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# The existence of periodic solutions to doubly degenerate Allen-Cahn equation with Neumann boundary condition

Raad Awad Hameed[a], Maan A. Rasheed[b], Hekmat Sh. Mustafa[c], Faez N. Ghaffoori[b]

[a] Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq
[b] Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq
[c] Department of Mathematics, College of Education, Al- Hamdaniya University, Mosul, Iraq

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**Abstract**

This work is concerned with the periodic solution of a doubly degenerate Allen-Cahn equation with nonlocal terms associated with Neumann boundary conditions. Firstly, we define a new associated auxiliary problem. Secondly, the topological degree theorem is applied to prove the existence of a limit point to the auxiliary problem, where this limit point represents a nontrivial nonnegative time-periodic solution of the main studied problem. It is observed that the topological degree theorem technique plays an important role in proving the desired results. Furthermore, this technique can be applied to other similar equations with homogeneous Dirichlet or Neumann boundary conditions.

*Keywords:* Degenerate Allen-Cahn equation; Neumann boundary conditions; Time-periodic solution; Topological degree theorem
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## 1. Introduction

In this work, we consider the following time-dependent problem, which is a doubly degenerate Allen-Cahn equation with nonlocal terms associated with Neumann boundary conditions:

*Email addresses:* dr.raad.awad@tu.edu.iq (Raad Awad Hameed), maan.rasheed.edbs@uomustansiriyah.edu.iq (Maan A. Rasheed), hekmat78@uohamdaniya.edu.iq (Hekmat Sh. Mustafa), fng-2022@uomustansiriyah.edu.iq (Faez N. Ghaffoori)

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$$\frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v^m|^{p-2}\nabla v) = h(v^3 - v), \qquad (x,t) \in S_T, \tag{1.1}$$
$$\frac{\partial v}{\partial \eta} = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$
$$v(x,0) = v(x,T), \qquad x \in \Omega, \tag{1.3}$$

Where $m \geq 1, p \geq 2$ $\Omega \subset R^n$ is a bounded domain with smooth boundary; the outward normal vector is denoted by $\eta$; $S_T = \Omega \times (0,T)$, the function $v$ represents the spatial densities of the species at $u(x,t)$ and the second term on the left-hand side in (1.1); $\operatorname{div}(|\nabla v^m|^{p-2}\nabla v)$ represents the nonlinear diffusion term. Since the last decades, time-dependent partial differential equations have played important roles in describing many phenomena in different scientific fields, such as physics, chemistry and engineering, see for instance [10, 11, 12, 15]. One of the most common-studied problems is Allen-Cahn equation, which is considered a simple type of nonlinear reaction-diffusion equations. It is commonly used to represent the interface motion in time, for instance, it is used in phase separation in alloys. Moreover, this equation has several applications in many areas such as material sciences, plasma physics, quantum mechanics, geology, image processing, as well as mathematical biology.

During the past fifty years, many authors have worked on the linear parabolic equation with nonlocal terms, see for example [1, 2, 3, 4]. In [1], Allegretto and Nistri have considered a special model, namely they studied the following equation:

$$\frac{\partial v}{\partial t} - \Delta v = f(x,t,m,\Phi[v],v), \tag{1.4}$$

Due to the realistic needs, some authors have concerned with nonlinear diffusion equations with nonlocal terms, such as the porous equation [7, 8]:

$$\frac{\partial v}{\partial t} - \Delta v = f(x,t,m,\Phi[v],v), \tag{1.5}$$

and the $p$-Laplacian equation [6]:

$$\frac{\partial v}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v) + (m - \Phi[v])v. \tag{1.6}$$

It is clear that equation (1.5) and (1.6) are degenerate if $(m > 1)$ and if $(p > 2)$ , respectively, while, they are singular if $(0 < m < 1)$and $(1 < p < 2)$, respectively. Few exceptions are considered to the cases $m > 1$ , and $p > 2$. In fact, equations (1.5) and (1.6) are studied with Dirichlet boundary conditions, which refer that the boundary is lethal to the species. In each of these references, the topological degree theory was used to prove the existence of nontrivial nonnegative time-periodic solution. Recently, in [13], the following periodic Neumann-boundary value problem has been studied:

$$\begin{aligned}
&\frac{\partial v}{\partial t} - \Delta v^m = (m - \Phi[v])v, && (x,t) \in S_T,\\
&\frac{\partial v}{\partial n} = 0, && (x,t) \in \partial\Omega \times (0,T),\\
&v(x,0) = v(x,T), && x \in \Omega,
\end{aligned}$$

where $m > 1$. The existence of the nontrivial nonnegative periodic solutions to the above problem was established by using the topological degree theory and the parabolic regularized method. In fact, considering a doubly degenerate Allen-Cahn equation with isolated Neumann condition leads to additional difficulties in establishing the priori estimates comparing with studying the a doubly degenerate Allen-Cahn equation with the Dirichlet boundary condition. Furthermore, for problem (1.1)-(1.3), an auxiliary problem needs to be considered for applying the topological degree theory. Other varies types of Allen-Cahn equation with (without) non-local terms have been considered in our recent works, such as linear diffusion Allen-Cahn equation, p-Laplacian Allen-Cahn equation, and a quasi-linear parabolic Allen-Cahn equation. The main goal of these works is to prove the existence of periodic solutions to these problems. This work is considered a continuation of these works. Namely, the aim of this work is to prove the existence of non-trivial nonnegative periodic solution to problem (1.1)-(1.3).

This article is divided into four sections. In the second section, the necessary preliminaries and the auxiliary problem is given. In the third section, the necessary priori-estimations of the solutions of the auxiliary problem are established. Moreover, the existence of non-trivial nonnegative periodic solution to problem (1.1)-(1.3) is proved. Finally, some conclusions are stated in the last section.

## 2. Preliminaries

In this work, it is assumed that:

**F1)** The functional $\Psi[\cdot] : L^2_+(\Omega) \to \mathbb{R}^+$ is bounded, continuous and satisfying the inequality:

$$0 \le \Psi[v] \le E\|v\|^k_{L^k(\Omega)}, \qquad k > 0$$

where $\Psi[v] = h(v^3 - v)$ and E is a positive constant, and it does not depend on $v$; where, $\mathbb{R}^+ = [0, +\infty)$, $L^p_+(\Omega) = \{v \in L^p(\Omega)|v \ge 0, \text{a. e. in } \Omega\}$.

**F2)** $h(x,t) \in C_T(\overline{S}_T)$ and it satisfies that $\{x \in \Omega : \frac{1}{T}\int_0^T h(x,t) > 0\} \ne \emptyset$, where $C_T(\overline{S}_T)$ is the set of $T$-periodic function with respect to $t$ and continuous in $\overline{\Omega}\times\mathbb{R}$ By (2), there exists $x_0 \in \Omega, h_0 > 0$ such that $\frac{1}{T}\int_0^T h(x,t)dt \ge h_0$ for all $x \in B(x_0, \delta)$

Due to the degeneracy of equation (1.1) at the points, where $v = 0$, the problem (1.1)- (1.3) does not have generally a classical solution. Therefore, we shall discuss rather the solutions of problem (1.1)-(1.3) in a weak sense .

**Definition 2.1.** *We say that the function $v$ is a weak solution to problem (1,1)-(1,3), if $v \in L^2(0,T; H^1(T)) \cap C_T(\overline{S}_T)$ and satisfies*

$$\iint_{S_T} \left(-v\frac{\partial\vartheta}{\partial t} + |\nabla v^m|^{p-2}\nabla v^m\nabla\vartheta - h(t)(v^2-1)v\vartheta\right) dxdt = 0, \tag{2.1}$$

*for any $\vartheta \in C^1(\overline{S}_T)$ with the periodic initial value $\vartheta(x,t) = \vartheta(x, t+T)$.*

Since equation (1.1) is degenerate, we need to define the following regularized problem:

$$\frac{\partial v_\tau}{\partial t} - \operatorname{div}((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)\nabla v_\tau) = h(t)(v_\tau^2 - 1)v_\tau^+, \qquad (x,t) \in S_T, \tag{2.2}$$

$$\frac{\partial v_\tau}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.3}$$

$$v_\tau(x,t) = v_\tau(x,t+T), \qquad x \in \Omega, \tag{2.4}$$

where $\tau \in R^+$ and it is sufficiently small; and $s^+ = \max\{0, s\}$ and $A(v_\tau) = mv_\tau^{m-1} + \tau, \tau$ is a sufficiently small positive constant .In this work, we shall discuss the existence of the limit point to the solutions of problem (2.2) - (2.4) , which represents the weak solution to problem (1.1)- (1.3). Next, we introduce a map as follows:

$$\frac{\partial v_\tau}{\partial t} - \operatorname{div}((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)\nabla v_\tau) = f, \qquad (x,t) \in S_T, \tag{2.5}$$

$$\frac{\partial v_\tau}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.6}$$

$$v_\tau(x,t) = v_\tau(x, t+T), \qquad x \in \Omega, \tag{2.7}$$

Moreover, the map is defined as $v_\tau = Qf$ where $Q : [0,1] \times C_T(\overline{S}_T) \to C_T(\overline{S}_T)$.By classical estimates (see [9]), it follows that $v_\tau$ is Holder-continuous in $S_T$ and $\|v_\tau\|_{L^\infty(S_T)}$ is bounded by $\|f\|_{L^\infty(S_T)}$. Let Then by the Arzela-Ascoli theorem, the map $Q$ is compact. Thus, it is compact continuous. $f(v_\varepsilon) = (m((v_\tau^+)^2 - 1)v_\varepsilon^+)$ where $v_\tau^+ = \max\{u_\tau, 0\}$.It follows that the nonnegative solution $v_\tau$ of problem (2.2) - (2.4) is a nonnegative fixed point of the map $v_\tau = Q(1, (h((v_\tau^+)^2 - 1)v_\tau^+))$. as well. So, one can study the existence of nonnegative fixed points of the map: $v_\tau = Q(1, (h((v_\tau^+)^2 - 1)v_\tau^+))$ rather than looking for a nonnegative solution of problem (2.2) - (2.4).

## 3. The Main Results

By applying the same technique used in [16]:, we can show that the solutions of problem (2.2) - (2.4) are nonnegative.

**Lemma 3.1.** *let $v_\tau \in C_T(\overline{S}_T)$ be a non-trivial solution for $v_\tau = Q(1, h((v_\tau^+)^2 - 1)v_\tau^+)$, then*

$$v_\tau(x,t) > 0, \ \textit{for all} \ (x,t) \in \overline{S}_T.$$

Next, by applying the Moser iterative technique, we obtain a priori estimate for the upper bound for the nonnegative periodic-solutions of problem (2.2) - (2.4). For simplicity, the $L^p(\Omega)$ norm is denoted by $\|.\|_p$ $(1 \le p \le \infty)$ .

**Lemma 3.2.** *Let $\gamma \in [0,1]$, $v_\tau(x,t)$ be a nonnegative periodic function, which solves $v_\tau = Q(1, \gamma h((v_\tau^+)^2 - 1)v_\tau^+)$, then there exists a constant,which is independent of $\gamma$, such that*

$$\|v(t)\|_\infty < R, \tag{3.1}$$

*where $v(t) = v(\cdot, t)$.*

**Proof .** If we multiply equation (2.5) by $v_\tau^{s+1} (s \ge 0)$ and integrate it over $\Omega$, it follows that

$$\frac{1}{s+2}\frac{d}{dt}\|v_\tau(t)\|_{s+2}^{s+2} + \frac{m^{p-1}(s+1)p^p}{[m(p-1)+s+1]^p}\|\nabla(v_\tau^{\frac{m(p-1)+s+1}{p}}(t))\|_p^p \le \|h(x,t)\|_{L^\infty(S_T)}\|v_\tau(t)\|_{s+2}^{s+4},$$

and hence

$$\frac{1}{s+2}\frac{d}{dt}\|v_\tau(t)\|_{s+2}^{s+2} + \frac{E_1}{[m(p-1)+s+1]^p}\|\nabla(v_\tau^{\frac{m(p-1)+s+1}{p}}(t))\|_p^p \le E_2(s+1)\|v_\tau(t)\|_{m+2}^{m+4}, \tag{3.2}$$

where for $j = 1, 2, E_i$ are positive constants independent of $v_\tau$ and $m$. We suppose that $\|v(t)\|_\infty \neq 0$ set

$$v_k(t) = v_\tau^{\frac{m(p-1)+s+1}{p}}, \qquad \alpha_k = \frac{p(s_k+2)}{m(p-1)+s+1}, \qquad s_k = \frac{p^k - p}{p-1} \qquad (k = 1, 2, ...),$$

then , $s_k = p^k + m - \frac{p}{p-1}$.

For simplicity, a positive constant, which is independent of $k$ and $m$ and it takes different values, is denoted by $E$. By (3.2), it follows that

$$\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} + E\|\nabla v_k(t)\|_p^p \leq E(s+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}. \tag{3.3}$$

Applying the Gagliardo-Nirenberg inequality, yields that

$$\|v_k(t)\|_{\alpha_k} \leq E\|\nabla v_k(t)\|_p^{\theta_k}\|v_k(t)\|_1^{1-\theta_k}, \tag{3.4}$$

with

$$\theta_k = \frac{s_k(p-1)+p}{s_k+2} \cdot \frac{N}{(p-1)N+p} \in (0, 1).$$

By inequalities (3.3), (3.4) and the fact that $\|v_k(t)\|_1 = \|v_{k-1}(t)\|_{\alpha_{k-1}}^{\alpha_{k-1}}$, we obtain the following differential inequality:

$$\begin{aligned}\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} \leq& - E\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}}\|v_k(t)\|_1^{\frac{p(\theta_k-1)}{\theta_k}} + E(s_k+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}\\ \leq& - E\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}}\|v_{k-1}(t)\|_{\alpha_{k-1}}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}} + E(s_k+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}.\end{aligned}$$

Let

$$\gamma_k = \max\{1, \sup_t \|v_k(t)\|_2\},$$

we have

$$\begin{aligned}\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} \leq (s_k+1)^{-(p-2)}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k(s_k+1)}{s_k+2}} \{-E\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}-\frac{\alpha_k(s_k+1)}{s_k+2}}\\ \chi_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}} + E(s_k+1)^{p-1}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k+2(s_k+2)}{s_k+2}}\}.\end{aligned} \tag{3.5}$$

By young's inequality

$$cd \leq \epsilon c^{p'} + \epsilon^{-\frac{q'}{p'}} d^{q'},$$

where $c > 0, d > 0, q' > 1, p' > 1, \epsilon > 0$ and $\frac{1}{p'} = \frac{q'-1}{q'}$. Set

$$a = \|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k+2(s_k+2)}{s_k+2}}, \qquad b = (s_k+1)^{p-1}, \qquad \epsilon = \frac{1}{2}\gamma_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}},$$

$$p' = l_k = \left(\frac{p}{\theta_k} - \frac{\alpha_k(s_k+1)}{s_k+2}\right) \cdot \left(\frac{s_k+2}{\alpha_k+2(s_k+2)}\right),$$

then we obtain

$$(s_k+1)^{p-1}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k+2(s_k+2)}{r_k+2}} \leq \frac{1}{2}\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}-\frac{\alpha_k(s_k+1)}{s_k+2}} \gamma_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}} + E(s_k+1)^{p-1\frac{l_k}{l_k-1}}\gamma_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}\frac{1}{l_k-1}}. \tag{3.6}$$

It is easy to show the fact that $p' = l_k > s > 1$ for some $s$ independent of $k$. Thus $\lim_{k\to\infty} l_k = +\infty$

$$\lim_{k\to\infty} l_k = +\infty.$$

Denote

$$c_k = \frac{l_k(p-1)}{l_k-1}, \qquad d_k = \frac{p(\theta_k-1)}{\theta_k}\frac{\alpha_{k-1}}{l_k-1},$$

and combining (3.5) with (3.6) we have

$$\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} \leq (s_k+1)^{-(p-2)}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k(s_k+1)}{s_k+2}}\{-\frac{E}{2}\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}-\frac{\alpha_k(s_k+1)}{s_k+2}}\gamma_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}} + E(s_k+1)^{c_k}\gamma_{k-1}^{d_k}\}. \tag{3.7}$$

Then

$$(s_k+1)\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{s_k+2}} \leq \frac{-E}{2}\|v_k(t)\|_{\alpha_k}^{\frac{p}{\theta_k}-\frac{\alpha_k(s_k+1)}{s_k+2}}\gamma_{k-1}^{\frac{p(\theta_k-1)}{\theta_k}\alpha_{k-1}} + E(s_k+1)^{c_k}\gamma_{k-1}^{d_k}. \tag{3.8}$$

Since $v_k(t)$ is periodic, there exists $t_0$ such that$\|v_k(t)\|_{\alpha_k}$ takes the maximum value at this point. Therefore, the left hand side of (3.7) is vanished. It follows that

$$\|v_k(t)\|_{\alpha_k} \leq \{E[(s_k+1)^{c_k}\gamma_{k-1}^{d_k+\frac{p(1-\theta_k)}{\theta_k}\alpha_{k-1}}]\}^{\frac{1}{\Upsilon_k}},$$

where

$$\Upsilon_k = \frac{p}{\theta_k} - \frac{\alpha_k(s_k+1)}{s_k+2} = \frac{\alpha_k l_k}{s_k+2}.$$

Therefore we conclude that

$$\|v_k(t)\|_{\alpha_k} \leq \{E(s_k+1)^{c_k}\gamma_{k-1}^{d_k+\frac{\alpha_{k-1}p(1-\theta_k)}{\theta_k}}\}^{\frac{1}{E_k}} = \{E(s_k+1)^{a_k}\}^{\frac{m_k+2}{\alpha_k l_k}}\gamma_{k-1}^{\frac{\alpha_{k-1}p(1-\theta_k)(m_k+2)}{(l_k-1)\theta_k\alpha_k}}.$$

Since $\frac{s_k+2}{(l_k-1)\theta_k} = \frac{\alpha_k}{p-\theta_k\alpha_k}$ and $\frac{s_k+2}{\alpha_k l_k}$ and $\alpha_k$ are bounded, we get

$$\|v_k(t)\|_{\alpha_k} \leq Ep^{a' k}\gamma_{k-1}^{\frac{(1-\theta_k)\alpha_{k-1}p}{p-\theta_k\alpha_k}},$$

where the constant $a'$ does not depend on $k$.
As $\alpha_k = \frac{p(s_k+2}{s_k+p} < p$ implies that

$$\|v_k(t)\|_{\alpha_k} \leq ED^k\gamma_{k-1}^p,$$

or

$$\ln\|v_k(t)\|_{\alpha_k} \leq \ln\gamma_k \leq \ln E + k\ln D + p\ln\gamma_{k-1},$$

where $D = p^{a'} > 1$. Thus

$$\ln \|v_k(t)\|_{\alpha_k} \leq \ln E \sum_{j=0}^{k-2} p^j + p^{k-1} \ln \gamma_1 + \ln D(\sum_{i=0}^{k-2}(k-i)p^i)$$
$$\leq \frac{(p^{k-1}-1)}{p-1} \ln E + p^{k-1} \ln l_1 + f(k) \ln D,$$

or

$$\|v_k(t)\|_{s_{k+2}} \leq \{E^{\frac{p^{k-1}-1}{p-1}} l_1^{p^{k-1}} Df(k)\}^{\frac{p}{s_k+p}},$$

where

$$f(k) = \frac{k-(k+1)p-p^{k-1}+2p^k}{(p-1)^2}.$$

Letting $k \to \infty$, we obtain

$$\|v_\tau(t)\|_\infty \leq E\gamma_1^{p-1} \leq E(\max\{1, \sup_t \|v_\tau(t)\|_2\})^{p-1}. \tag{3.9}$$

On the other hand, it follows from (3.2) with $s = 0$ that

$$\frac{d}{dt}\|v_\tau(t)\|_2^2 + E_1\|\nabla v_\tau(t)\|_p^p \leq E_2\|u_\tau(t)\|_2^4. \tag{3.10}$$

Applying the Sobolev's theorem and Holder's inequality, yields that

$$\|v_\tau(t)\|_2 \leq \mid \Omega \mid^{\frac{1}{2}-\frac{1}{p}} \|v_\tau(t)\|_p \leq E \mid \Omega \mid^{\frac{1}{2}-\frac{1}{p}} \|\nabla v_\tau(t)\|_p. \tag{3.11}$$

Combined with (3.9), it yields that

$$\frac{d}{dt}\|v_\varepsilon(t)\|_2^2 + E_1\|\nabla v_\tau(t)\|_2^p \leq E_2\|v_\tau(t)\|_2^4.$$

By applying the Young's inequality, we obtain that

$$\frac{d}{dt}\|v_\tau(t)\|_2^2 + E_1\|\nabla v_\tau(t)\|_2^p \leq E_2. \tag{3.12}$$

for $E_j (j = 1, 2)$ are constants independent of $v$. since $v$ is periodic,by (3.11) it follows that

$$\|v_k(t)\|_2 \leq E,$$

Finally, the by last inequality and (3.8) we obtain (3.1), So, the theorem is proved.
□

**Corollary 3.3.** *There is $0 < R$ , independent of $\tau$, such that such that*

$$\deg(I - Q(1, h((v_\tau^+)^2 - 1)v_\tau^+), B_R, 0) = 1,$$

*where $B_R$ is a ball with the origin center and radius $R$ in $L^\infty(S_T)$.*

**Proof .** By Lemma (3.2), there exists $0 < R$, which is independent on $\tau$ , such that

$$v_\tau \neq Q(\gamma(h((v_\tau^+)^2 - 1)v_\tau^+)), \qquad \forall v_\tau \in \partial B_R, \gamma \in [0,1].$$

Therefore, the degree is well defined on $B_R$.

Based on the homotopy invariance of the topological degree, it follows that:

$$\deg(1 - Q(1, (h((v_\tau^+)^2 - 1)v_\tau^+), B_R, 0) = \deg(1 - Q(0), B_R, 0).$$

Since the existence of a unique solution to $u_\tau = Q(0)$ is guaranteed, it follows that $\deg(1 - Q(0), B_R, 0) = 1$. That is ,
$\deg(1 - Q(1, (h((v_\tau^+)^2 - 1)v_\tau^+), B_R, 0)) = 1$. The proof is completed. □

**Lemma 3.4.** *There exist constants $s_0 > 0$ and $\tau_0 > 0$, such that for any $s < s_0$, $\tau < \tau_0$, $Q(\tau, h((v_\tau^+)^2 - 1)v_\tau^+ + (1 - \gamma)), \gamma \in [0,1]$ does not have non-trivial solution $v_\tau$ satisfy*

$$0 < \|v_\tau\|_{L^\infty(S_T)} \leq s$$

*where $s > 0$ is independent of $\tau$.*

**Proof .** We shall proceed by contradiction, let $v_\tau$ be a non-trivial solution of $Q(\tau, h((v_\tau^+)^2 - 1)v_\tau^+ + (1 - \gamma)), \gamma \in [0,1]$ satisfying $0 < \|v_\tau\|_{L^\infty(S_T)} \leq s$. For any given $\phi(x) \in C_0^\infty(\Omega)$, multiplying (2.5) by $\frac{\phi^2}{v_\tau}$ and integrating over $S_T^* = B_\delta(x_0) \times (0,T)$, we obtain

$$\begin{aligned}&\iint_{S_T^*} \frac{\phi^2}{v_\tau}\frac{\partial v_\tau}{\partial t} dtdx + \iint_{S_T^*} \left((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)\nabla v_\tau \nabla\left(\frac{\phi^2}{v_\tau}\right)\right) dtdx \\ &= \iint_{S_T^*} \frac{\phi^2}{v_\tau}(h(v_\tau^2 - \tau - 1))v_\tau + (1-\gamma)\frac{\phi^2}{v_\tau}) dtdx.\end{aligned} \tag{3.13}$$

since $v_\tau$, is periodic, in the left-hand side of (3.13) the first term is zero. By [7], in the left-hand side of (3.13), the second term can be rewritten as follows:

$$\begin{aligned}&\iint_{S_T^*} \left((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)\nabla v_\tau \nabla\left(\frac{\phi^2}{v_\tau}\right)\right) dtdx \\ &= \iint_{S_T^*} ((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)|\nabla\phi|^2) dtdx \\ &- \iint_{S_T^*} ((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau) v_\tau^2 \left|\nabla\left(\frac{\phi}{v_\tau}\right)\right|^2) dtdx\end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), we obtain

$$\begin{aligned}&\iint_{S_T^*} ((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau)|\nabla\phi|^2) dtdx - \iint_{S_T^*} (\phi^2(h(v_\tau^2 - \tau - 1)) dtdx = \\ &\iint_{S_T^*} (\frac{\phi^2}{v_\tau}(1-\gamma) dtdx + \iint_{S_T^*} ((|A(v_\tau)\nabla v_\tau|^2 + \tau)^{\frac{p-2}{2}} A(v_\tau) v_\tau^2 |\nabla(\frac{\phi}{v_\tau})|^2) dtdx \geq 0.\end{aligned} \tag{3.15}$$

From Theorem 5.1 and also some remarks in [[3]:.pp.238, 243], it follows that there exists a constant $\gamma = \gamma(N,p)$ such that

$$\sup_{[(x_0,t_0)+s(\frac{1}{2}s_0,\frac{1}{2}p)]} |A(v_\tau)\nabla v_\tau| = c(N,p,x_0,h_0,\mu_1)\left(\iint_{[(x_0,t_0)+s(\frac{1}{2}s_0,\frac{1}{2}p)]} |A(v_\tau)\nabla v_\tau|^p dtdx\right)^{\frac{1}{2}} \wedge \frac{1}{2}\left(\frac{h_0}{4\mu_1}\right)^{\frac{1}{2-p}}.$$

For any $(x_0,t_0) \in S^*_{(T,3T)} = \Omega \times (T,3T), [(x_0,t_0)+(s_0,p)] \subset S^*_{(T,3T)}$ and $p = \min\left\{T, \frac{\sqrt{h_0 s_0}}{2^{\frac{p+6}{2}}}\right\}$. On the other hand, by (2.2) - (2.4) , we have

$$\iint_{S^*_T} |A(v_\tau)\nabla v_\tau|^p dtdx \le \max_{S^*_T} |h(x,t)| \iint_{S^*_T} (|v_\tau|^{m+1} + |v_\tau|^2) dtdx.$$

$\sup_{[(x_0,t_0)+s(\frac{1}{2}s_0,\frac{1}{2}p)]} |A(v_\tau)\nabla v_\tau| = c(N,p,x_0,h_0,\mu_1)\left(\iint_{S^*_T}(|v_\tau|^{m+1}+|v_\tau|^2)\right)^{\frac{1}{2}} dtdx \wedge \frac{1}{2}\left(\frac{h_0}{4\mu_1}\right)^{\frac{1}{2-p}}$
which implies
$\|A(v_\tau)\nabla v_\tau\|_{L^\infty_{B(x_0,s_0)\times(0,T)}} = C\left(\|v_\tau\|^{\frac{m+1}{2}}_{L^\infty(S^*_T)} + \|v_\tau\|_{L^\infty(S^*_T)}\right)^{\frac{1}{2}} \wedge \frac{1}{2}\left(\frac{h_0}{4\mu_1}\right)^{\frac{1}{2-p}}$
where $C$ is a constant independent of $\tau$ , from $\tau \in (0,\frac{1}{2})$ we have

$$A(v_\tau) = mv_\tau^{m-1} + \tau \le mv_\tau^{m-1} + \frac{1}{2}$$

By the approximating process, we can let $\phi_1 = \phi$ is the positive eigenfunction of the first eigenvalue $\mu_1$ , then we have

$$\iint_{S^*_T} ((|A(v_\tau)\nabla v_\tau|^2+\tau)^{\frac{p-2}{2}} A(v_\tau)|\nabla\phi|^2) dtdx \ge \iint_{S^*_T} (\phi^2(h(v_\tau^2-\tau-1)) dtdx.. \tag{3.16}$$

and then

$$\begin{aligned}\iint_{B(x_0,\frac{1}{2}t_0)\times(0,T)} (\phi_1^2(h(v_\tau^2-\tau-1))dtdx &\le \iint_{B(x_0,\frac{1}{2}t_0)\times(0,T)} ((|A(v_\tau)\nabla v_\tau|^{p-2}+(\tau)^{\frac{p-2}{2}})A(v_\tau)|\nabla\phi|^2)dtdx \\ &\iint_{B(x_0,\frac{1}{2}t_0)\times(0,T)} (c(s^{\frac{s+1}{2}}+\tau)^{p-2}\wedge\frac{h_0}{4\mu_1}+(\tau)^{\frac{p-2}{2}})(ms^{m-1}+\frac{1}{2})|\nabla\phi|^2dtdx \\ &= (c\mu_1(s^{\frac{s+1}{2}}+\tau)^{p-2}\wedge\frac{h_0}{4\mu_1}+\mu_1(\tau)^{\frac{p-2}{2}})(ms^{m-1}+\frac{1}{2}t_0)\int_{B(x_0,\frac{1}{2}t_0)}\phi_1^2dx.\end{aligned}$$

On the other hand

$$\iint_{B(x_0,\frac{1}{2}t_0)\times(0,T)} (\phi_1^2(h(v_\tau^2-\tau-1))dtdx \le \int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 \int_0^T h(\|v_\tau\|_{L^\infty(S^*_T)}-\tau-1)dtdx.$$

By the assumption $0 < \|v_\tau\|_{L^\infty(S^*_T)} \le s$ and also we use $|\Omega|$ to denote the Lebesgue measure of the domain $\Omega$, we obtain

$$0 \leq (c\mu_1(s^{\frac{s+1}{2}}+\tau)^{p-2} \wedge \frac{h_0}{4\mu_1} + 2^{\frac{p-2}{2}}\mu_1(\tau)^{\frac{p-2}{2}})(ms^{m-1}+\frac{1}{2})\int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 dx$$
$$-\iint_{B(x_0,\frac{1}{2}t_0)\times(0,T)} (\phi_1^2(h(v_\tau^2-\tau-1))dtdx$$

we get

$$0 \leq (c\mu_1(s^{\frac{s+1}{2}}+\tau)^{p-2} \wedge \frac{h_0}{4\mu_1} + 2^{\frac{p-2}{2}}\mu_1(\tau)^{\frac{p-2}{2}})(ms^{m-1}+\frac{1}{2})\int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 dx$$
$$-\int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 \int_0^T h(\|v_\tau\|_{L^\infty(S_T^*)}-\tau-1)dtdx$$

and then

$$0 \leq (c\mu_1(s^{\frac{s+1}{2}}+\tau)^{p-2} \wedge \frac{h_0}{4\mu_1} + 2^{\frac{p-2}{2}}\mu_1(\tau)^{\frac{p-2}{2}})(ms^{m-1}+\frac{1}{2})\int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 dx$$
$$-h_0(|\Omega|s^2-\tau-1)\int_{B(x_0,\frac{1}{2}t_0)} \phi_1^2 dtdx.$$

We get
$\tau \leq \frac{1}{2}(\frac{h_0}{4\mu_1})^{\frac{2}{p-2}}, s = \min\left\{ \sqrt[m-1]{\frac{1}{2m}}, (\frac{h_0}{4C_2})^{\frac{1}{2}}, \frac{1}{2}(\frac{h_0}{4\mu_1 C})^{\frac{1}{p-2}}, 1\right\}$
we can get

$$h_0 \leq \frac{h_0}{4} + (\frac{h_0}{4} \wedge \frac{h_0}{4}) + \frac{h_0}{4} = \frac{3h_0}{4}$$

This inequality does not hold. Therefore there exists a positive constant $s$ such that no nontrivial solutions $v_\tau$ of the equation of
$v_\tau = Q(\tau, h((v_\tau^+)^2-1)v_\tau^+ + (1-\gamma)), \gamma \in [0,1]$ satisfying $0 < \|v_\tau\|_{L^\infty(S_T)} \leq s$
Thus we complete the proof. □

**Corollary 3.5.** *There is a small constant; $0 < s < R$ which does not depend on $\tau$, such that*

$$\deg(I - Q(1, (h((v_\tau^+)^2-1)v_\tau^+), B_s, 0) = 0,$$

*where $B_s$ is a ball with zero center and radius $s$ in $L^\infty(S_T^*)$.*

**Proof .** By following the same technique of Lemma (3.4), it follows that there is $s \in (0, R)$ , which does not depend on $\tau$, such that

$$v_\tau \neq Q(\gamma, h(h((v_\tau^+)^2-1)v_\tau^+) + 1 - \gamma), \ \forall v \in \partial B_s, \ \gamma \in [0,1].$$

So the degree is well defined on $B_s$. Based on the homotopy invariance of the topological degree, it follows that:

$$\deg(I - Q(1, (h((v_\tau^+)^2-1)v_\tau^+)), B_s, 0) = \deg(1 - Q(1), B_s, 0).$$

By Lemma 3.4 , $v_\tau = Q(1)$ does not have a non-trivial solution in $B_s$.

Clearly, $v_\tau = 0$ cannot be a solution to $v_\tau = Q(1)$. It follows that $\deg(1 - Q(1), B_s, 0) = 0$, which leads to

$$\deg(I - Q(1, (h((v_\tau^+)^2 - 1)v_\tau^+)), B_s, 0) = 0.$$

So, the corollary is proved. □

**Theorem 3.6.** *If assumptions(F1)and(F2)are satisfied, then problem (1.1)-(1.3) has a non-trivial non-negative periodic solution.*

**Proof .** Based on Corollaries (3.3) and (3.5), it follows that.

$$\deg(1 - Q(f(.)), \Xi, 0) = 1,$$

where $\Xi = B_R \setminus B_s, B_\rho$ is a ball with zero center and radius $\rho \in L^\infty(S_T^*)$,

Since $0 < s \leq R$, by Lemma (3.1) and topological degree theorem, it follows that problem (2.2) - (2.4) has a non-negative non-trivial solution $v_\tau$.

Based on Lemma 3.4 and by applying a similar technique used in [13], it follows that

$\|\nabla v_\tau\|_{L^p(S_T)} \leq C$ $\|\frac{\partial v_\tau}{\partial t}\| \leq C$

If we combine the regularity results of [3] with a similar argument, given in [14], one can easily show that the limit function of $v_\tau$ is a non-negative non-trivial periodic solution of problem (1.1)-(1.3). □

## 4. Conclusions

This work is devoted to study a nonlinear diffusion Allen-Cahn equation with isolated Neumann boundary condition. For this problem, we have proved the existence of nontrivial periodic solutions by using the topological degree theorem. Namely, we show that the solution of problem (2.2) - (2.4) has a limit point, which is also considered the nonnegative nontrivial periodic solution to problem (1.1)-(1.3 in infinite space. It is observed that the topological degree theorem technique plays an important role in proving the desired results. Furthermore, this technique can be applied to other similar equations with homogeneous Dirichlet or Neumann boundary conditions.

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