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Uniform stability of integro-differential inequalities with nonlinear control inputs and delay

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Abstract

In this paper, the uniform stability for the solution of integro-differential inequalities, with nonlinear control inputs and delay functions, is investigated by using some inequality estimator conditions. Moreover, we apply the obtained results on the solutions of some proposed classes of integro-differential inequalities with nonlinear control input functions as problem formulations examples. The results show that the stability technique used in this work is efficient and robust and it can be applied to a general class and various types of integro-differential inequalities.

Keywords: Uniform stability; Integro-differential inequalities; Nonlinear control inputs; Delay functions.
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1. Introduction

Let us consider the following first order differential equation:

$$\dot{y}\,(t) = f\,(t, y(t))\,, \tag{1.1}$$

where the right hand side f is a continuous function in an open region J, and suppose that $w(t,\ t_0, y_0)$ is the unique solution of (1.1), which passes through a point $(t_0, y_0) \in I \subseteq J$, and it is defined in the interval I. Let $g(t)$ be a continuous function on the interval J, such that $g\,(t) \leq \mathrm{w}\,(t,\ t_0, y_0)\,,$

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for $t \in \mathrm{I} \bigcap \mathrm{J}$. Under the preceding assumptions on equation (1.1), a differentiable function g satisfies the inequality:

$$\dot{g}(\mathrm{t}) \leq f\,(t, g(t)) \tag{1.2}$$

Therefore, g is decreasing with respect to equation (1.1). Moreover, it is considered a sub-solution to the differential equation (1.1), [13].

The theory of integro-differential inequalities has many applications in life and in various scientific fields such as information theory, control theory, mechanics, chemistry, physics, and so on. Moreover, it has many applications in the field of ordinary and partial differential equations of parabolic and hyperbolic types, such as: estimating solutions of differential equations, estimating the existence domain of solutions, estimating the difference between two solutions, studying the criteria of the uniqueness of solutions, estimating of the error for approximate solutions, and studying the stability, see [6, 9, 16].

Over the years, integral and integro-differential inequalities have become a major tool in the analysis of differential and integro-differential equations. Therefore, they have been studied by many authors [3, 4, 5, 8, 10, 13, 14]. For instance, J. Szarski [13] has presented the theoretical results and applications of some differential inequalities, G.Ladas and I. P. Stavroulakis, [10] have studied Delay differential inequalities of first order. B. C. Dhage and S. B.Dhage, [4], have studied the differential inequalities of nonlinear first order Volterra integro-differential Equations, T. Hara, T. Yoneyama, and R. Miyazaki [3], have given some results on Voltera integro-differential inequality. S. Q. Hasan, [5], has investigated some of estimators of some inequality dynamical system. The uniform stability of nonlinear delay differential equations has been studied by A.T. Ademola, and P.O.Arawomo, [1], also the uniform stability and exponential stability of delay integro-differential equations have been studied by some authors, see [2, 7, 11, 12, 15, 17].

In this work, we consider the following types of integro-differential inequalities with nonlinear control inputs and delay functions:

$$\begin{aligned}\dot{x}\,(t) \leq f\,(t) + &\int_0^{\alpha(t)} B_1\,(t-\tau)\,u_1\,(\tau)\,w\,(x\,(\tau))\,d\tau + \int_0^{\alpha(t)} B_2(t-\tau)u_2\,(x(\tau))\,w\,(x\,(\tau))\,d\tau \\ &+ \int_0^{\alpha(t)} B_3\,(t-\tau)\,u_3\,(\tau)\,w\,(x\,(\tau))\,d\tau \int_0^t g\,(t,s)\,w\,(x\,(s))\,ds,\end{aligned}$$

$$\begin{aligned}\dot{x}\,(t)\,\dot{x}\,(t) \leq f\,(t) + &\int_0^{\alpha(t)} B_1\,(t-\tau)\,u_1\,(\tau)\,w\,(x\,(\tau))\,d\tau \\ &+ \int_0^{\alpha(t)} B_2\,(t-\tau)\,u_2\,(\tau)\,w\,(x\,(\tau))\,d\tau \int_0^t g\,(t,s)\,w\,(x\,(s))\,ds,\end{aligned}$$

$$\dot{x}\,(\mathrm{t}) \leq \left(f_1\,(\mathrm{t}) + \int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\mathrm{u}_1\,(\tau)\,\mathrm{w}\,(\mathrm{x}\,(\tau))\,\mathrm{d}\tau\right)\left(f_2\,(t) + \int_0^{\alpha(t)} B_2\,(t-\tau)\,u_2\,(\tau)\,w\,(x\,(\tau))\,d\tau\right),$$

$$\begin{aligned}\dot{x}\,(t) \leq f\,(t) + &\int_0^{\alpha(t)} B_1(t-\tau)u_1\,(\tau)\,x\,(\tau)\,d\tau + \int_0^{\alpha(t)} B_2(t-\tau)u_2\,(x(\tau))\,x\,(\tau)\,d\tau \\ &+ \int_0^{\alpha(t)} B_3\,(t-\tau)\,u_3\,(\tau)\,x\,(\tau)\,d\tau \int_0^t g\,(t,s)\,x\,(\tau)\,ds,\end{aligned}$$

where $B_1,\ B_2\ ,\ B_3$ are nonnegative, differentiable delay functions and $g \in C\,(R_+, R_+)$; and $\partial_t g \in C\,(R_+, R_+)$; and u_1, u_2, u_3, are the control functions; and $f,\ f_1,\ f_2,\ \ w \in C\,(R_+, R_+)$; $\alpha \in C^1\,(\mathrm{R}_+, \mathrm{R}_+)$ is nondecreasing functions, $t \in J = [0, \infty]\ , \alpha(t) \leq t$.

The aim of this paper is to study and investigate the uniform stability for the solution of these types of integro-differential inequalities, with nonlinear control inputs and delay functions. Moreover, we apply the obtained results on some classes of integro-differential inequalities as illustrative application examples. This paper is organized as follows, in section two, we state the main theorem and some corollaries regarding the uniform stability for some types of integro-differential inequalities with nonlinear control inputs and delay. As applications, some examples are given in section three. The last section is devoted to state some conclusions one can observe based on the main results, also some future research plans are stated in the last section.

2. The Main Results

In this section, we prove the main stability theorem and present some of its corollaries for some types of integro-differential inequalities with nonlinear control inputs and Delay.

Theorem 2.1. *Let $x \in C^1(R_+, R_+)$ be a solution of the following differential inequality:*

$$\dot{x}(t) \leq f(t) + \int_0^{\alpha(t)} B_1(t-\tau)\, u_1(\tau)\, w(x(\tau))\, d\tau + \int_0^{\alpha(t)} B_2(t-\tau) u_2(x(\tau))\, w(x(\tau))\, d\tau$$

$$+ \int_0^{\alpha(t)} B_3(t-\tau)\, u_3(\tau)\, w(x(\tau))\, d\tau \int_0^t g(t,s)\, w(x(s))\, ds \quad , \; x(0) = x_0 \; , \tag{2.1}$$

where B_1, B_2 , B_3 are nonnegative, differentiable delay functions and $g \in C(R_+, R_+)$; and $\partial_t g \in C(R_+, R_+)$; and the control functions: $u_1, u_3 \in L^2(R_+, R_+)$, $u_2 \in L^2\left(R_+ \to R_+, R_+\right)$ and f, $w \in C(R_+, R_+)$; $\alpha \in C^1(\mathrm{R}_+, \mathrm{R}_+)$ is nondecreasing functions, $t \in J = [0, \infty]$, $\alpha(t) \leq t$.
If $w\left(\mathrm{x}_0 + \int_0^T f(s)ds\right) \int_0^\tau \tilde{Q}(s)\, \tilde{R}(s)\, ds \; < 1$, *for $t \geq 0$, where*

$$Q(t) = exp\left(\int_0^{\alpha(t)} B_1(t-\tau) u_1(\tau)\, d\tau\right)$$

$$\tilde{R}(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} B_2(t-\tau) f_1(\tau) d\tau + \left(\left(\int_0^{\alpha(t)} B_3(t-\tau) u_3(\tau)\, d\tau\right) \int_0^t g(t,s)\, w(x(s))\, ds\right)\right)$$

for $t \in J$ *such that* $\frac{w\left(x_0 + \int_0^T f(s)ds\right)\tilde{Q}(\tau)}{1 - w\left(x_0 + \int_0^T f(s)ds\right)\int_0^\tau \tilde{Q}(s)\tilde{R}(s)ds} \in \mathrm{Dom}(w^{-1})$, *then*

$$x(t) \leq w^{-1}\left[\frac{w\left(\mathrm{x}_0 + \int_0^T f(s)\, ds\right) \tilde{Q}(\tau)}{1 - w\left(\mathrm{x}_0 + \int_0^T f(s)\, ds\right) \int_0^\tau \tilde{Q}(s)\, \tilde{R}(s)\, ds}\right]$$

Proof . From (2.1) and for $t \in J$, $t \geq 0$, it follows that

$$y(t) = \int_0^{\alpha(t)} B_1(t-\tau) u_1(\tau)\, w(x(\tau))\, d\tau + \int_0^{\alpha(t)} B_2(t-\tau) u_2(x(\tau))\, w(x(\tau))\, d\tau$$

$$+ \int_0^{\alpha(t)} B_3(t-\tau)\, u_3(\tau)\, w(x(\tau))\, d\tau \int_0^t g(t,s)\, w(x(s))\, ds \tag{2.2}$$

From (2.2), we get $\dot{x}(t) \leq f(t) + y(t)$, therefore x is non-degreasing on R_+. By taking the derivative of equation (2.2), for $t \in J$, we obtain

$$\begin{aligned}
\dot{y}(t) &= \mathrm{B}_1(t-\alpha(\mathrm{t}))\,\mathrm{u}_1(\alpha(\mathrm{t}))\,\mathrm{w}(\mathrm{x}(\alpha(\mathrm{t})))\,\dot{\alpha}(\mathrm{t}) + \int_0^{\alpha(\mathrm{t})} \partial_t \mathrm{B}_1(t-\tau)\,\mathrm{u}_1(\tau)\,\mathrm{B}_1\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau + \\
&+ \mathrm{B}_2(t-\alpha(\mathrm{t}))\,\mathrm{u}_2(\mathrm{x}(\alpha(\mathrm{t})))\,\mathrm{w}(\mathrm{x}(\alpha(\mathrm{t})))\,\dot{\alpha}(\mathrm{t}) + \int_0^{\alpha(\mathrm{t})} \partial_t \mathrm{B}_2(t-\tau)\,\mathrm{u}_2(\mathrm{x}(\alpha(\tau)))\,\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau \\
&+ \begin{bmatrix} \mathrm{B}_3(t-\alpha(\mathrm{t}))\,\mathrm{u}_3(\mathrm{x}(\alpha(\mathrm{t})))\,\mathrm{w}(\mathrm{x}(\alpha(\mathrm{t})))\,\dot{\alpha}(\mathrm{t}) + \\ \int_0^{\alpha(\mathrm{t})} \partial_t \mathrm{B}_3(t-\tau)\mathrm{u}_3(\mathrm{x}(\alpha(\tau)))\,\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau \end{bmatrix} \int_0^{\mathrm{t}} \mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{s}))\,\mathrm{ds} \\
&+ \int_0^{\alpha(\mathrm{t})} \mathrm{B}_3(t-\tau)\,\mathrm{u}_3(\tau)\,\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau \left[\mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{t}))\,\dot{\alpha}(\mathrm{t}) + \int_0^{\mathrm{t}} \partial_t \mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{s}))\,\mathrm{ds}\right] \\
&\leq w\left(\mathrm{x}_0 + \int_0^t f(s)ds\right)\left(\frac{d}{dt}\int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\,\mathrm{d}\tau\right) \\
&+ w^2\left(\mathrm{x}_0 + \int_0^t f(s)ds\right)\left(\frac{d}{dt}\int_0^{\alpha(\mathrm{t})} \mathrm{B}_2(t-\tau)\mathrm{f}_1(\tau)\mathrm{d}\tau\right) \\
&+ w^2\left(x_0 + \int_0^t f(s)ds\right)\left(\frac{d}{dt}\int_0^{\alpha(\mathrm{t})} B_3(t-\tau)u_3(\tau)\,d\tau\right)\int_0^t g(t,s)\,w(x(s))\,ds \\
&+ \leq w\left(\mathrm{x}_0 + \int_0^t f(s)\,ds\right)\left[\frac{d}{dt}\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\,\mathrm{u}_1(\tau)\,\mathrm{d}\tau\right)\right] \\
&+ w^2\left(x_0 + \int_0^t f(s)ds\right)\frac{d}{dt}\begin{pmatrix} \int_0^{\alpha(t)} B_2(t-\tau)\,u_1(\tau)\,d\tau + \\ \left(\left(\int_0^{\alpha(t)} B_3(t-\tau)u_3(\tau)\,d\tau\right)\int_0^t g(t,s)\,w(x(s))\,ds\right) \end{pmatrix} \\
&w\left(\mathrm{x}_0 + \int_0^T f(s)\,ds\right)\left[\frac{d}{dt}\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\,\mathrm{u}_1(\tau)\,\mathrm{d}\tau\right)\right] \\
&w^2\left(\mathrm{x}_0 + \int_0^T f(s)ds\right)\frac{d}{dt}\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_2(t-\tau)\mathrm{u}_1(\tau)\mathrm{d}\tau\right. \\
&\left.+\left(\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_3(t-\tau)\mathrm{u}_3(\tau)\,\mathrm{d}\tau\right)\int_0^{\mathrm{t}} \mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{s}))\,\mathrm{ds}\right)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\dot{y}(t)}{w^2\left(\mathrm{x}_0 + \int_0^T f(s)ds\right)} - \frac{1}{w\left(\mathrm{x}_0 + \int_0^T f(s)ds\right)}\left[\frac{d}{dt}\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\,\mathrm{d}\tau\right)\right] \\
&\leq \frac{d}{dt}\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_2(t-\tau)\mathrm{f}_1(\tau)\mathrm{d}\tau + \left(\left(\int_0^{\alpha(\mathrm{t})} \mathrm{B}_3(t-\tau)\mathrm{u}_3(\tau)\,\mathrm{d}\tau\right)\int_0^{\mathrm{t}} \mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{s}))\,\mathrm{ds}\right)\right)
\end{aligned}$$

Consider $z(t) = \frac{1}{w\left[\mathrm{x}_0 + \int_0^T f(s)ds + \int_0^t y(s)ds\right]}$

$$\tilde{q}(t) = \int_0^{\alpha(\mathrm{t})} \mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\,\mathrm{d}\tau$$

$$Q\left(t\right)=exp\left(\int_{0}^{\alpha(\mathrm{t})}\mathrm{B}_1(t-\tau)\mathrm{u}_1\left(\tau\right)\mathrm{d}\tau\right)$$

$$\tilde{R}\left(t\right)=\frac{d}{dt}\left(\int_{0}^{\alpha(\mathrm{t})}\mathrm{B}_2(t-\tau)\mathrm{f}_1(\tau)\mathrm{d}\tau+\left(\left(\left(\int_{0}^{\alpha(\mathrm{t})}\mathrm{B}_3(t-\tau)\mathrm{u}_3\left(\tau\right)\mathrm{d}\tau\right)\int_{0}^{\mathrm{t}}\mathrm{g}\left(\mathrm{t},\mathrm{s}\right)\mathrm{w}\left(\mathrm{x}\left(\mathrm{s}\right)\right)\mathrm{ds}\right)\right)\right)$$

Thus

$$\dot{z}\left(t\right)+z\left(t\right)\left(\frac{d}{ds}\tilde{q}\left(t\right)\right)\geq -\tilde{R}\left(t\right)$$

$$\dot{z}\left(t\right)e^{\tilde{q}(t)}+z\left(t\right)e^{\tilde{q}(t)}\left(\frac{d}{dt}\tilde{q}\left(t\right)\right)\geq -\tilde{R}\left(t\right)e^{\tilde{q}(t)}$$

$\frac{d}{dt}\left(z\left(t\right)\tilde{Q}\left(t\right)\right)\geq -\tilde{Q}\left(t\right)\tilde{R}\left(t\right)$, we obtain
$z\left(t\right)\tilde{Q}\left(t\right)\geq z\left(0\right)-\int_0^t\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds$, $0\leq t\leq T$, we get that,

$$z\left(t\right)=\frac{1}{w\left[\mathrm{x}_0+\int_0^T f(s)ds+\int_0^t y(s)ds\right]}\geq\left[\frac{1}{w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]}-\int_0^t\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds\right]\frac{1}{\tilde{Q}\left(t\right)}$$

$$=\frac{1-w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\int_0^t\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds}{w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\tilde{Q}\left(t\right)}$$

For $0\leq t\leq T$. Let $t=\tau$ and since $w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\int_0^\tau\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds\quad<1$

Then, $w\left[\mathrm{x}_0+\int_0^T f(s)ds+\int_0^T y(s)ds\right]\leq\frac{w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\tilde{Q}(T)}{1-\ w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\int_0^T\tilde{Q}(s)\tilde{R}(s)ds}$. Therefore,

$$x\left(t\right)\leq w^{-1}\left[\frac{w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\tilde{Q}\left(T\right)}{1-\ w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\int_0^T\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds.}\right]$$

□

Corollary 2.2. *Let $x\in C^1\left(R_+,R_+\right)$ be a solution of the following integro-differential inequality with nonlinear control inputs and delay:*

$$\dot{x}\left(t\right)\leq f\left(t\right)+\int_0^{\alpha(t)}B_1\left(t-\tau\right)u_1\left(\tau\right)w\left(x\left(\tau\right)\right)d\tau$$
$$+\int_0^{\alpha(t)}B_2\left(t-\tau\right)u_2\left(\tau\right)w\left(x\left(\tau\right)\right)d\tau\int_0^t g\left(t,s\right)w\left(x\left(s\right)\right)ds,$$

$$x\left(0\right)=x_0,$$

where $B_1,\ B_2,g,\alpha,w,\ f$ are defined as in the integro-differential inequality (2.1) *and the control functions $u_1,\ u_2\in L^2\left(R_+,R_+\right)$. If $w\left[x_0+\int_0^T f(s)ds\right]\int_0^\tau\tilde{Q}\left(s\right)\tilde{R}\left(s\right)ds$ for $t\in R_+,$ where*

$$\tilde{Q}\left(t\right)=e^{\int_0^{\alpha(\mathrm{t})}\mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\mathrm{d}\tau},$$

$$\tilde{R}(t)=\frac{d}{dt}\left[\left(\int_0^{\alpha(t)}\mathrm{B}_2(t-\tau)\,\mathrm{u}_2(\tau)\,\mathrm{d}\tau\int_0^{\mathrm{t}}\mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{ds}\right)\right],$$

then

$$x(t)\le w^{-1}\left[\frac{w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\tilde{Q}(\tau)}{1-w\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\int_0^\tau\tilde{Q}(s)\,\tilde{R}(s)\,ds}\right],t\in J.$$

Corollary 2.3. *Let* $x\in C^1(R_+,R_+)$ *be a solution of the following integro-differential inequality with nonlinear control inputs and delay:*

$$\dot{x}(t)\le\left(f_1(t)+\int_0^{\alpha(t)}B_1(t-\tau)u_1(\tau)\,w(x(\tau))\,d\tau\right)\left(f_2(t)+\int_0^{\alpha(t)}B_2(t-\tau)\,u_2(\tau)\,w(x(\tau))\,d\tau\right),\tag{2.3}$$

$$x(0)=x_0,$$

where B_1, B_2, g, α, w *are defined as in the integro-differential inequality* (2.1)*; the control functions* u_1, $u_2\in L^2(R_+,R_+)$ *and* f_1, $f_1\in C(R_+,R_+)$. *If* $w\left(x_0+\int_0^T f_1(s)\,f_2(s)\,ds\right)\int_0^t\tilde{Q}(s)\,\tilde{R}(s)\,ds\quad<1$ *for* $t\in R_+$, *where* $\tilde{Q}(t)=e^{\int_0^{\alpha(t)}\mathrm{f}_2(\mathrm{t})\mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\mathrm{w}(\mathrm{x}(\tau))\mathrm{d}\tau+\int_0^{\alpha(t)}\mathrm{f}_1(\mathrm{t})\mathrm{B}_2(t-\tau)\mathrm{u}_2(\tau)\mathrm{w}(\mathrm{x}(\tau))\mathrm{d}\tau}$,

$$\tilde{R}(t)=\frac{d}{dt}\left[\int_0^{\alpha(t)}\mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\,\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau\int_0^{\alpha(t)}\mathrm{B}_2(t-\tau)\,\mathrm{u}_2(\tau)\,\mathrm{w}(\mathrm{x}(\tau))\,\mathrm{d}\tau\right],$$

then

$$x(t)\le w^{-1}\left[\frac{w\left(\mathrm{x}_0\int_0^T\mathrm{f}_1(\mathrm{s})\,\mathrm{f}_2(\mathrm{s})\,ds\right)\tilde{Q}(\tau)}{1-w\left(\mathrm{x}_0+\int_0^T\mathrm{f}_1(\mathrm{s})\,\mathrm{f}_2(\mathrm{s})\,ds\right)\int_0^\tau\tilde{Q}(s)\,\tilde{R}(s)\,ds}\right],t\in J,$$

Corollary 2.4. *Let* $x\in C^1(R_+,R_+)$ *be a solution of the following integro-differential inequality with nonlinear control inputs and delay:*

$$\begin{aligned}\dot{x}(t)\le f(t)+&\int_0^{\alpha(t)}B_1(t-\tau)u_1(\tau)\,x(\tau)\,d\tau+\int_0^{\alpha(t)}B_2(t-\tau)u_2(x(\tau))\,x(\tau)\,d\tau\\&+\int_0^{\alpha(t)}B_3(t-\tau)\,u_3(\tau)\,x(\tau)\,d\tau\int_0^t g(t,s)\,x(\tau)\,ds\end{aligned}\tag{2.4}$$

$$x(0)=x_0,$$

where B_1, B_2, B_3, g, α, w, f *are defined as in the integro-differential inequality* (2.1)*; and the control functions:* u_1, $u_2, u_3\in L^2(R_+,R_+)$.

If $\left(\mathrm{x}_0+\int_0^T\mathrm{f}(\mathrm{s})ds\right)\int_0^t\tilde{Q}(s)\,\tilde{R}(s)\,ds\quad<1$ *for* $t\in R_+$,

where

$$Q(t)=exp\left(\int_0^{\alpha(t)}\mathrm{B}_1(t-\tau)\mathrm{u}_1(\tau)\,\mathrm{d}\tau\right)$$

$$\tilde{R}(t)=\frac{d}{dt}\left(\int_0^{\alpha(t)}\mathrm{B}_2(t-\tau)\,\mathrm{f}_1(\tau)\,\mathrm{d}\tau+\left(\left(\int_0^{\alpha(t)}\mathrm{B}_3(t-\tau)\,\mathrm{u}_3(\tau)\,\mathrm{d}\tau\right)\int_0^{\mathrm{t}}\mathrm{g}(\mathrm{t},\mathrm{s})\,\mathrm{w}(\mathrm{x}(\mathrm{s}))\,\mathrm{ds}\right)\right)$$

then

$$x(t)\le\left[\frac{\left[\mathrm{x}_0+\int_0^T f(s)ds\right]\tilde{Q}(\tau)}{1-\left(\mathrm{x}_0+\int_0^T f(s)ds\right)\int_0^\tau\tilde{Q}(s)\,\tilde{R}(s)\,ds}\right],$$

3. Some Applications of Problem Formulations

The applications of uniform stability for some integro-differential inequalities with nonlinear control inputs and delay are presented in this section.

Proposition 3.1. *Consider that the assumptions of corollary 2.2 hold, $x \in C^1(R_+, R_+)$ is a solution of the integro-differential inequality with nonlinear control inputs and delay:*

$$\dot{x}(t) \leq \left(f_1(t) + \int_0^{\alpha(t)} B_1(t-\tau)\, u_1(\tau)\, x(\tau)\, d\tau\right)\left(f_2(t) + \int_0^{\alpha(t)} B_2(t-\tau)u_2(x(\tau))\, x(\tau)\, d\tau\right) \tag{3.1}$$

If $\left|x_0 + \int_0^t f_1(s)\, f_2(s)\, ds\right| \leq L_1$, where $L_1 > 0$. *Moreover,*

$$L_2 = \lim_{t\to 0} \int_{t_0}^{t} \int_0^{\alpha(r)} \left[f_2(t)\, B_1(t-s)\, u_1(s)\, ds + f_1(t)\, B_2(t-s)u_2(s)\, ds\right]\, dr < \infty$$

$$L_3 = \lim_{t\to 0} \int_{t_0}^{t} \int_0^{\alpha(r)} B_1(r-s)\, u_1(s)\, x(s)\, ds \int_0^{\alpha(r)} B_2(r-s)u_2(x(s))\, x(s)\, ds dr,$$

where L_2 , L_3, are nonnegative constants and $L_1\ L_3 e^{L_2} < 1$. Then for $t \geq 0$, $|\mathrm{x}(\mathrm{t})| \leq \frac{L_1\ e^{L_2}}{1-L_1\ L_3 e^{L_2}}$.

Proposition 3.2. *In the proposition 3.1, if the condition: $\left|x_0 + \int_0^t f_1(s)\, f_2(s)\, ds\right| \leq L_1$ is replaced by*

$$\left|x_0 + \int_0^t f_1(s)\, f_2(s)\, ds\right| \leq L_1 e^{-t} \quad on\ R_+,$$

then $x(t) \to 0$ as $t \to \infty$

Example 3.3. *Consider the following integro-differential inequality with nonlinear control inputs and delay:*

$$\dot{x}(t) \leq \int_{t_0}^{t} B_1(t-\tau)u_1(\tau)\, x(\tau) d\tau + \int_{t_0}^{t} B_2(t-\tau)u_2(x(\tau))\, x(\tau)\, d\tau \tag{3.2}$$

where B_1, $B_2 \in C(R_+, R_+)$, and the control functions: $u_1 \in L^2(R_+, R_+)$, $u_2 \in L^2\left(R_+ \to R_+, R_+\right)$, for $t \geq 0$ and $\tau \in C^1(\dot{R}_+, R_+)$ such that $\tau(t) \leq t$. If $\alpha(t) = t - \tau(t)$ is an increasing of R_+ and

$$|B_1(t)u_1(\tau)\, x(\tau)| \leq a(t)\, |u_1(t)|\, |x(t)|, \quad |B_2(t)u_2(x(\tau))\, x(\tau)| \leq b(t)\, |f_1(t)|\, |x(t)|^2,$$

and

$$M_1 = \int_0^{\infty} a(s)\, |u_1(s)|\, ds < \infty, \ M_2 = \int_0^{\infty} b(s)\, |f_1(s)|\, ds < \infty$$

where M_1, M_2 are nonnegative constants, then the solution of (3.2) is uniformly stable on R_+.

Proof . Let $x(t)$ be the solution od (2.4) with the initial condition: $x(t_0) = x_0$, hence from (2.4) , we obtain

$$\dot{x}(t) \leq \int_{t_0}^{t} B_1(t-\tau)u_1(\tau)\, x(\tau) d\tau + \int_{t_0}^{t} B_2(t-\tau)u_2(x(\tau))\, x(\tau)\, d\tau$$

$$\leq \int_{t_0}^{t} B_1(\alpha(s))u_1(s)\, x(s) ds + \int_{t_0}^{t} B_2(\alpha(s))u_2(x(s))\, x(s)\, ds$$

$$\leq \int_{\alpha(t_0)}^{\alpha(t)} \frac{B_1(r)u_1\left(\alpha^{-1}(r)\right)x\left(\alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}dr + \int_{\alpha(t_0)}^{\alpha(t)} \frac{B_2(r)u_2\left(x(\alpha^{-1}(r))\right)x\left(\alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}dr,$$

$$0 \leq t_0 \leq t \leq \infty$$

So,

$$|x(t)| \leq \int_0^{\alpha(t)} \frac{a(r)\left|u_1\left(\alpha^{-1}(r)\right)\right|\left|x\left(\alpha^{-1}(r)\right)\right|dr}{\left|\alpha'\left(\alpha^{-1}(r)\right)\right|} + \int_0^{\alpha(t)} \frac{b(r)\left|f_1\left(\alpha^{-1}(r)\right)\right|\left|x(\alpha^{-1}(r))\right|^2}{\left|\alpha'\left(\alpha^{-1}(r)\right)\right|}dr$$

Then

$$|x(t)| \leq \left[\frac{|x_0|\,\tilde{Q}(\tau)}{1-|x_0|\int_0^{\tau}\tilde{Q}(s)\,\tilde{R}(s)\,ds}\right],$$

where

$$Q(t) = exp\left(\int_0^{\alpha(t)} \frac{a(r)\left|\mathrm{u}_1\left(\alpha^{-1}(r)\right)\right|\mathrm{dr}}{\left|\alpha'\left(\alpha^{-1}(r)\right)\right|}\right), \quad \tilde{R}(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} \frac{\mathrm{b}(r)\left|\mathrm{f}_1\left(\alpha^{-1}(r)\right)\right|}{\left|\alpha'\left(\alpha^{-1}(r)\right)\right|}dr\right)$$

From corollary2.4, we get

$$|x(t)| \leq \left[\frac{|x_0|\,exp\left(\int_0^{\mathrm{t}} a(u)\left|\mathrm{u}_1(\mathrm{u})\right|du\right)}{1-|x_0|\int_0^t exp\left(\int_0^{\mathrm{r}} a(u)\left|\mathrm{u}_1(\mathrm{u})\right|du\right)\frac{d}{dt}\left(\int_0^r \mathrm{b}(u)\left|\mathrm{f}_1(\mathrm{u})\right|du\right)dr}\right], \quad 0 \leq t_0 \leq t \leq \infty$$

$$|x(t)| \leq \left[\frac{|x_0|\,expM_1}{1-|x_0|\int_0^t expM_1\frac{d}{dt}(M_2)\,dr}\right]$$

$$|x(t)| \leq |x_0|\,expM_1$$

□

Example 3.4. *Consider the following integro-differential inequality with nonlinear control inputs and delay:*

$$\begin{aligned}\dot{x}(t) \leq f(t) + \int_0^t B_1(t-\tau)u_1(\tau)\,x(\tau)\,d\tau + \int_0^t B_2(t-\tau)u_2(x(\tau))\,x(\tau)\,d\tau \\ + \int_0^t B_3(t-\tau)\,u_3(\tau)\,x(\tau)\,d\tau \int_0^t g(t,s)\,x(s)\,ds,\ t \geq 0 \end{aligned} \tag{3.3}$$

where $B_1,\ B_2,\ g \in C(R_+, R_+)$, for $t \geq 0$, the control functions: $u_1 \in L^2(R_+, R_+)$, $u_2 \in L^2(R_+ \to R_+, R_+)$, for $t \geq 0$ and $\tau \in C^1(R_+, R_+)$ such that $\tau(t) \leq t$. If $\alpha(t) = t - \tau(t)$ is an increasing of R_+ and

$$|B_1(t)u_1(\tau)\,x(\tau)| \leq a(t)\,|u_1(t)|\,|x(t)|\,, |B_2(t)u_2(x(\tau))\,x(\tau)| \leq b(t)\,|f_1(t)|\,|x(t)|^2,$$

$|B_3(t)\,u_3(\tau)\,x(\tau)| \leq c(t)\,|u_3(t)|\,|x(t)|$ *and* $|g(t,s)\,x(s)| \leq d(s)\,|x(s)|$,

$$M_1 = \int_0^{\infty} a(t)\,|u_1(t)|\,ds < \infty\ ,\ M_2 = \int_0^{\infty} b(t)\,|u_2(t)|\,ds < \infty$$

$M_3 = \int_0^{\infty} c(t)\,|u_3(t)|\,ds < \infty$, $M_4 = \int_0^{\infty} d(t)\,ds < \infty$ *and* $M_5 = \int_0^{\infty} \mathrm{f(s)}ds$, *where* M_1, M_2 , M_3, M_4 *and* M_5 *are nonnegative constants, then the solution of* (3.3) *is uniformly stable on* R_+.

Proof . Let $x(t)$ be the solution of (3.3) with the initial condition: $x(t_0) = x_0$, hence from (3.3), $x(t)$ satisfies

$$\begin{aligned}
\dot{x}(t) &\leq f(t) + \int_0^t B_1(t-\tau)u_1(\tau)\,x(\tau)\,d\tau + \int_0^t B_2(t-\tau)u_2(x(\tau))\,x(\tau)\,d\tau \\
&\quad + \int_0^t B_3(t-\tau)\,u_3(\tau)\,x(\tau)\,d\tau \int_0^t g(t,s)\,x(s)\,ds,\ t \geq 0 \\
&= f(t) + \int_{t_0}^t B_1(\alpha(s))u_1(s)\,x(s)ds + \int_{t_0}^t B_2(\alpha(s))u_2(x(s))\,x(s)\,ds \\
&\quad + \int_{t_0}^t B_3(\alpha(s))\,u_3(s)\,x(s)\,d\tau \int_0^t g(t,s)\,x(s)\,ds \\
&= f(t) + \int_{\alpha(t_0)}^{\alpha(t)} \frac{B_1(r)u_1(\alpha^{-1}(r))\,x(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}dr + \int_{\alpha(t_0)}^{\alpha(t)} \frac{B_2(r)u_2(x(\alpha^{-1}(r)))\,x(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}dr \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \frac{B_3(r)u_3(\alpha^{-1}(r))\,x(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}dr \int_0^t g(t,s)\,x(s)\,ds \\
&\leq |f(t)| + \int_{\alpha(t_0)}^{\alpha(t)} \frac{a(r)\,|u_1(\alpha^{-1}(r))|\,|x(\alpha^{-1}(r))|\,dr}{|\alpha'(\alpha^{-1}(r))|} + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b(r)\,|f_1(\alpha^{-1}(r))|\,|x(\alpha^{-1}(r))|\,dr}{|\alpha'(\alpha^{-1}(r))|} \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \frac{c(r)\,|u_3(\alpha^{-1}(r))|\,|x(\alpha^{-1}(r))|\,dr}{|\alpha'(\alpha^{-1}(r))|} \int_0^t d(s)\,|x(s)|\,ds.
\end{aligned}$$

From Corollary 2.4, we have that

$$x(t) \leq \left[\frac{\left[\mathrm{x}_0 + \int_0^T f(s)ds\right]\tilde{Q}(\tau)}{1 - \left(\mathrm{x}_0 + \int_0^T f(s)ds\right)\int_0^\tau \tilde{Q}(s)\,\tilde{R}(s)\,ds}\right],$$

where

$$Q(t) = exp\left(\int_0^{\alpha(t)} \frac{a(r)\,|\mathrm{u}_1(\alpha^{-1}(r))|\,|\mathrm{x}(\alpha^{-1}(r))|\,\mathrm{dr}}{|\alpha'(\alpha^{-1}(r))|}\right)$$

and

$$\tilde{R}(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} \frac{b(r)\,|\mathrm{f}_1(\alpha^{-1}(r))|\,\mathrm{dr}}{|\alpha'(\alpha^{-1}(r))|} + \left(\int_0^{\alpha(t)} \frac{c(r)\,|\mathrm{u}_3(\alpha^{-1}(r))|\,\mathrm{dr}}{|\alpha'(\alpha^{-1}(r))|} \int_0^{\mathrm{t}} d(s)\mathrm{ds}\right)\right).$$

Then

$$\begin{aligned}
|x(t)| &\leq \frac{\left[|x_0| + \left|\int_0^t \mathrm{f(s)}ds\right|\right]\tilde{Q}(t)}{1 - \left[|x_0| + \left|\int_0^t \mathrm{f(s)}ds\right|\right]\int_0^t \tilde{Q}(s)\,\tilde{R}(s)\,ds} \\
&\leq \frac{\left[|x_0| + \left|\int_0^t \mathrm{f(s)}ds\right|\right] exp\left(\int_0^t a(s)\,|\mathrm{u}_1(\mathrm{s})|\,ds\right)}{1 - |x_0| + \left|\int_0^t \mathrm{f(s)}ds\right| \int_0^t \frac{d}{dt}\left[\int_0^r b(s)\,|\mathrm{f}_1(\mathrm{s})|\,ds + \int_0^r c(s)\,|\mathrm{u}_3(\mathrm{s})|\,ds \int_0^{\mathrm{t}} d(s)\mathrm{ds}\right] exp\left(\int_0^r a(s)\,|\mathrm{u}_1(\mathrm{s})|\,ds\right) dr} \\
&\leq \frac{\left[|x_0| + \left|\int_0^t \mathrm{f(s)}ds\right|\right] e^{M_1}}{1 - \left[|x_0| + \left|\int_0^t \mathrm{f(s)}ds\right|\right]\int_0^t \frac{d}{dt}\left[M_2 + M_3M_4\right]e^{M_1}dr} \leq \left[|x_0| + M_5\right]e^{M_1}
\end{aligned}$$

□

4. Conclusions and Future Plans

In this work, firstly, we have studied and investigated the uniform stability for the solution of some types of integro-differential inequalities with nonlinear control inputs and delay functions. Secondly, we have applied the obtained results on some classes of integro-differential inequalities as illustrative application examples. The results show that the stability technique used in this work is efficient and robust and it can be applied to a general class and various types of integro-differential inequalities. As future research plans, one may study the existence-domain of solutions, uniqueness of solutions, and stability for some classes of fractional order integro-differential inequality. As well as, we may be concerned with introducing some classes of stochastic integro-differential inequality and investigating their theoretical properties such as the existence of solutions and stability.

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