involves multiple objective functions together with semidefinite decision matrix instead of nonnegative decision vector. In mathematical terms, it can be formulated as:

$$\begin{aligned} min \quad & (C_1 \bullet X, C_2 \bullet X, ..., C_k \bullet X) \\ & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \\ & X \succeq 0. \end{aligned}$$

For these problems, rarely exists a solution that optimizes all of the objectives, simultaneously. In this case we ask efficient solutions. A solution is called efficient, if none of the objective functions can be improved in value without degrading some of the other objectives. In linear case, a feasible point X_0 is said to be an extreme efficient solution if it is extreme and efficient.

3. General formulation and Theorems

In this section, we propose an approach to solve problem (2.3) using multiobjective programming. To solve problem (2.3), we can solve the following nonlinear semidefinite programming, instead:

$$\begin{array}{rl} min & C \bullet X + c^T x \\ s.t. & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \\ & Bx = f, \\ & X \succeq 0, \quad x \geq 0, \\ & x_q(1 - x_q) = 0, \quad q \in Q, \\ & X_{ij}(1 - X_{ij}) = 0, \quad ij \in L. \end{array} \tag{3.1}$$

Note that the feasible region of problem (3.1) is nonconvex, so it is difficult to solve it. To resolve this difficulty, we introduce a multiobjective semidefinite programming problem and show that an special set of efficient solutions of this problem are exactly the optimal solutions of problem (3.1). Consider the following multiobjective semidefinite programming problem:

$$\begin{array}{rl} min & (C \bullet X + c^T x, x_Q, 1 - x_Q, X_L, 1 - X_L) \\ s.t. & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \\ & Bx = f, \\ & X \succeq 0, \quad x \geq 0. \end{array} \tag{3.2}$$

where $x_Q = (x_{i_1}, x_{i_2}, ..., x_{i_{|Q|}})$ and $X_L = (X_{j_1}, X_{j_2}, ..., X_{j_{|L|}})$. Now we introduce the set Γ as follows:

$$\Gamma = \{(x, X) : (x, X) \text{is efficient and for problem (3.2)}, \;\; x_q(1 - x_q) = 0, \; \forall q \in Q, X_{ij}(1 - X_{ij}) = 0, \; \forall ij \in L\},$$

then the following lemma is derived.

Lemma 3.1. *Γ is a closed set.*

Proof . According to [4], we know the the set of all efficient solutions is closed.
Now assume that $\{(x^k, X^k)\}$ be an elements in Γ which converges to some point (x^0, X^0). Since the product of two continoues function is a continoues function, then we have

$$\begin{aligned} x_q^0(1 - x_q^0) &= \lim_{k\to\infty} x_q^k \lim_{k\to\infty}(1 - x_q^k) = \lim_{k\to\infty} x_q^k(1 - x_q^k) = \lim_{k\to\infty} 0 = 0, \\ X_{ij}^0(1 - X_{ij}^0) &= \lim_{k\to\infty} X_{ij}^k \lim_{k\to\infty}(1 - X_{ij}^k) = \lim_{k\to\infty} X_{ij}^k(1 - X_{ij}^k) = \lim_{k\to\infty} 0 = 0. \end{aligned}$$

Hence (x^0, X^0) belongs to Γ and hence Γ is closed. $\square$

Corollary 3.2. *If the feasible set of problem (2.3) is bounded, then $C \bullet X + c^T x$ gets its minimum value.*

Theorem 3.3. *Suppose that the problem (2.3) is bounded from below. Also assume that (x^0, X^0) be a point in Γ by which $C \bullet X + c^T x$ gets its minimum value on Γ. Then (x^0, X^0) is the optimal solution for problem (2.3).*

Proof . Since $x_q(1-x_q)=0$ and $X_{ij}(1-X_{ij})=0$, then the point X^0 is a feasible point for problem (2.3). Assume that (x^0,X^0) is not an optimal point for problem (2.3) according to the assumption, the feasible space of this problem is compact and hence there exists an optimal solution (x^1,X^1), therefore $C\bullet X^1+c^Tx^1<C\bullet X^0+c^Tx^0$. If (x^1,X^1) is efficient, then $(x^1,X^1)\in\Gamma$ and hence it is in contrary with the assumption for (x^0,X^0). If (x^1,X^1) is not efficient, so there exist (x^2,X^2) such that

$$C\bullet X^2+c^Tx^2\le C\bullet X^1+c^Tx^1, \tag{3.3}$$
$$x_q^2\le x_q^1, X_L^2\le X_L^1, \tag{3.4}$$
$$1-x_q^2\le 1-x_q^1, 1-X_L^2\le 1-X_L^1 \tag{3.5}$$

and at least one of them is strict. From (3.4) and (3.5) we have $x_q^2=x_q^1, X_L^2=X_L^1$, therefore $(x^2,X^2)\in\Gamma$. On the other hand $C\bullet X^2+c^Tx^2<C\bullet X^1+c^Tx^1$ must be held, again this is in contrary with the assumption for (x^0,X^0). □ Now, we are going to derive another theorem which shows that there is an optimal solution for problem (2.3) that is efficient for problem (3.2).

Theorem 3.4. *Let (x^0,X^0) be an optimal solution for problem (2.3). Then $(x^0,X^0)\in\Gamma$.*

Proof . First, we consider the following axillary problem:

$$\begin{array}{ll} Min & C\bullet X+c^Tx \\ s.t. & A_i\bullet X=b_i,\quad i=1,2,...,m, \\ & Bx=f, \\ & C\bullet X+c^Tx\le C\bullet X^0+c^Tx^0, \\ & x_q\le x_q^0,\ \ 1-x_q\le 1-x_q^0, \\ & X_L\le X_L^0,\ \ 1-X_L\le 1-X_L^0, \\ & X\succeq 0,\ \ x\ge 0. \end{array} \tag{3.6}$$

This is obvious that (x^0,X^0) is an optimal solution of (3.6). So $(x^0,X^0)\in\Gamma$ which completes the proof. □ But, sometimes problem (3.1) might not be bounded. To raise this issue, we derive the following theorem which investigates the unboundedness and infeasibility of problem (3.2).

Theorem 3.5. *If $\Gamma=\varnothing$, problem (3.1) is either unbounded or infeasible.*

Proof . Suppose that $\Gamma=\varnothing$ and problem (3.1) be feasible. If the problem is bounded from below, then from Theorem 3.3 there exists an optimal solution belong to Γ and therefore, $\Gamma\neq\varnothing$ which is a contradiction. Hence, the problem should be unbounded. □ In linear case, we will show that it is not required to search through all the solutions, while it is sufficient to only check the extreme efficient points to find the optimal solution. Consider the following binary linear programming:

$$\begin{array}{ll} min & c^Tx \\ s.t. & Bx=f, \\ & x_q\in\{0,1\},\quad q\in Q \\ & x\ge 0. \end{array} \tag{3.7}$$

Theorem 3.6. *The optimal solution of problem (3.7) is happened in one of the extreme efficient points of the following problem*

$$\begin{aligned} min \quad & (c^T x, x_Q, 1 - x_Q) \\ s.t. \quad & Bx = f, \\ & x_q \leq 1, \quad q \in Q \\ & x \geq 0. \end{aligned} \tag{3.8}$$

which belongs to

$$\Gamma = \{x : x \textit{ is efficient for problem } (3.8) \textit{ and } x_q(1 - x_q) = 0, \ \forall q \in Q\}.$$

Proof . We know that at least one of the optimal solutions of problem (3.7) belongs to the set Γ. Suppose that none of these optimal solutions is an extreme point and let z be an optimal solution for problem (3.7) which belongs to Γ but is not an extreme point for problem (3.8). By the representation theorem there are real numbers λ_i, $i = 1, 2, ..., n$ and extreme points x^i of feasible set of (3.8), such that:

$$z = \sum_{i=1}^{l} \lambda_i x^i \tag{3.9}$$

$$\sum_{i=1}^{l} \lambda_i = 1, \qquad \lambda_i > 0 \ \ i = 1, 2, ..., l.$$

For an arbitrary $q \in Q$, it is easy to show that if $z_q = 0$ then $x_q^i = 0, \quad \forall i = 1, 2, ..., l$, and if, $z_q = 1$ since $x_q^i \leq 1$, then $x_q^i = 1, \quad \forall i = 1, 2, ..., l$. Therefore $x_q^i(1 - x_q^i) = 0, \quad \forall i = 1, 2, ..., l$. Since $z = \sum_{i=1}^{l} \lambda_i x^i$ then $c^T z = \sum_{i=1}^{l} \lambda_i c^T x^i$ and from optimality of z we have $c^T z = c^T x^i$. Now, let y be a feasible solution for problem (3.8) such that y dominates z. From $y_q \leq z_q$ and $1 - y_q \leq 1 - z_q$ for $q \in Q$, we conclude that $y_q = z_q$, then y is feasible for (3.7). Finally, the optimality of z indeed a contradiction. $\square$

Theorem 3.6 says to seek a minimizer on Γ we can explore the extreme efficients of problem (3.8). Therefore the examined set is smaller and we can use some available algorithms such as the algorithms mentioned in [3] and [11].

In [3] a nonadjacent extreme-point search algorithm is presented for finding a globally optimal solution for a linear programming problem . The algorithm finds an exact extreme-point optimal solution for the problem after a finite number of iterations. It can be implemented using only linear programming methods.

4. Example

In order to test our algorithm, we generate some random problem in which the dimension of matrix B and vector f are also random numbers. The set of binary variables and its cardinal number is shown by Q and $|Q|$, respectively. we use weighted sum method for scalarization of objective vector in which the weights are chosen by genetic algorithm ([10]). Then we use existence algorithm (CPLEX solver), heuristic algorithm (genetic algorithm) and our algorithm. Some results are shown in Table 1. In this table, k is the number of constraints, l is the size of variables, x_q is binary for Q is a subset of $q \in Q \subseteq \{1, 2, ..., n\}$.

As it is seen, for small l, the three algorithms work the same, but for large l the existing algorithms cannot be reached the optimal solution, although heuristic algorithm has some solution which we do not know that is optimal or not. In Figure 1 the CPU time of three algorithms are illustrated.

Table 1: CPU Time (s)

l	k	$\|Q\|$	Existence algorithm	New Algorithm	Heuristic algorithm [10]
10	5	3	15.7512	15.7512	15.7512
20	5	5	18.2155	18.2155	18.2155s
50	20	10	73.4570	83.5788	73.4570
100	20	20	105.3266	160.0112	107.9997
1000	50	50	209.5488	483.9156	220.4588
1000	50	100	261.3657	520.4820	266.4874
2000	100	200	309.3657	698.0125	327.0198
2000	200	400	507.6497	1029.1349	507.6497
2000	100	1000	-	11588.2458	798.1407
2000	200	1000	-	9852.6951	1198.9803

As it is seen, for small l, the most algorithms work well, but for large l exact algorithms can not be reached the optimal solution, although heuristic algorithm has some solution which we do not know that is optimal or not. In Figure 1 we illustrate the CPU time of three algorithms.

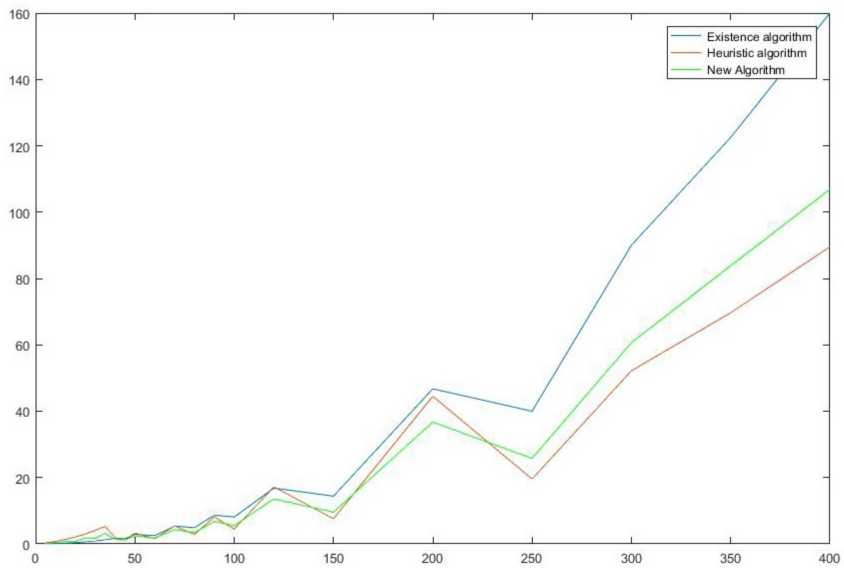

Figure 1: The comparison of CPU time

5. Concluding remarks

In this paper, we have converted binary semidefinite programming to continuous multiobjective programming and have proved that two problems are equivalent. Then we have sought a special global minimizer of the resulting continuous optimization problem to find the optimal solution to the main problem. The weighted sum approach is used for obtaining efficient solutions. The efficiency of our algorithm is illustrated by generating and solving random problems. We hope to develop this algorithm in our feature works, specifically using a combination of $\epsilon-$ constraint and weighting min-max method.

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