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Approximate analytic solution of composed linear descriptor operator system using functional analysis approach

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Abstract

This paper focuses on the solvability and approximate analytic solution of composed linear descriptor operator with constant coefficient, using functional (Variational) approach. This approach is based on finding a suitable functional form whose critical points are the solution of the proposed problem and the solution of a proposed problem is a critical point of the obtained variational functional defined on suitable reflexive Hilbert space since the existence of this approach is based on the symmetry and positivity of the composed linear descriptor operator on Hilbert space. The necessary mathematical requirements are derived and proved. A step-by-step computational algorithm is proposed. Illustration and computation with the giving exact solution are also proposed.

Keywords: Descriptor System, Differential-Algebraic Operator equation, Functional Analysis, Variational Formulation, {1}-Inverse Matrix, Ritz Method.

1. Introduction

The linear descriptor system is a linear system expressed in descriptor form, which is capable of handling systems that are not possible to be formed as state-space expression, such as those involving order changes and algebraic constraints among internal variables. The descriptor systems are also called differential-algebraic systems, due to the present of both differential and algebraic equations. These equations may also contain hidden constraints, consistency requirements for initial conditions, and unexpected regularity requirements, [3, 12, 14, 11].

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A large number of physical problem of constrained dynamical systems are modeled as differential-algebraic equations, there are numerous applications in the modeling and simulation of DAE's, such as electrical circuits[8], mechanical multibody systems [13], the approximation of singular perturbation problems arising in fluid dynamics, mathematical economics the analysis of chemical processes, biotechnology, radio physics, circuit theory, and chemical and biological kinetics, for more application of DAE's, we refer to [21, 23, 10, 26, 9].

Many considerable progress has been made to investigating in the descriptor system, for linear descriptor system and for nonlinear system see surveys [7, 15], on the investigation of stability, solvability and controllability of general descriptor system many result have been derived [9, 19, 20].

The linear differential-algebraic equations are also arised typically from the linearization of non-linear DAE systems along a reference trajectory, see [5].
The solvability of constant square singular matrix system of differential algebraic equation has been developed by [5], the author used the application of Drazin inverse theory and core-nillpotent decomposition, to formulate the compact form solution of the descriptor system. Composed linear descriptor operator which is an (n-coupled pair) mixing the descriptor operator together with the initial condition operator. The normal solvability and some functional properties of this composed descriptor operator have been developed in [24, 18].

In this paper, A descriptor - composed operator is proposed. The solvability and approximate analytic solution in operator form are proved. numerical illustration using the theoretical results are provided. This approach will help to solve a large numbers of applications easily. The following preliminary are needed later on.

2. Preliminary

The following mathematical concepts are needed:

Definition 2.1. *[22] A given linear operator $D : dom\,(D) \subset X \to Y$, where X, Y are suitable spaces, is*

(i) Symmetric with respect to the chosen bilinear form $\langle x,\ y\rangle$ if the condition $\langle Dx_1,\ x_2\rangle = \langle Dx_2, x_1\rangle$ is satisfied for every $x_1,\, x_2 \in dom(D)$.

(ii) Positive in its domain, if it is symmetric and $\langle Dx,\ x\rangle \geq 0, \forall x(t) \in dom\,(D)$ and $\langle Dx,\ x\rangle = 0 \Leftrightarrow x = 0,\ \forall x \in dom\,(D)$, hold .

Definition 2.2. *[1] For any square or rectangular matrix E of real or complex number, there exist unique matrix E^+ satisfies the following*

(i) $EE^+E = E$,

(ii) $E^+EE^+ = E^+$,

(iii) $(E^+E)^* = E^+E$,

(iv) $(EE^+)^* = EE^+$

If the matrix E^+ satisfies (i), it is called {1}-inverse of the matrix E, if it is satisfied (i)-(ii), then it is called {1-2}- inverse, if it is satisfy (i)-(iii), it is called {1-2-3}- inverse, and if it is satisfied the conditions (i)-(iv), then it is known as the Moore-Penrose inverse of E.

3. Statement of the Problem

Consider the linear constant coefficient descriptor system of the form

$$E\frac{d}{dt}x(t) + \ Ax(t) = f(t), t \in I \tag{3.1}$$

where $0 < rank\,(E) < m$, wher $m > n$ and E is singular, define

$$Dx \triangleq E\frac{d}{dt}x(t) + \ Ax(t), \ x(t) \in domD \subset X \tag{3.2}$$

where $D : dom\,(D) \subset X \to im\,(D) \subseteq Y$, X, Y are separable Hilbert space, the matrices $E, A \in \mathcal{L}\,(R^m, R^n)$ are $m \times n$, the compact interval I is defined by $I = [t_0, t_1]$, and for given functions $f(t) \in im\,(D) \subseteq Y$ being at least continuous on the interval $I \subseteq R$, with the initial condition

$$Ex(t_0) = d \in R^n, \tag{3.3}$$

which is assumed to be consistent. The consistency of (3.3) may be obtained using the following requirement $E^+Ed = d$, For some $E^+ \in \mathcal{L}\,(R^n, R^m)$ is $\{1\}-$ invers of the matrix E, the solution of (3.3) can be obtained as

$$x(t_0) \ = \ \mathrm{E}^+d + \ (I_{n\times n} \ - E^+E\ )y(t_0)$$

for arbitrary y $(t_0) \in \ R^n$,

$$Ex(t_0) = EE^+d + \ E\left(I_{n\times n} \ - E^+E\right)y(t_0) = EE^+d + E\ y(t_0) - EE^+Ey(t_0) \ = d$$

The initial value problem corresponding to (3.1) - (3.3) is redefined as descriptor - composed operator is given by:

$$(Dx(t), Ex(t_0)) \triangleq (f, d)\,, \ x(t) \in dom\,(D)\,, \ x(t_0) = x_0 \in R^m \tag{3.4}$$

Let $X \overset{\text{def}}{=} L_2 \ \ (I, R^m) \times R^m$, $Y \overset{\text{def}}{=} L_2 \ \ (I, R^n) \times R^n$ are the Catesian product of Hilbert spaces, for all pairs (x, x_0), where $x \in L_2 \ \ (I, R^m)\,, \ x_0 \in R^m$, under the operations

$$(x, x_0) + (y, y_0) = (x + y, x_0 + y_0)$$
$$\alpha\,(x, x_0) = (\alpha x, \alpha x_0)\,,$$

and endowed inner product defined as follow:

$$\langle (x(t)\ , x(t_0))\,, (y(t)\ , y(t_0))\rangle_Y \overset{\text{def}}{=} \ \langle x(t), y(t)\rangle_{L_2 \ \ (I,R^n)} + \langle x(t_0), y(t_0)\rangle_{R^n}$$
$$= \int_{t_0}^{t_1} x^T(t)y(t)dt + x^T(t_0)y(t_0), \tag{3.5}$$

$$\|(x\ , x_0)\|_Y = \|x\|_{L_2(I,R^n)} + \|x(t_0)\|_{R^n} = \left(\int_{t_0}^{t_1} |x(t)|^2_{R^n}\,dt\right)^{\frac{1}{2}} + \sqrt{|x(t_0)|^2_{R^n}}\ , \ |x(t)|_{R^n}$$
$$= \sqrt{\sum_{i=1}^{m} |x_i(t)|^2} \tag{3.6}$$

Now let us define the set

$G^E \overset{\text{def}}{=} \{(x, x(t_0)) \in Y : Ex(t) \in \ W_2^m, Ex_0 = d, \ E^+Ed = d\}\,, E^+$ is $\{1\}$-invers of E where W_2^m denotes the collection of all absolutely continuous vector functions from $L_2 \ \ (I, R^m)$ whose derivatives

lie in $L_2 \ (I, R^m)$ and $Ex(t)$ is an m-vector function whose i^{th} component is a linear combination of components of $x(t)$, with coefficients $\{E_{\text{ij}}\}_{j=1}^{n}$. The density of the set G^E is coming from the fact that the set of all polynomials are dense in W_2^m, with $W_2^m \subset G^E$.
Then the densely define linear descriptor-composed operator $(D, E) : G^E \subseteq X \rightarrow im\left((D, E)\right) \subseteq Y$ can be defined as:

$$(Dx(t), Ex(t_0)) \overset{\text{def}}{=} (f, d) \in Y, \quad \text{for} \quad (x(t)\ , x(t_0)) \in G^E \tag{3.7}$$

4. **The solvability of densely define linear descriptor-composed operator**

The solvability of the linear descriptor-composed operator defined by (3.4)-(3.7) by Variational formulation is proposed on the fact that the linear descriptor-composed operator is symmetric with respect to usual endowed inner product form (3.5). The following lemma shows that the operator defined by (3.4)-(3.7) is not symmetric with respect to usual endowed inner product (3.5).

Lemma 3.1. *The linear descriptor-composed operator* $(D, E) \ : G^E \rightarrow im\left((D, E)\right) \subseteq Y$ *given by* (3.4)-(3.7) *is not symmetric with respect to usual endowed inner product* (3.5)*.*
Proof . *Let* $(z(t)\ , z(t_0))$ *and* $(x(t)\ , x(t_0)) \in G^E, t \in [t_0\ ,\ \ t_1\ ]$

$$\begin{aligned} \langle (Dx(t), Ex(t_0)), (z(t)\ , z(t_0))\rangle_Y &= \int_{t_0}^{t_1} \left(E\frac{Dx(t)}{dt} + \ Ax(t)\right)^T z(t)dt + \left(Ex(t_0)\right)^T z(t_0) \\ &= \int_{t_0}^{t_1} \left[\left(\frac{dx^T(t)}{dt} E^T E^{+T} \ \right) \ E^T z(t) + A^T x^T(t) z(t) \right] dt + \ x^T(t_0) E^T z(t_0) \end{aligned} \tag{3.8}$$

where E^+ *is* $\{1\}$*-invers of the matrix* E*, using integration by parts for the first part of (8),*

$$\begin{aligned} &= x^T(t) E^T E^{+T} \ E^T z(t)|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[x^T(t) E^T E^{+T} \ E^T \frac{dz(t)}{dt} + A^T x^T(t) z(t) \right] dt + \ x^T(t_0) E^T z(t_0) \\ &= \int_{t_0}^{t_1} x^T(t) \left(-E^T E^{+T} \ E^T \ \frac{dz(t)}{dt} + A^T z(t) \right) dt + x^T(t_0) E^T z(t_0) + x^T(t) E^T E^{+T} E^T z(t)|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} x^T(t) \ \left(-E^T \frac{dz(t)}{dt} + A^T z(t) \right) dt + \ x^T(t_0) E^T z(t_0) + x^T(t) E^T E^{+T} \ E^T z(t) \\ &+ x^T(t_0) E^T E^{+T} \ E^T z(t_0) \\ &= - \left\langle \left(D^T z(t)\ , E^T z(t_0) \right), (x(t), x(t_0)) \right\rangle_Y + x^T(t) E^T E^{+T} \ E^T z(t) + x^T(t_0) E^T E^{+T} \ E^T z(t_0) \end{aligned}$$

□

Remark 3.2. *Due to the present of the operator* $E\frac{Dx(t)}{dt}$*, where* $0 < rank(E) < m$*, the descriptor-composed operator* (3.4)-(3.7) *is not symmetric with respect to usual endowed inner product* (3.5)*. Even when is full rank E (is invertible), the system is still not symmetric linear ordinary equation [24], in [24] the author suggest to used convolution bilinear form with zero initial condition to insure the symmetry. while in [16] the author have shown that there exist infinity of bilinear forms so that the linear operator is symmetric. In this work a class of descriptor-composed operator is considered, and a composed Cartesian product bilinear form is defined to ensure the symmetry and positivity of the descriptor-composed operator* (3.4)-(3.7)*. This will help us to study the solvability of the composed operator and then the approximate analytic solution is obtained.*

let $(x(t), x(t_0)), (y(t), y(t_0)) \in G^E$, the composed Cartesian product bilinear form will defined as follows:

$$\langle (x(t), x(t_0)), (y(t), y(t_0)) \rangle_{(Y,New)} \overset{\text{def}}{=} [(x(t), x(t_0)), (\mathrm{Dy}(t), Ey(t_0))], \tag{3.9}$$

which makes the given linear operator $(D, E) : G^E \to \mathrm{im}((D, E)) \subseteq Y$ symmetric and positive with respect to (3.9).

Lemma 3.3. *The linear descriptor-composed operator* $(D, E) : G^E \to im((D, E)) \subseteq Y$ *given by* (3.4)-(3.7) *is symmetric and positivity with respect to composed Cartesian product bilinear form* (3.9).
Proof . *For any* $(x(t), x(t_0)) \in G^E$

$$\begin{aligned}
&\langle (Dx(t), Ex(t_0)), (y(t), y(t_0)) \rangle_{(Y,New)} \\
&= [(Dx(t), Ex(t_0)), (Dy(t), Ey(t_0))] \\
&= \int_{t_0}^{t_1} \left(E\frac{Dx(t)}{dt} + \ Ax(t)\right)^T \left(E\frac{dy(t)}{dt} + \ Ay(t)\right) dt + (Ex(t_0))^T Ey(t_0) \\
&= \int_{t_0}^{t_1} \left(E\frac{dy(t)}{dt} + \ Ay(t)\right)^T \left(E\frac{dx(t)}{dt} + \ Ax(t)\right) dt + (Ey(t_0))^T Ex(t_0) \\
&= [(Dy(t), Ey(t_0)), (Dx(t), Ex(t_0))] = \langle (Dy(t), Ey(t_0)), (x(t), x(t_0)) \rangle_{(Y,New)}.
\end{aligned}$$

Then it is symmetric.
Now to prove the positivity of the descriptor-composed operator,

$$\begin{aligned}
&\langle (Dx(t), Ex(t_0)), (x(t), x(t_0)) \rangle_{(Y,New)} \\
&= [(Dx(t), Ex(t_0)), (Dx(t), Ex(t_0))] = [(Dx(t), Ex(t_0)), (Dx(t), Ex(t_0))] \\
&= \langle Dx(t), Dx(t) \rangle + \langle Ex(t_0), Ex(t_0) \rangle = \|Dx\|^2 + \|Ex_0\|^2 > 0, \ \forall (x(t), x(t_0)) \in G^E, \ t \in [t_0 \ , \ \ t_1 \ ]
\end{aligned}$$

By the continuity and positivity of the norm,

$$\begin{aligned}
&\|Dx\|^2 + \|Ex_0\|^2 = 0 \Leftrightarrow Dx(t) = 0 \ and \ Ex(t_0) = 0 \Rightarrow [(Dx(t), Ex(t_0)), (Dx(t), Ex(t_0))] = (0, 0) \\
&\Rightarrow \ \langle (Dx(t), Ex(t_0)), (x(t), x(t_0)) \rangle_{(Y,New)} = \langle 0, 0 \rangle \Rightarrow (x(t), x(t_0)) = 0, \ \forall (x(t), x(t_0) \in G^E, t \in [t_0, t_1].
\end{aligned}$$

□

The following theorem is a generalization of the Minimum of a Quadratic Functional Theorem [5].

Theorem 3.4. *Consider the problem formulation* (3.1)-(3.7), *If the operator* $(D, E) : G^E \to Im((D, E)) \subseteq Y$ *satisfying Lemma 3.3, then minimum values* $(\bar{x}(t), \bar{x}(t_0)) \in G^E$ *of the functional*

$$\begin{aligned}
\phi[(x(t), x(t_0)] = \ &\frac{1}{2} \langle (Dx(t), Ex(t_0)), (x(t), x(t_0)) \rangle_{(Y,New)} - \ \langle (f, d), (x(t), x(t_0)) \rangle_{(Y,New)} \\
&= \frac{1}{2} \left[\int_{t_0}^{t_1} \left(E\frac{dx(t)}{dt} + \ Ax(t)\right)^T \ \left(E\frac{dx(t)}{dt} + \ Ax(t)\right) dt + (Ex(t_0))^T (Ex(t_0)) \right] \\
&- \int_{t_0}^{t_1} (f)^T \ \left(E\frac{dx(t)}{dt} + \ Ax(t)\right) dt + d^T (Ex(t_0))
\end{aligned} \tag{3.10}$$

is a solution of

$$(D\bar{x}(t), E\bar{x}(t_0) = (f, d) \ \text{in } Y \tag{3.11}$$

and vice versa.

Proof . *let* $(\bar{x}(t), \bar{x}(t_0) \in G^E$ *be a solution such that that* $(D\bar{x}(t), E\bar{x}(t_0)) = (f, d)$. *from functional* (3.10)*, we obtain,*

$$\phi\left[(x(t), x(t_0)\right] = \frac{1}{2}\left\langle (Dx(t), Ex(t_0)), (x(t), x(t_0)\right\rangle_{(Y,New)} - \left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)}$$

$$\begin{aligned}\phi\left[(x(t), x(t_0)\right] = &\frac{1}{2}\left\langle (Dx(t), Ex(t_0)), (x(t), x(t_0)\right\rangle_{(Y,New)} - \frac{1}{2}\left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)} \\ &- \frac{1}{2}\left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)}\end{aligned}$$

from lemma 3.3, one gets

$$\begin{aligned}&= \frac{1}{2}\Big[\left\langle (Dx(t), Ex(t_0)), (x(t), x(t_0)\right\rangle_{(Y,New)} - \left\langle (Dx(t), Ex(t_0)), ((\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} \\ &- \left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)}\Big]\end{aligned} \tag{3.12}$$

adding and subtracting $\left\langle (D\bar{x}(t), \bar{x}(t_0)), (E\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}$*in* (3.12)

$$\begin{aligned}&= \frac{1}{2}\Big[\left\langle (Dx(t), Ex(t_0)), (x(t), x(t_0)\right\rangle_{(Y,New)} - \left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)} \\ &- \left\langle (D\bar{x}(t), E\bar{x}(t_0)), ((x(t), x(t_0))\right\rangle_{(Y,New)} + \left\langle (D\bar{x}(t), E\bar{x}(t_0)), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} \\ &- \left\langle (D\bar{x}(t), E\bar{x}(t_0)), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}\Big] \\ &= \frac{1}{2}\Big[\left\langle (Dx(t), Ex(t_0)) - (D\bar{x}(t), E\bar{x}(t_0)), (x(t), x(t_0)) - (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} \\ &- \left\langle (D\bar{x}(t), E\bar{x}(t_0)), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}\Big]\end{aligned} \tag{3.13}$$

From Lemma (3.3) *and the independence* $\left\langle (D\bar{x}(t), E\bar{x}(t_0)), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}$ *from* $(x(t), x(t_0))$ *which remains this constant and not effective on the minimum value of the functional, for the first term of the right- hand side of* (3.13) *we obtained that*

$$\begin{aligned}\phi\left[(x(t), x(t_0)\right] = &\frac{1}{2}\Big[\left\langle (Dx(t), Ex(t_0)) - (D\bar{x}(t), E\bar{x}(t_0)), (x(t), x(t_0)) - (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}\Big] \\ &\geq 0 \quad \text{for evrey } (x(t), x(t_0)) \in G^E\end{aligned} \tag{3.14}$$

with

$$\begin{aligned}&\frac{1}{2}\Big[\left\langle (Dx(t), Ex(t_0)) - (D\bar{x}(t), E\bar{x}(t_0)), (x(t), x(t_0)) - (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}\Big] = 0 \\ &\Longleftrightarrow (x(t), x(t_0)) - (\bar{x}(t), \bar{x}(t_0)) = 0 \ \text{in } \ G^E.\end{aligned}$$

Form (3.14) *one can get*

$$\phi\left[(x(t), x(t_0))\right] \geq \phi\left[(\bar{x}(t), \bar{x}(t_0))\right] \quad \text{for evrey } (x(t), x(t_0)) \in G^E,$$

with

$$\phi\left[(x(t), x(t_0))\right] = \phi\left[(\bar{x}(t), \bar{x}(t_0))\right] \iff (x(t), x(t_0)) = (\bar{x}(t), \bar{x}(t_0)) \quad \text{in } \ G^E.$$

Conversely, let $\phi\left[(x(t), x(t_0))\right]$ be a minimum functional for the element $(\bar{x}(t), \bar{x}(t_0))$, that means if we choose an arbitrary element $(v(t), v(t_0)) \in G^E$ and an arbitrary ϵ, so that $(\bar{x}(t), \bar{x}(t_0))+\epsilon\,(v(t), v(t_0)) \in G^E$, and we have

$$\phi\left[(\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right] \geq \phi\left[(\bar{x}(t), \bar{x}(t_0))\right]$$

when

$$\begin{aligned}
&\phi\left[(\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right]\\
&= \frac{1}{2}\left\langle\; ((D\bar{x}(t), E\bar{x}(t_0)) + \epsilon\,(Dv(t), Ev(t_0)))\,, (\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right\rangle_{(Y,New)}\\
&-\;\left\langle (f,d), (\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right\rangle_{(Y,New)}\\
&= \frac{1}{2}\Big[\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + \left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, \epsilon\,(v(t), v(t_0))\right\rangle_{(Y,New)}\\
&+ \left\langle \epsilon\,(Dv(t), Ev(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + \left\langle \epsilon\,(Dv(t), Ev(t_0))\,, \epsilon\,(v(t), v(t_0))\right\rangle_{(Y,New)}\Big]\\
&-\;\left\langle (f,d), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} - \left\langle (f,d), \epsilon\,(v(t), v(t_0))\right\rangle_{(Y,New)}\\
&= \frac{1}{2}\Big[\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + \epsilon\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\\
&+\epsilon\left\langle (Dv(t), Ev(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + \epsilon^2\left\langle (Dv(t), Ev(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\Big]\\
&- \left\langle (f,d), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} - \epsilon\left\langle (f,d), (v(t), v(t_0))\right\rangle_{(Y,New)}.
\end{aligned}$$

From Lemma 3.3

$$\begin{aligned}
&\left\langle (Dv(t), Ev(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} = \left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}, \quad \textit{thus}\\
&\phi\left[(\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right]\\
&= \frac{1}{2}\Big[\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + \epsilon\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\\
&+\epsilon\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)} + \epsilon^2\left\langle (Dv(t), Ev(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\Big]\\
&-\;\left\langle (f,d), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} - \epsilon\left\langle (f,d), (v(t), v(t_0))\right\rangle_{(Y,New)}\\
&= \frac{1}{2}\Big[\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)} + 2\epsilon\left\langle (D\bar{x}(t), E\bar{x}(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\\
&+\epsilon^2\left\langle (Dv(t), Ev(t_0))\,, (v(t), v(t_0))\right\rangle_{(Y,New)}\Big] - \left\langle (f,d), (\bar{x}(t), \bar{x}(t_0))\right\rangle_{(Y,New)}\\
&- \epsilon\left\langle (f,d), (v(t), v(t_0))\right\rangle_{(Y,New)}
\end{aligned}$$

It comes from the functional $\phi\left[\bar{x} + \epsilon v\right]$ has a local minimum at $\epsilon = 0$, and first derivative is equal to zero at $\epsilon = 0$,

$$\frac{d}{dt}\phi\left[(\bar{x}(t), \bar{x}(t_0)) + \epsilon\,(v(t), v(t_0))\right]\Bigg|_{\epsilon=0} = 0, \;\textit{for an arbetrary element}\; (v(t), v(t_0)) \in G^E.$$

Or from (3.14)

$$\begin{aligned}
&\frac{d}{dt}\phi\left[(\bar{x}(t),\bar{x}(t_0))+\epsilon\,(v(t),v(t_0))\right]\Big|_{\epsilon=0}\\
&=\frac{1}{2}\left[2\left\langle(D\bar{x}(t),E\bar{x}(t_0)),(v(t),v(t_0))\right\rangle_{(Y,New)}+2\epsilon\left\langle(D\bar{x}(t),E\bar{x}(t_0)),(v(t),v(t_0))\right\rangle_{(Y,New)}\right]\\
&-\ \left\langle(f,d),(v(t),v(t_0))\right\rangle_{(Y,New)}\\
&=\frac{1}{2}\left[2\left\langle(D\bar{x}(t),E\bar{x}(t_0)),(v(t),v(t_0))\right\rangle_{(Y,New)}\right]-\ \left\langle(f,d),(v(t),v(t_0))\right\rangle_{(Y,New)}
\end{aligned}$$

that means

$$\begin{aligned}
&\left\langle(D\bar{x}(t),E\bar{x}(t_0))-(f,d),(v(t),v(t_0))\right\rangle_{(Y,New)}=0\\
&\Rightarrow(D\bar{x}(t),E\bar{x}(t_0))-(f,d)=0\in im((D,E))\subseteq Y.
\end{aligned}\tag{3.15}$$

The element $(v(t),v(t_0))\in G^E$ *is fixed and arbitrary, so that* (3.15) *is true for every* $(v(t),v(t_0))\in G^E$*, using the fundamental variatuinal lemma [20], one gets*

$$(D\bar{x}(t),E\bar{x}(t_0))-(f,d)=0.$$

Thus $(\bar{x}(t),\bar{x}(t_0))$ *is a solution of* $(D\bar{x}(t),E\bar{x}(t_0))-(f,d)$ *in* $im((D,E))\subseteq Y$ *, of the equation* $(D\bar{x}(t),E\bar{x}(t_0))=(f,d)$*, which was to be proved.* □

To make the variational approach is a suitable for some type of application, one can mix this approach with Ritz method [5], and using the fact that the Hilbert space $L_2\quad(I,R^m)\times R^m$ is separable space, that implies the existence of a (at most countable) base

$$(\psi_i(t),\psi_i(t_0))\quad\text{for}\quad i=0,1,2,\ldots,n\tag{3.16}$$

Now find an approximate solution for the descriptor-composed operator system (3.4)-(3.7), which is near enough to exact solution, one can follow the next algorithm:

Algorithm (1)

Step 0: Input $E\frac{dx(t)}{dt}+Ax(t)=f(t)$, satisfies the problem formulation where

$$E=\begin{bmatrix}e_{11}&e_{12}&\ldots&e_{1n}\\e_{21}&e_{22}&\ldots&e_{2n}\\e_{31}&e_{32}&\ldots&e_{3n}\\\vdots&\vdots&\ddots&\vdots\\e_{m1}&e_{m2}&\ldots&e_{mn}\end{bmatrix},\quad A=\begin{bmatrix}a_{11}&a_{12}&\ldots&a_{1n}\\a_{21}&a_{22}&\ldots&a_{2n}\\a_{31}&a_{32}&\ldots&a_{3n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\ldots&a_{mn}\end{bmatrix},\quad f=\begin{bmatrix}f_1\\f_2\\f_3\\\vdots\\f_n\end{bmatrix},\ t\in[t_0,t_1],\quad x=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\\vdots\\x_m(t)\end{bmatrix}$$

with the initial condition $\qquad Ex(t_0)=d$

$$\begin{bmatrix}e_{11}&e_{12}&\ldots&e_{1n}\\e_{21}&e_{22}&\ldots&e_{2n}\\e_{31}&e_{32}&\ldots&e_{3n}\\\vdots&\vdots&\ddots&\vdots\\e_{m1}&e_{m2}&\ldots&e_{mn}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\\vdots\\x_m(t)\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\\\vdots\\d_n\end{bmatrix}$$

Step 1: Input $Dx = E\frac{dx(t)}{dt} + Ax(t); x = [x_1(t), x_2(t), \ \ldots, \ x_m(t)]^T, \ \frac{dx(t)}{dt} = [\dot{x}_1(t), \dot{x}_2(t), \ldots, \ \dot{x}_m(t)]^T$.

Step 2: Define the Cartesian product bilinear form:

$$\langle (x(t), x(t_0)), (y(t), y(t_0)) \rangle_{(Y,New)} \stackrel{\text{def}}{=} [(x(t), x(t_0)), (\text{Dy}(t), Ey(t_0))]$$
$$= \int_I \left(E\frac{dx(t)}{dt} + \ Ax(t) \right)^T \left(E\frac{\text{dy}(t)}{dt} + \ Ay(t) \right) dt + \left(Ex(t_0)^T Ey(t_0) \right), \quad I = [t_0, t_1], \ t_0, t_1$$

are real numbers and $t_1 > t_0$.

Step 3: the varational functional is defined as:

$$\phi [(x(t), x(t_0)] = \frac{1}{2} \langle (Dx(t), Ex(t_0)), (x(t), x(t_0)) \rangle_{(Y,New)} - \ \langle (f, d), (x(t), x(t_0)) \rangle_{(Y,New)}$$
$$= \frac{1}{2} \left[\int_{t_0}^{t_1} \left(E\frac{dx(t)}{dt} + \ Ax(t) \right)^T dt + (Ex(t_0))^T Ex(t_0) \right]$$
$$- \left[\int_{t_0}^{t_1} f^T \ \left(E\frac{dx(t)}{dt} + \ Ax(t) \right) dt + d^T Ex(t_0) \right]$$

Step 4: parameterization the solution of linear independent set of basis defined on reflexive Hilbert space $L_2 \quad (I, R^m) \times R^m$ by

$$(\widehat{x}_j(t), \widehat{x}_j(t_0)) = \left(\sum_{i=0}^{N_j} a_i^j \psi_i^j(t), \sum_{i=0}^{N_j} a_i^j \psi_i^j(t_0) \right), \qquad j = 1, 2, \ldots, m,$$

where $\psi_i^j(t)$ is a linearly independent set of basis, $N = N_1 + N_2 + \ldots + N_m$ is the number of the selected basis, and

Step 5: Set

$$\phi [(\widehat{x}(\overrightarrow{a}, t), \ \widehat{x}(\overrightarrow{a}, t_0))]$$
$$= \frac{1}{2} \left[\int_{t_0}^{t_1} \left(E\frac{d\widehat{x}(\overrightarrow{a}, t)}{dt} + A \ \widehat{x}(\overrightarrow{a}, t_0) \right)^T \left(E\frac{d\widehat{x}(\overrightarrow{a}, t)}{dt} + A \ \widehat{x}(\overrightarrow{a}, t_0) \right) \ dt \right.$$
$$\left. + (E\widehat{x}(\overrightarrow{a}, t_0))^T E\widehat{x}(\overrightarrow{a}, t_0) \right] - \left[\int_{t_0}^{t_1} f^T \left(E\frac{d\widehat{x}(\overrightarrow{a}, t)}{dt} + A \ \widehat{x}(\overrightarrow{a}, t_0) \right) dt + \left(d^T E\widehat{x}(\overrightarrow{a}, t_0) \right) \right]$$

with $\overrightarrow{a} = \left(a_0^1, a_1^1, \ldots, a_{N_1}^1, a_0^2, a_1^2, \ldots, a_{N_2}^2, \ldots, a_0^m, a_1^m, \ldots, a_{N_m}^m \right)$, and $(\widehat{x}(\overrightarrow{a}, t), \ \widehat{x}(\overrightarrow{a}, t_0)) = (\widehat{x}_j(t), \widehat{x}_j(t_0))$

Step 6: Calculate $\frac{\partial \phi}{\partial \overrightarrow{a}} = 0 \Longleftrightarrow \frac{\partial \phi}{\partial a_i^j} = 0$, where $i = 0, 1, \ldots, N_j$ and $j = 1, 2, \ldots, m$

Step 7: Since the functional $\phi [(\widehat{x}(\overrightarrow{a}, t), \ \widehat{x}(\overrightarrow{a}, t_0))]$ is quadratic functional, $\frac{\partial \phi}{\partial \overrightarrow{a}} = 0$ leads to the following solution of the linear system of the algebraic equation

$$\begin{pmatrix} \left\langle \left(D\psi_0^1(t), E\psi_0^1(t_0) \right), \left(\psi_0^1(t), \psi_0^1(t_0) \right) \right\rangle_{(Y,New)} a_0^1 + \ldots + \left\langle \left(D\psi_0^1(t), E\psi_0^1(t_0) \right), \left(\psi_N^1(t), \psi_N^1(t_0) \right) \right\rangle_{(Y,New)} a_N^1 = \left\langle (f, d), \left(\psi_0^1(t), \psi_0^1(t_0) \right) \right\rangle_{(Y,New)} \\ \left\langle \left(D\psi_0^1(t), E\psi_0^1(t_0) \right), \left(\psi_1^1(t), \psi_1^1(t_0) \right) \right\rangle_{(Y,New)} a_0^1 + \ldots + \left\langle \left(D\psi_1^1(t), E\psi_1^1(t_0) \right), \left(\psi_N^1(t), \psi_N^1(t_0) \right) \right\rangle_{(Y,New)} a_N^1 = \left\langle (f, d), \left(\psi_1^1(t), \psi_1^1(t_0) \right) \right\rangle_{(Y,New)} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \left\langle \left(D\psi_0^1(t), E\psi_0^1(t_0) \right), \left(\psi_N^1(t), \psi_N^1(t_0) \right) \right\rangle_{(Y,New)} a_0^1 + \ldots + \left\langle \left(D\psi_N^1(t), E\psi_N^1(t_0) \right), \left(\psi_N^1(t), \psi_N^1(t_0) \right) \right\rangle_{(Y,New)} a_n^1 = \left\langle (f, d), \left(\psi_N^1(t), \psi_N^1(t_0) \right) \right\rangle_{(Y,New)} \end{pmatrix}$$

$$\begin{pmatrix} \left\langle \left(D\psi_0^2(t), E\psi_0^2(t_0)\right), \left(\psi_0^2(t), \psi_0^2(t_0)\right)\right\rangle_{(Y,New)} a_0^2 + \ldots + \left\langle \left(D\psi_0^2(t), E\psi_0^2(t_0)\right), \left(\psi_N^2(t), \psi_N^2(t_0)\right)\right\rangle_{(Y,New)} a_N^2 = \left\langle (f,d), \left(\psi_0^2(t), \psi_0^2(t_0)\right)\right\rangle_{(Y,New)} \\ \left\langle \left(D\psi_0^2(t), E\psi_0^2(t_0)\right), \left(\psi_1^2(t), \psi_1^2(t_0)\right)\right\rangle_{(Y,New)} a_0^2 + \ldots + \left\langle \left(D\psi_1^2(t), E\psi_1^2(t_0)\right), \left(\psi_N^2(t), \psi_N^2(t_0)\right)\right\rangle_{(Y,New)} a_N^2 = \left\langle (f,d), \left(\psi_1^2(t), \psi_1^2(t_0)\right)\right\rangle_{(Y,New)} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \left\langle \left(D\psi_0^2(t), E\psi_0^2(t_0)\right), \left(\psi_N^2(t), \psi_N^2(t_0)\right)\right\rangle_{(Y,New)} a_0^2 + \ldots + \left\langle \left(D\psi_N^2(t), E\psi_N^2(t_0)\right), \left(\psi_N^2(t), \psi_N^2(t_0)\right)\right\rangle_{(Y,New)} a_N^2 = \left\langle (f,d), \left(\psi_N^2(t), \psi_N^2(t_0)\right)\right\rangle_{(Y,New)} \end{pmatrix}$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

$$\begin{pmatrix} \langle (D\psi_0^m(t), E\psi_0^m(t_0)), (\psi_0^m(t), \psi_0^m(t_0))\rangle_{(Y,New)} a_0^m + \ldots + \langle (D\psi_0^m(t), E\psi_0^m(t_0)), (\psi_N^m(t), \psi_N^m(t_0))\rangle_{(Y,New)} a_N^m = \langle (f,d), (\psi_0^m(t), \psi_0^m(t_0))\rangle_{(Y,New)} \\ \langle (D\psi_0^m(t), E\psi_0^m(t_0)), (\psi_1^m(t), \psi_1^m(t_0))\rangle_{(Y,New)} a_0^m + \ldots + \langle (D\psi_1^m(t), E\psi_1^m(t_0)), (\psi_N^m(t), \psi_N^m(t_0))\rangle_{(Y,New)} a_N^m = \langle (f,d), (\psi_1^m(t), \psi_1^m(t_0))\rangle_{(Y,New)} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \langle (D\psi_0^m(t), E\psi_0^m(t_0)), (\psi_N^m(t), \psi_N^m(t_0))\rangle_{(Y,New)} a_0^m + \ldots + \langle (D\psi_N^m(t), E\psi_N^m(t_0)), (\psi_N^m(t), \psi_N^m(t_0))\rangle_{(Y,New)} a_N^m = \langle (f,d), (\psi_N^m(t), \psi_N^m(t_0))\rangle_{(Y,New)} \end{pmatrix}$$

Step 8: the solution of the system in step 7 nominated the critical pointes (varational pointes), hence Hessian matrix required to decide the local minimum points as

$$H_{(\phi)_{N_m \times N_m}} = \frac{\partial^2 \phi}{\partial \overrightarrow{a}^2}$$

Step 9: determinate the eigenvalues (λ_i) of the matrix H_ϕ , if $\lambda_i \geq 0$, where $\lambda_i \in \sigma(H_\phi)$,

$\sigma(H_\phi) = \left\{\lambda \in \mathbb{C} : \left|\lambda \mathbb{I}_{N_m \times N_m} - H_{(\phi)_{N_m \times N_m}}\right| = 0\right\}$, then the local minimum at $(\widehat{x}(\overrightarrow{a}, t),\ \widehat{x}(\overrightarrow{a}, t_0))$ is obtained otherwise go to Step (13)

Step 10: Solving the system in step 7 leads to linear algebraic system

$$\left[\frac{\partial \phi}{\partial \overrightarrow{a}}\right] [\overrightarrow{a}]^T = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \Leftrightarrow [\overrightarrow{a}]^T = \left[\frac{\partial \phi}{\partial \overrightarrow{a}}\right]^{-1} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Step 11: Set the solution $(\widehat{x}_j(t), \widehat{x}_j(t_0)) = (\widehat{x}(\overrightarrow{a}, t),\ \widehat{x}(\overrightarrow{a}, t_0))$,

Step 12: Out put the approximate solution

$$((\widehat{x}_1(t), \widehat{x}_1(t_0)),\ (\widehat{x}_2(t), \widehat{x}_2(t_0)),\ \ldots, (\widehat{x}_m(t), \widehat{x}_m(t_0)))$$

Step 13: Stop.

Example 3.5. *Consider the model of a chemical reactor in which a first order isomerization reaction takes place and which is externally cooled. Denoting by c_0 the given feed reactant concentration, by t_0 the initial temperature, by c(t), T(t).*
The concentration and temperature at time t, and by R(t) the reaction rate per unit volume, the model takes the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{dM}{dt} \\ \frac{dT}{dt} \\ \frac{dR}{dt} \end{bmatrix} = \begin{bmatrix} c_0 & 0 & 0 \\ 0 & T_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M \\ T \\ R \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

where $f(t) = \begin{bmatrix} \sin t \\ \cos t \\ \sin t + \cos t \end{bmatrix}$ *refers to* T_c *is cooling temperature.*
Take $c_0 = 2$ *and* $T_0 = 1$ *we get*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{dM}{dt} \\ \frac{dT}{dt} \\ \frac{dR}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M \\ T \\ R \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} M(t) \\ T(t) \\ R(t) \end{bmatrix}, \quad 0 \le t \le 1 \tag{3.17}$$

with the initial condition

$$Ex(t_0) = d, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M(0) \\ T(0) \\ R(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{3.18}$$

and exact solution

$$\begin{aligned} M(t) &= \frac{-2}{5}\ sint - \frac{1}{5}\ cost + \frac{6}{5}\ e^{2t}, \\ T(t) &= \sin t - 2\cos t \\ R(t) &= -\sin t - \cos t, \end{aligned} \tag{3.19}$$

Solution:
According to the data of the example we have:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ f = \begin{bmatrix} \sin t \\ \cos t \\ \sin t + \cos t \end{bmatrix}, t_0 = 0,\ t_1 = 1, .$$

The linear operator

$$Dx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{dM}}{dt} \\ \frac{dT}{dt} \\ \frac{\mathrm{dR}}{dt} \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} M(t) \\ T(t) \\ R(t) \end{bmatrix}, \tag{3.20}$$

and

$$(\widehat{x}_j(t), \widehat{x}_j(t_0)) = \left(\sum_{i=0}^{N_j} a_i^j \psi_i^j(t), \sum_{i=0}^{N_j} a_i^j \psi_i^j(t_0) \right), \qquad j = 1, 2, 3$$

where $N_1 = 1,\ N_2 = 2\ ,\ N_3 = 3\ \ and\ \ N = N_1 + N_2 + N_3 = 12, \qquad \psi_i(t) = t^i,\ i = 0, 1, 2, 3$

$$\begin{aligned} (\widehat{x}_1(t), \widehat{x}_1(t_0)) &= \left(a_0^1 + a_1^1 t + a_2^1 t^2 + a_3^1 t^3,\ a_0^1\right) \\ (\widehat{x}_2(t), \widehat{x}_2(t_0)) &= \left(a_0^2 + a_1^2 t + a_2^2 t^2 + a_3^2 t^3, a_0^2\right) \\ (\widehat{x}_3(t), \widehat{x}_3(t_0)) &= \left(a_0^3 + a_1^3 t + a_2^3 t^2 + a_3^3 t^3, a_0^3\right) \end{aligned}$$

Define the Cartesian product bilinear form:

$$\begin{aligned} &\langle (x(t), x(t_0)), (y(t), y(t_0)) \rangle_{(Y,New)} \overset{\text{def}}{=} [(x(t), x(t_0)), (\mathrm{Dy}(t), Ey(t_0))] \\ &= \int_I \left(E\frac{dx(t)}{dt} + \ Ax(t) \right)^T \left(E\frac{\mathrm{dy}(t)}{dt} + \ Ay(t) \right) dt + (Ex(t_0))^T\, \mathrm{Ey}(t_0), \quad I = [0, 1]\,. \end{aligned}$$

Then, the variational functional will be:

$$\begin{aligned}
&\phi\left[\left(\widehat{x}\left(\overrightarrow{a},t\right),\ \widehat{x}\left(\overrightarrow{a},t_0\right)\right)\right] \\
&=\frac{1}{2}\left[\int_{t_0}^{t_1}\left(E\frac{d\widehat{x}\left(\overrightarrow{a},t\right)}{dt}+A\,\widehat{x}\left(\overrightarrow{a},t_0\right)\right)^T\left(E\frac{d\widehat{x}\left(\overrightarrow{a},t\right)}{dt}+A\,\widehat{x}\left(\overrightarrow{a},t_0\right)\right)\,dt\right. \\
&\left.+\left(E\widehat{x}\left(\overrightarrow{a},t_0\right)\right)^T E\widehat{x}\left(\overrightarrow{a},t_0\right)\right]-\left[\int_{t_0}^{t_1} f^T\left(E\frac{d\widehat{x}\left(\overrightarrow{a},t\right)}{dt}+A\,\widehat{x}\left(\overrightarrow{a},t_0\right)\right)dt+\left(d^T E\widehat{x}\left(\overrightarrow{a},t_0\right)\right)\right]
\end{aligned}$$

Solving the system in step (7) of the algorithm (1) for $m=3$. Where the operator D is given by (3.20), $f(t)$ is given by (3.17), d is given by (3.18), $N=12$, $\overrightarrow{a}\in R^{12}$. Hence 12×12 linear system of algebraic equation for the unknown parameter $\overrightarrow{a}\in R^{12}$ is obtained, the following table shows the numerical solutions of the parameters $\overrightarrow{a}$:

Table 1

a_0^1	a_1^1	a_2^1	a_3^1	a_0^2	a_1^2	a_2^2	a_3^2	a_0^3	a_1^3	a_2^3	a_3^3
2.3219	***0.3110***	***4.5831***	***1.1258***	***0.7455***	***-0.0788***	***-1.0154***	***0.5683***	***0.0654***	***0.9742***	***-2.0148***	***-1.9992***

To ensure the local minimum solution, Hessian matrix is derived, where

$$\begin{aligned}
&h_0=\frac{\partial\phi}{\partial a_0^1},\ h_1=\frac{\partial\phi}{\partial a_1^1},\ h_2=\frac{\partial\phi}{\partial a_2^1}, h_3=\frac{\partial\phi}{\partial a_3^1},\ h_4=\frac{\partial\phi}{\partial a_0^2},\ h_5=\frac{\partial\phi}{\partial a_1^2}, \\
&h_6=\frac{\partial\phi}{\partial a_2^2},\ h_7=\frac{\partial\phi}{\partial a_3^2},\ h_8=\frac{\partial\phi}{\partial a_0^3},\ h_9=\frac{\partial\phi}{\partial a_1^3},\ h_{10}=\frac{\partial\phi}{\partial a_2^3}, h_{11}=\frac{\partial\phi}{\partial a_3^3}\quad\text{and}\quad H_{(\phi)_{N_3\times N_3}}=\frac{\partial h_i}{\partial a_i^j},
\end{aligned}$$

$$H_\phi=\begin{bmatrix}
0.3333 & 0 & -0.2000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.1333 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0 & -0.667 & 0 & 0\\
-0.2000 & 0.1667 & 0.3714 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0000 & 0 & 0\\
0 & 0 & 0 & 0.3333 & 0.2500 & 0.2000 & -0.5000 & -0.667 & -0.7500 & 0 & 0.5000 & 0\\
0 & 0 & 0 & 0.2500 & 0.200 & 0.1667 & -0.3333 & -0.5000 & -0.6000 & 0 & 0.3333 & 0\\
0 & 0 & 0 & 0.2000 & 0.1667 & 0.1429 & -0.2500 & -0.4000 & -0.5000 & 0 & 0.2500 & 0\\
0 & 0 & 0 & -0.5000 & -0.3333 & -0.2500 & 1.3333 & 1.2500 & 1.2000 & 0 & -1.0000 & 0.5000\\
0 & 0 & 0 & -0.6667 & -0.5000 & -0.4000 & 1.2500 & 1.5333 & 1.6667 & 0 & -1.0000 & 0.3333\\
0 & 0 & 0 & -0.7500 & -0.6000 & -0.5000 & 1.2000 & 1.6667 & 1.9429 & 0 & -1.0000 & 0.2500\\
0 & -0.6667 & -1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0\\
0 & 0 & 0 & 0.5000 & 0.3333 & 0.2500 & -1.0000 & -1.0000 & -1.0000 & 0 & 1.0000 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0.5000 & 0.3333 & 0.2500 & 0 & 0 & 1.0000
\end{bmatrix}$$

The eigenvalues of the Hessian matrix are presented in the Table (2), which shows the positivity of the matrix H_ϕ

Table 2

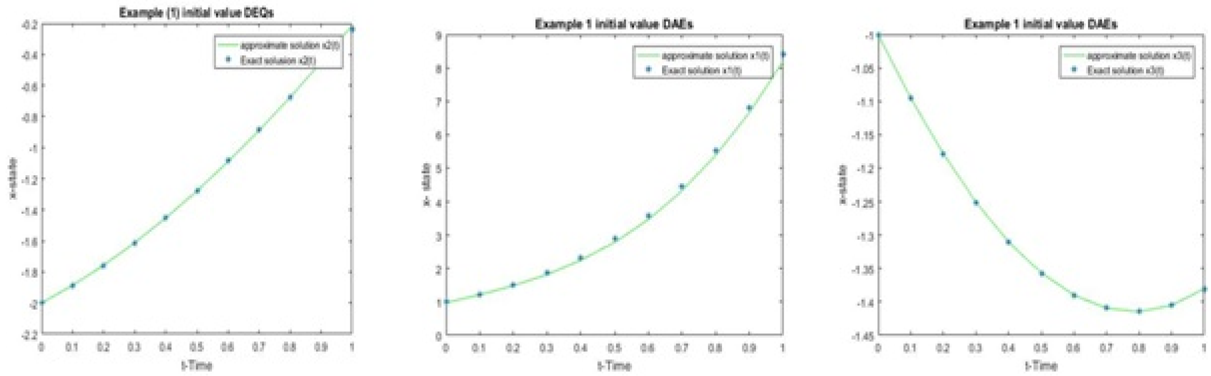
λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}
0.0000	**0.0002**	**0.0021**	**0.0123**	**0.0196**	**0.0612**	**0.0652**	**0.4622**	**0.5510**	**1.1115**	**5.2984**	**5.7401**

The comparisons between the approximate solution of the proposed approach and the exact solution (3.19) are presented in the following table:

Table 3

Time	V.M. $\hat{x}_1(t)$	Exact $x_1(t)$	Abso. Error	V.M. $\hat{x}_2(t)$	Exact $x_2(t)$	Abso. Error	V.M. $\hat{x}_3(t)$	Exact $x_3(t)$	Abso. Error
0	0.9742	1	0.0258	-2.0149	-2	0.0149	-0.9992	-1	0.0008
0.1	1.2141	1.2267	0.0126	-1.8950	-1.8902	0.0048	-1.0950	-1.0948	0.0002
0.2	1.4877	1.5147	0.027	-1.7606	-1.7615	0.0009	-1.1790	-1.1787	0.0003
0.3	1.8225	1.8773	0.0548	-1.6122	-1.6152	0.003	-1.2509	-1.2509	0
0.4	2.2460	2.3307	0.0847	-1.4503	-1.4528	0.0025	-1.3102	-1.3105	0.0003
0.5	2.7858	2.8947	0.1089	-1.2754	-1.2757	0.0003	-1.3567	-1.3570	0.0003
0.6	3.4693	3.5932	0.1239	-1.0880	-1.0860	0.002	-1.3897	-1.3900	0.0003
0.7	4.3240	4.4556	0.1316	-0.8885	-1.8855	0.997	-1.4091	-1.4091	0
0.8	5.3774	5.5174	0.14	-0.6774	-0.6766	0.0008	-1.4143	-1.4141	0.0002
0.9	6.6570	6.8219	0.1649	-0.4552	-0.4599	0.0047	-1.4051	-1.4049	0.0002
1	8.1903	8.4222	0.2319	-0.2223	-0.2401	0.0178	-1.3809	-1.3818	0.0009

plots of state compared with the given exact solution are shown in the following plots



As one can see from Table (3) and the plots, the good accuracy is obtained even when simple numbers of the basis function (N) , $\psi_i(t) = t^i$ are nominated. to insure this accuracy, the number of selected basis function and their type are suggested to be modified.

4. Conclusions

The approximate analytic solution to class of descriptor-composed operator have been proposed. this approach is based on the theoretical results. A step-by-step algorithm is derived to simplify the numerical computation. the approximate solution is based on the type of selection basis function and their numbers. the unknown parameters of the representation solution are esaily obtained as a result of solvable linear algebraic system.
The future work is looking for generalized these results to include non-linear system of composed differential-algebraic system with and without control inputs.

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