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On existence of solutions for some functional integral equations in Banach algebra by fixed point theorem

Manochehr Kazemi*

Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran

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Abstract

In this research, we analyze the existence of solution for some nonlinear functional integral equations using the techniques of measures of noncompactness and the Petryshyn's fixed point theorem in Banach space. The results obtained in this paper cover many existence results obtained by numerous authors under some weaker conditions. We also give an example satisfying the conditions of our main theorem but not satisfying the conditions described by other authors.

Keywords: Functional integral equations, Existence of solution, Measures of noncompactness, Petryshyn's fixed point theorem.
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1. Introduction

The concept of a measure of noncompactness was introduced for the first time by Kuratowski [22] in 1930. The theory of measure of noncompactness and densifying operators has applications in general topology, geometry of Banach spaces, and the theory of integral equations and differential equations. Nonlinear integral equations have arisen in many branches of science [12, 17] such as in the theory of optimal control, mathematical physics, population dynamics, economics etc. [30, 7, 20, 3, 39, 28, 9]. Recently, there have been several successful attempt to apply the concept of measure of noncompactness in the study of the existence of solutions of nonlinear integral equations [37, 36, 38, 1, 2, 13, 25, 31, 27]. In this paper, we present and prove a new existence theorem for

*Corresponding author
Email address: univer_ka@yahoo.com, m.kazemi@aiau.ac.ir (Manochehr Kazemi*)

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solution of nonlinear functional integral equations which contains several functional integral equations as a special case and is in the following form:

$$x(t) = \Big(q(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) + F\big(t, x(\tau_1(t)), x(\tau_2(t)), \int_0^{\varphi(t)} u(t,s,x(\theta_1(s)))ds\big)\Big)$$
$$\times \Big(g(t, x(\beta_1(t)), x(\beta_2(t))) + G\big(t, x(\upsilon_1(t)), x(\upsilon_2(t)), \int_0^{a} v(t,s,x(\theta_2(s)))ds\big)\Big),\ t \in I_a = [0,a]. \tag{1.1}$$

Numerous authors have carried out some successful efforts to solve many functional integral equations by applying Darbo condition which is a powerful tool to study these equations [1, 2, 13, 25, 31, 27, 32, 33, 16, 4, 24, 14, 15]. For the existence of solutions of integral equation (1.1), we use the Petryshyn fixed point theorem [35] (instead of Darbo's theorem) that has been analyzed as a generalization of Darbo's fixed theorem [5]. The existence result proved in this paper generalizes several ones obtained earlier by other authors (cf. [25, 31, 27, 32, 33, 16, 4, 24, 14, 15, 10, 34, 23], for example). The idea of using the Petryshyn fixed point theorem in order to investigate the existence of solution of nonlinear functional integral equations for the first time was introduced in [21] by Kazemi et al. The following statements describe the main reasons why we use Eq. (1.1) and what is the excellence of our work: The first is that the conditions in many papers will be simplified. The second reason is that this paper unifies the relevant work in this field. The next reason is that bounded condition (H3) of Theorem 3.1, shows that the "sublinear condition" that has been discussed in several literature (see e.g. (C6) below and [25, 16, 10, 34, 23, 26, 8]) have not a significant role.
The paper is organized as five sections including the introduction. In Section 2, we introduce some preliminaries and use them to obtain our aims in Section 3. Section 3 is devoted to state and prove existence theorem for equations involving condensing operators using the Petryshyn's fixed point theorem. In Section 4, we provide some examples that verifies the applications of these kind of nonlinear functional-integral equations in nonlinear analysis. Finally Section 5, concludes the paper.

2. Preliminaries

Throughout the paper, we have the following assumptions:

- E: Real Banach space;
- $\bar{B}_r$: Closed ball with center 0 and radius r ;
- $\partial\bar{B}_r$: Sphere in E around 0 with radius $r > 0$;
- $ConvM$: Convex hull of a subset M of E;
- $Conv\bar{M}$: Closed convex hull of a set M;
- $\mathfrak{m}_E$: Set of all bounded subsets of E;
- $\mathfrak{n}_E$: Set of all relatively compact subsets of E.

Definition 2.1 ([22]). *If M is a bounded subset of a Banach space E, let $\alpha(M)$ denote the (Kuratowski) measure of noncompactness of M, that is,*

$$\alpha(M) = \inf\{\varepsilon > 0 : M \text{ may be covered by finitely many sets of diameter} \leq \varepsilon\}. \tag{2.1}$$

Other measures of noncompactness were introduced by Gol'denšte˘ın.

Definition 2.2 ([18]). *The Hausdorff (or ball) measure of noncompactness*

$$\mu(M) = \inf\{\varepsilon > 0 : \text{there exists a finite } \varepsilon\text{-net for } M \text{ in } E\}, \tag{2.2}$$

where by a finite ε-net for M in E we mean, as usual, a set $\{d_1, d_2, ..., d_m\} \subset E$ such that the balls $B_\varepsilon(E; d_1), B_\varepsilon(E; d_2), ..., B_\varepsilon(E; d_m)$ over M. These measures of noncompactness are mutually equivalent in the sense that

$$\mu(M) \leq \alpha(M) \leq 2\mu(M)$$

for any bounded set $M \subset E$.

It is easy to see that the following basic results hold for any measure of noncompactness

Theorem 2.3 ([35]). *Let E be a Banach space, $\lambda \in \mathbb{R}$ and $M, N \in \mathfrak{m}_E$ bounded. Then*

(i) $\mu(M) = 0$ *if and only if* $M \in \mathfrak{n}_E$;
(ii) $M \subseteq Y$ *implies* $\mu(M) \leq \mu(N)$;
(iii) $\mu(\bar{M}) = \mu(ConvM) = \mu(M)$;
(iv) $\mu(M \cup N) = \max\{\mu(M), \mu(N)\}$;
(v) $\mu(\lambda M) =\mid \lambda \mid \mu(M)$, *where* $\lambda M = \{\lambda m : m \in M, \lambda \in \mathbb{R}\}$;
(vi) $\mu(M + N) \leq \mu(M) + \mu(N)$, *where* $M + N = \{m + n : m \in M, n \in N\}$.

In what follows, we focus on the Banach space $E = C([0, a], \mathbb{R})$ consisting of all real-valued functions and continuous on the interval $[0, a]$. The space $C[0, a]$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, a]\}.$$

Let M be a nonempty bounded subset of $E = C([0, a], \mathbb{R})$ and for $u \in M, \varepsilon > 0$, the modulus of continuity $\omega(u, \varepsilon)$ is given by:

$$\omega(u, \varepsilon) = \sup\{|u(x) - u(y)| : |x - y| \leq \varepsilon, x, y \in [0, a]\}.$$

Further,

$$\omega(M, \varepsilon) = \sup\{\omega(u, \varepsilon), u \in M\}, \quad \omega_0(M) = \lim_{\varepsilon \to 0} \omega(M, \varepsilon)$$

It may be shown [6] that $\omega_0(M)$ is regular measure of noncompactness in C[a, b].

Theorem 2.4 ([18]). *On the space $C[0, a]$, the measures of noncompactness* (2.2) *is equivalent to*

$$\mu(M) = \lim_{\varepsilon \to 0} \sup_{u \in M} \omega(u, \varepsilon) = \omega_0(M) \tag{2.3}$$

for all bounded sets $M \subset C[0, a]$.

For our purpose we use equation (2.3) in the rest of the paper. Closely associated with the measures of noncompactness is the concept of k-set contraction.

Definition 2.5. *[29] Let $\Gamma : E \to E$ be a continuous mapping of E. Γ is called a k-set contraction if for all $B \subset E$ with B bounded, $\Gamma(B)$ is bounded and $\beta(\Gamma B) \leq k\beta(B), 0 < k < 1$. if*

$$\beta(\Gamma B) < \beta(B), \text{ for all } \beta(B) > 0,$$

then Γ is called densifying or condensing map. A k-set contraction with $k \in (0, 1)$, is densifying, but converse is not true.

Theorem 2.6 ([35], see also [40]). *Assume that $\Gamma : \bar{B}_r \to E$ be a densifying mapping which satisfying the boundary condition,*

$$\text{If } \Gamma(x) = kx, \text{ for some } x \text{ in } \partial B_r \text{ then } k \le 1, \tag{2.4}$$

then the set of fixed points of Γ in $\bar{B}_r$ is nonempty. This is known by Petryshyn's fixed point theorem.

This property allows us to characterize solution of the integral Eq. (1.1) and will be used in the next section.

3. Main results

In this section, we will study the existence of the nonlinear functional Eq. (1.1) for $x \in C[0, a]$ under the following assumptions:

(H1) $x, q \in C(I_a, \mathbb{R}), f, g \in C(I_a \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), F, G \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), u \in C(I_a \times [0, B] \times \mathbb{R}, \mathbb{R}), v \in C(I_a \times I_a \times \mathbb{R}, \mathbb{R})$,
Also,
the functions $\alpha_i, \tau_i, \beta_i, \upsilon_i, \theta_i : I_a \to I_a, i = 1, 2$ and $\varphi : I_a \to R^+$ are continuous such that $\varphi(t) \le B, k = \sup |q(t)|$ for each $t \in I_a$;

(H2) There exist nonnegative constants $c, c', k, k', 2c + 2k, 2c' + 2k' < 1$, such that
$|f(t, x_1, x_2) - f(t, \bar{x}_1, \bar{x}_2)| \le c(|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2|)$;
$|g(t, x_1, x_2) - g(t, \bar{x}_1, \bar{x}_2)| \le c'(|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2|)$;
$|F(t, x_1, x_2, x_3) - F(t, \bar{x}_1, \bar{x}_2, \bar{x}_3)| \le k(|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2| + |x_3 - \bar{x}_3|)$;
$|G(t, x_1, x_2, x_3) - G(t, \bar{x}_1, \bar{x}_2, \bar{x}_3)| \le k'(|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2| + |x_3 - \bar{x}_3|)$;

(H3) (Bounded condition) There exists $r_0 \ge 0$ such that the following bounded condition is satisfied
$\sup\{(k + A_1 + B_1) \times (A_2 + B_2)\} \le r_0$,
where,
$A_1 = \sup\{|f(t, x_1, x_2)| : \text{for all } \ t \in I_a \ , \ \text{and} \ x_1, x_2 \in [-r_0, r_0]\}$.
$B_1 = \sup\{|F(t, x_1, x_2, x_3)| : \text{for all } \ t \in I_a \ , \ \text{and} \ x_1, x_2 \in [-r_0, r_0], -M_1 B \le x_3 \le M_1 B\}$.
$M_1 = \sup\{|u(t, s, x)| : \text{for all } \ t \in I_a, s \in [0, B], \ , \ \text{and} \ x \in [-r_0, r_0]\}$.
$A_2 = \sup\{|g(t, x_1, x_2)| : \text{for all } \ t \in I_a \ , \ \text{and} \ x_1, x_2 \in [-r_0, r_0]\}$.
$B_2 = \sup\{|G(t, x_1, x_2, x_3)| : \text{for all } \ t \in I_a \ , \ \text{and} \ x_1, x_2 \in [-r_0, r_0], -M_2 a \le x_3 \le M_2 a\}$.
$M_2 = \sup\{|v(t, s, x)| : \text{for all } \ t, s \in I_a \ , \ \text{and} \ x \in [-r_0, r_0]\}$.

Theorem 3.1. *Under the hypothesis (H1)-(H3), Eq.* (1.1) *has at least one solution in the Banach space $E = C(I_a)$.*

Proof . To prove this result using Theorem 2.6 as our main tool. Let the operators $P, Q : B_{r_0} \to E$ and Ω are defined on the x such as $\Omega x = (Px) \times (Qx)$, where,

$$Px(t) = \left(q(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) + F\big(t, x(\tau_1(t)), x(\tau_2(t)), \int_0^{\varphi(t)} u(t, s, x(\theta_1(s)))ds\big)\right) \tag{3.1}$$

$$Qx(t) = \left(t, g(x(\beta_1(t)), x(\beta_2(t))) + G\big(t, x(\upsilon_1(t)), x(\upsilon_2(t)), \int_0^a v(t, s, x(\theta_2(s)))ds\big)\right), \tag{3.2}$$

for $t \in I_a$.
Now, we show that the operator P is continuous on the ball B_{r_0}. To do this, consider $\varepsilon > 0$ and take arbitrary $x, y \in B_{r_0}$ such that $\| x - y \| \leq \varepsilon$. Then for $t \in I_a$, we get

$$\begin{aligned}
& |(Px)(t) - (Py)(t)| \\
= & \ |\Big(q(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) + F\big(t, x(\tau_1(t)), x(\tau_2(t)), \int_0^{\varphi(t)} u(t, s, x(\theta_1(s)))ds\big)\Big) \\
- & \ \Big(q(t) + f(t, y(\alpha_1(t)), y(\alpha_2(t))) + F\big(t, y(\tau_1(t)), y(\tau_2(t)), \int_0^{\varphi(t)} u(t, s, y(\theta_1(s)))ds\big)\Big)| \\
\leq & \ c(|x(\alpha_1(t) - y(\alpha_1(t)| + |x(\alpha_2(t) - y(\alpha_2(t)|) \\
+ & \ |F\big(t, x(\tau_1(t)), x(\tau_2(t)), \int_0^{\varphi(t)} u(t, s, x(\theta_1(s)))ds\big) \\
- & \ F\big(t, y(\tau_1(t)), y(\tau_2(t)), \int_0^{\varphi(t)} u(t, s, y(\theta_1(s)))ds\big)| \\
\leq & \ c(|x(\alpha_1(t) - y(\alpha_1(t)| + |x(\alpha_2(t) - y(\alpha_2(t)|) \\
+ & \ k(|x(\tau_1(t) - y(\tau_1(t)| + |x(\tau_2(t) - y(\tau_2(t)|) \\
+ & \ k \int_0^{\varphi(t)} |u(t, s, x(\theta_1(s))) - u(t, s, y(\theta_1(s))))ds| \\
\leq & \ (2c + 2k)\|x - y\| + kB\omega(u, \varepsilon)
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
& |(Qx)(t) - (Qy)(t)| \\
= & \ \Big|\Big(q(t) + g(t, x(\beta_1(t)), x(\beta_2(t))) + G\big(t, x(\upsilon_1(t)), x(\upsilon_2(t)), \int_0^{a} v(t, s, x(\theta_1(s)))ds\big)\Big) \\
- & \ \Big(q(t) + g(t, y(\beta_1(t)), y(\beta_2(t))) + G\big(t, y(\upsilon_1(t)), y(\upsilon_2(t)), \int_0^{a} v(t, s, y(\theta_1(s)))ds\big)\Big)\Big| \\
\leq & \ c'(|x(\beta_1(t) - y(\beta_1(t)| + |x(\beta_2(t) - y(\beta_2(t)|) \\
+ & \ |G\big(t, x(\upsilon_1(t)), x(\upsilon_2(t)), \int_0^{a} v(t, s, x(\theta_1(s)))ds\big) \\
- & \ G\big(t, y(\upsilon_1(t)), y(\upsilon_2(t)), \int_0^{a} v(t, s, y(\theta_1(s)))ds\big)| \\
\leq & \ c'(|x(\beta_1(t) - y(\beta_1(t)| + |x(\beta_2(t) - y(\beta_2(t)|) + k'(|x(\upsilon_1(t) - y(\upsilon_1(t)| \\
+ & \ |x(\upsilon_2(t) - y(\upsilon_2(t)|) + k' \int_0^{a} |v(t, s, x(\theta_1(s))) - v(t, s, y(\theta_1(s))))ds| \\
\leq & \ (2c' + 2k')\|x - y\| + k'B\omega(v, \varepsilon)
\end{aligned}$$

where for $\varepsilon > 0$ we define

$$\omega(u, \varepsilon) = \sup\{|u(t, s, x) - u(t, s, y)| : t \in I_a, s \in [0, B], x, y \in [-r_0, r_0], \|x - y\| \leq \varepsilon\}$$
$$\omega(v, \varepsilon) = \sup\{|v(t, s, x) - v(t, s, y)| : t, s \in I_a, x, y \in [-r_0, r_0], \|x - y\| \leq \varepsilon\}$$

Since we know that $u = u(t, s, x)$ and $v = v(t, s, x)$ are uniformly continuous on the subset $[0, a] \times [0, B] \times \mathbb{R}$ and $[0, a] \times [0, a] \times \mathbb{R}$, respectively, we infer that $\omega(u, \varepsilon) \to 0$ and $\omega(v, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus, the above estimates show that the operator P, Q are continuous on B_{r_0}. Hence, Ω is also continuous on B_{r_0}.
Further, we prove that P and Q satisfy the densifying condition with respect to the measure μ in the ball B_{r_0}. To do this, we choose a fixed arbitrary $\varepsilon > 0$. Let us take $x \in M$ and M is bounded subset of E, $t_1, t_2 \in I_a$ such that without loss of generality we may assume that $\varphi(t_1) \leq \varphi(t_2)$ with $t_2 - t_1 \leq \varepsilon$, we obtain

$$
\begin{aligned}
&|(Px)(t_2) - (Px)(t_1)| \\
&= |\Big(q(t_2) + f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2))) + F\big(t_2, x(\tau_1(t_2)), x(\tau_2(t_2)), \int_0^{\varphi(t_2)} u(t_2, s, x(\theta_1(s)))ds\big)\Big) \\
&- \Big(q(t_1) + f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) + F\big(t_1, x(\tau_1(t_1)), x(\tau_2(t_1)), \int_0^{\varphi(t_1)} u(t_1, s, x(\theta_1(s)))ds\big)\Big) \\
&\leq \omega(q, \varepsilon) + |f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2)) - f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1))| \\
&+ |f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1)) - f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))| \\
&+ |F\big(t_2, x(\tau_1(t_2)), x(\tau_2(t_2)), \int_0^{\varphi(t_2)} u(t_2, s, x(\theta_1(s)))ds\big) \\
&- F\big(t_2, x(\tau_1(t_1)), x(\tau_2(t_1)), \int_0^{\varphi(t_1)} u(t_1, s, x(\theta_1(s)))ds\big)| \\
&+ |F\big(t_2, x(\tau_1(t_1)), x(\tau_2(t_1)), \int_0^{\varphi(t_1)} u(t_1, s, x(\theta_1(s)))ds\big) \\
&- F\big(t_1, x(\tau_1(t_1)), x(\tau_2(t_1)), \int_0^{\varphi(t_1)} u(t_1, s, x(\theta_1(s)))ds\big)| \\
&\leq \omega(q, \varepsilon) + c(|x(\alpha_1(t) - y(\alpha_1(t)| + |x(\alpha_2(t) - y(\alpha_2(t)|) + \omega^1(f, \varepsilon) \\
&+ k(|x(\tau_1(t) - y(\tau_1(t)| + |x(\tau_2(t) - y(\tau_2(t)|) \\
&+ k|\int_0^{\varphi(t_2)} u(t_2, s, x(\theta_1(s)))ds - \int_0^{\varphi(t_1)} u(t_1, s, x(\theta_1(s)))ds| + \omega^1(F, \varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \omega(q, \varepsilon) + c(\omega(x, \omega(\alpha_1, \varepsilon)) + \omega(x, \omega(\alpha_2, \varepsilon))) + \omega^1_{r_0}(f, \varepsilon) + k(\omega(x, \omega(\tau_1, \varepsilon)) + \omega(x, \omega(\tau_2, \varepsilon))) \\
&+ k\int_0^{\varphi(t_1)} |u(t_2, s, x(\theta_1(s))) - u(t_1, s, x(\theta_1(s)))|ds + k\int_{\varphi(t_1)}^{\varphi(t_2)} |u(t_2, s, x(\theta_1(s)))|ds + \omega^1_{r_0}(F, \varepsilon) \\
&\leq \omega(q, \varepsilon) + c(\omega(x, \omega(\alpha_1, \varepsilon)) + \omega(x, \omega(\alpha_2, \varepsilon))) + \omega^1_{r_0}(f, \varepsilon) + k(\omega(x, \omega(\tau_1, \varepsilon)) + \omega(x, \omega(\tau_2, \varepsilon))) \\
&+ kB\omega(u, \varepsilon) + kM_1\omega(\varphi, \varepsilon) + \omega^1_{r_0}(F, \varepsilon)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|(Qx)(t_2) - (Qx)(t_1)| = \\
&\Big|\big(g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2))) + G\big(t_2, x(\upsilon_1(t_2)), x(\upsilon_2(t_2)), \int_0^a v(t_2, s, x(\theta_2(s)))ds\big)\big) \\
- \quad &\big(g(t_1, x(\beta_1(t_1)), x(\beta_2(t_1))) + G\big(t_1, x(\upsilon_1(t_1)), x(\upsilon_2(t_1)), \int_0^a v(t_1, s, x(\theta_1(s)))ds\big)\big)\Big| \\
\leq \quad &c'(\omega(x, \omega(\beta_1, \varepsilon)) + \omega(x, \omega(\beta_2, \varepsilon))) + \omega^1_{r_0}(g, \varepsilon) \\
+ \quad &k'(\omega(x, \omega(\upsilon_1, \varepsilon)) + \omega(x, \omega(\upsilon_2, \varepsilon))) + k'a\omega(u, \varepsilon) + \omega^1_{r_0}(G, \varepsilon)
\end{aligned}
$$

Let:

$$\begin{aligned}
&\omega(q,\varepsilon) = \sup\{|q(t)-q(\bar t)| : |t-\bar t| \le \varepsilon, t, \bar t \in I_a\},\\
&\omega^1_{r_0}(u,\varepsilon) = \sup\{|u(t,s,x)-u(\bar t,s,x)| : |t-\bar t| \le \varepsilon, t \in I_a, s \in [0,B]\ x \in [-r_0,r_0]\},\\
&\omega^1_{r_0}(v,\varepsilon) = \sup\{|v(t,s,x)-v(\bar t,s,x)| : |t-\bar t| \le \varepsilon, t, s \in I_a,\ x \in [-r_0,r_0]\},\\
&\omega^1_{r_0}(f,\varepsilon) = \sup\{|f(t,x_1,x_2)-f(\bar t,x_1,x_2)| : |t-\bar t| \le \varepsilon, t \in I_a,\ x_1,x_2 \in [-r_0,r_0]\},\\
&\omega^1_{r_0}(g,\varepsilon) = \sup\{|g(t,x_1,x_2)-g(\bar t,x_1,x_2)| : |t-\bar t| \le \varepsilon, t \in I_a,\ x_1,x_2 \in [-r_0,r_0],\\
&\omega^1_{r_0}(\varphi,\varepsilon) = \sup\{|\varphi(t)-\varphi(\bar t)| : |t-\bar t| \le \varepsilon, t, \bar t \in I_a,\\
&\omega^1_{r_0}(F,\varepsilon) = \sup\{F(t,x_1,x_2,x_3)-F(\bar t,x_1,x_2,x_3)| : |t-\bar t| \le \varepsilon, t \in I_a,\ x_1,x_2 \in [-r_0,r_0],\\
&\quad -M_1B \le x_3 \le M_1B\}\\
&\omega^1_{r_0}(G,\varepsilon) = \sup\{G(t,x_1,x_2,x_3)-G(\bar t,x_1,x_2,x_3)| : |t-\bar t| \le \varepsilon, t \in I_a,\ x_1,x_2 \in [-r_0,r_0],\\
&\quad -M_2a \le x_3 \le M_2a\}
\end{aligned}$$

Then using above relation we obtain the estimate

$$\begin{aligned}
\omega(Px,\varepsilon) \le{}& \omega(q,\varepsilon) + c(\omega(x,\omega(\alpha_1,\varepsilon)) + \omega(x,\omega(\alpha_2,\varepsilon))) + \omega^1_{r_0}(f,\varepsilon) + k(\omega(x,\omega(\tau_1,\varepsilon))\\
&+ \omega(x,\omega(\tau_2,\varepsilon))) + kB\omega(u,\varepsilon) + kM_1\omega(\varphi,\varepsilon) + \omega^1_{r_0}(F,\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
\omega(Qx,\varepsilon) \le{}& c'(\omega(x,\omega(\beta_1,\varepsilon)) + \omega(x,\omega(\beta_2,\varepsilon))) + \omega^1_{r_0}(g,\varepsilon) + k'(\omega(x,\omega(\upsilon_1,\varepsilon))\\
&+ \omega(x,\omega(\upsilon_2,\varepsilon))) + k'a\omega(u,\varepsilon) + \omega^1_{r_0}(G,\varepsilon)
\end{aligned}$$

Taking limit as $\varepsilon \to 0$, we have

$$\mu(PM) \le (2c+2k)\mu(M).$$

Also,

$$\mu(QM) \le (2c'+2k')\mu(M).$$

This means Ω is a densifying map. Finally, investigation of condition (2.4) is remained. Suppose $x \in \partial \bar B_{r_0}$. If $\Gamma x = kx$ then we have $kr_0 = k\|x\| = \|\Omega x\|$ and by condition (H3) we concluded that

$$\begin{aligned}
|\Omega x(t)| = \bigg|&\Big(q(t) + f(t,x(\alpha_1(t)),x(\alpha_2(t))) + F\big(t,x(\tau_1(t)),x(\tau_2(t)),\int_0^{\varphi(t)} u(t,s,x(\theta_1(s)))ds\big)\Big)\\
&\times\Big(t, g(x(\beta_1(t)),x(\beta_2(t))) + G\big(t,x(\upsilon_1(t)),x(\upsilon_2(t)),\int_0^a v(t,s,x(\theta_2(s)))ds\big)\Big)\bigg| \le r_0,
\end{aligned}$$

for all $t \in I_a$, hence $\|\Omega x\| \le r_0$, so this shows $k \le 1$. The proof is complete. □

Corollary 3.2. *[27] Assume that*

(C1) $q \in C(I_a,\mathbb{R})$ *with* $q = \sup|q(t)| < \infty, t \in I_a$.
(C2) $f,g \in C([0,a]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $F,G \in C([0;a]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.
(C3) *There exists the continuous functions* $a_j : [0,a] \to [0,a]$, *for j = 1, 2,...,10 such that*
$|f(t,x_1,x_2) - f(t,\bar x_1,\bar x_2)| \le a_1(t)|x_1-\bar x_1| + a_2(t)|x_2-\bar x_2|$;
$|g(t,x_1,x_2) - g(t,\bar x_1,\bar x_2)| \le a_3(t)|x_1-\bar x_1| + a_4(t)|x_2-\bar x_2|$;
$|F(t,x_1,z_1,x_2) - F(t,\bar x_1,\bar z_1,\bar x_2)| \le a_5(t)|x_1-\bar x_1| + a_6(t)|z_1-\bar z_1| + a_7(t)|x_2-\bar x_2|$;
$|G(t,x_1,z_1,x_2) - G(t,\bar x_1,\bar z_1,\bar x_2)| \le a_8(t)|x_1-\bar x_1| + a_9(t)|z_1-\bar z_1| + a_{10}(t)|x_2-\bar x_2|$;
for all $t \in I_a$ *and* $x_1,\bar x_1,x_2,\bar x_2,\bar x_2,z_1,\bar z_1 \in \mathbb{R}$.

(C4) *The functions* $u = u(t,s,x(\theta_1(s)))$ *and* $v = v(t,s,x(\theta_2(s)))$ *are continuously from the set* $[0,a]\times[0,a]\times\mathbb{R}$ *into* $\mathbb{R}$*. Moreover, the functions* $\alpha_2, \tau_2, \beta_2, \upsilon_2, \theta_1$ *and* θ_2 *transform continuously the interval* $[0,a]$ *into itself.*

(C5) *There exists a nonnegative constant* K *such that* $K = \max\{a_j(t)|t \in [0,a]\}$ *for* $j = 1,2,...,10$.

(C6) *(Sublinear condition) There exist the constants* ξ *and* η *such that:*
$|u(t,s,x)| \le \xi + \eta|x|$,
$|v(t,s,x)| \le \xi + \eta|x|$
for all $t,s \in [0,a]$ *and* $x \in \mathbb{R}$,

(C7) *there exist nonnegative constants* l,m *such that*
$|g(t,0,0)| \le l$,
$|g(t,0,0)| \le l$,
$|F(t,0,0,0)| \le m$,
$|G(t,0,0,0)| \le m$,
for all $t \in [0,a]$.

(C8) $4bc < 1, a\eta > 1, b = 4K + Ka\eta, c = k + l + Ka\xi + m.$

Then the equation

$$x(t) = \Big(q(t) + f(t,x(t),x(\alpha_2(t))) + F\big(t,x(t),x(\tau_2(t)),\int_0^t u(t,s,x(\theta_1(s)))ds\big)\Big)$$
$$\times\Big(g(t,x(t),x(\beta_2(t))) + G\big(t,x(t),x(\upsilon_2(t)),\int_0^a v(t,s,x(\theta_2(s)))ds\big)\Big),\ t \in I_a = [0,a]. \tag{3.3}$$

has at least one solution in the Banach space $E = C(I_a)$.

Proof . Setting $\alpha_1(t) = \tau_1(t) = \beta_1(t) = \upsilon_1(t) = \varphi(t) = t$, Eq. (1.1) is reduces to the Eq. (3.3). It is check that (H 2) is conducted by (C3). Now we prove that (H3) is also holds. Setting $M_1 = \xi + \eta r_0, M_2 = \xi + \eta r_0$, then we get

$$\begin{aligned}
|x(t)| = |&\Big(q(t) + f(t,x(t),x(\alpha_2(t))) + F\big(t,x(t)),x(\tau_2(t)),\int_0^t u(t,s,x(\theta_1(s)))ds\big)\Big)\\
&\times\Big(g(t,x(t),x(\beta_2(t))) + G\big(t,x((t)),x(\upsilon_2(t)),\int_0^a v(t,s,x(\theta_2(s)))ds\big)\Big)\\
&\le \Big(k + f(t,x(t),x(\alpha_2(t))) - f(t,0,0)| + |f(t,0,0)|\\
&+ F\big(t,x(t)),x(\tau_2(t)),\int_0^t u(t,s,x(\theta_1(s)))ds\big) - F(t,0,0,0)| + |F(t,0,0,0)|\Big)\\
&\times\Big(g(t,x(t),x(\beta_2(t))) - g(t,0,0)| + |g(t,0,0)|\\
&+ G\big(t,x(t)),x(\upsilon_2(t)),\int_0^a v(t,s,x(\theta_2(s)))ds\big) - G(t,0,0,0)| + |G(t,0,0,0)|\Big)\\
&\le \Big(k + a_1(t)|x(t)| + a_2(t)|x(\alpha_2(t))| + l + a_3(t)|x(t)|\\
&+ a_6(t)\int_0^t u(t,s,x(\theta_1(s)))ds\big) + a_7(t)|x(\tau_2(t))| + m\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(a_3(t)|x(t)| + a_4(t)|x(\beta_2(t))| + l + a_8(t)|x(t)| \\
&+ a_9(t) \int_0^a u(t,s,x(\theta_2(s)))ds\Big) + a_{10}(t)|x(\upsilon_2(t))| + m \Big) \\
&\le \big(k + 4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\big).\big(4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\big) \\
&\le \big((4K + Ka\eta)\|x\| + k + l + Ka\xi + m\big)^2 \\
&\le \big(b\|x\| + c\big)^2
\end{aligned}
$$

for all $t \in I_a$. Hence, r_0 in (H3) is real number that satisfies

$$
\sup_{t\in I_a} |x(t)| \le \big(br_0 + c\big)^2 \le r_0. \tag{3.4}
$$

The inequality (3.4), has a solution in $[r_1, r_2]$, where

$$
\begin{aligned}
r_1 &= \frac{1 - 2bc - \sqrt{1-4bc}}{2b^2}, \\
r_2 &= \frac{1 - 2bc + \sqrt{1-4bc}}{2b^2}.
\end{aligned}
$$

Under the assumption (C8), we know that $1 - \sqrt{1-4bc} < 1$, so $r_0 = r_1$ is a positive real number. Now, the desired result obtained from Theorem 3.1. □

Corollary 3.3. *[32] Assume that*

(K1) $F, G \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and There exist positive constants* k *and* k' *such that*
$|F(t,x_1,x) - F(t,x_2,x)| \le k|x_1 - x_2|$,
$|G(t,x_1,x) - G(t,x_2,x)| \le k|x_1 - x_2|$,
$|F(t,x,x_1) - F(t,x,x_2)| \le k'|x_1 - x_2|$,
$|G(t,x,x_1) - G(t,x,x_2)| \le k'|x_1 - x_2|$,
for all $x, x_1, x_2, x, x_1, x_2 \in \mathbb{R}, t \in I = [0,1]$,

(K2) $u, v \in C(I \times I \times \mathbb{R}, \mathbb{R})$ *and there exist nonnegative constants* $\alpha_i, \beta_i, p_i; (i = 1,2)$ *such that*
$|u(t,s,x)| \le \alpha_1 + \beta_1|x|^{p_1}, |v(t,s,x)| \le \alpha_2 + \beta_2|x|^{p_2}$ *for all* $t, s \in I$ *and* $x \in \mathbb{R}$,

(K3) $\varphi, \tau_1, \upsilon_j, \theta_j \in C(I, I)$ *for* $j = 1,2$,

(K4) $(k\alpha_1 + m_1)m_2 > 0$, *where* m_1 *and* m_2 *are the constants such that*
$|F(t,0,0)| \le m_1, |G(t,0,0)| \le m_2$ *for all* $t \in I$,

(K5) $[k(\alpha_1 + \beta_1) + m_1 + k'][k(\alpha_2 + \beta_2) + m_2 + k'] < 1$,

(K6) $k'[(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + m_1 + m_2 + 2k'] + kM[k(\alpha_1 + \beta_1) + m_1 + k'] < 1$,

where M *is the nonnegative constant such that* $|v(t,s,x)| \le M$ *for all* $t, s \in I$ *and* $x \in [-1,1]$.
Then the equation

$$
x(t) = F\left(t, \int_0^{\varphi(t)} u(t,s,x(\theta_1(t))ds, x(\tau_1(t))\right) \times G\left(t, x(\upsilon_1(t)) \int_0^1 v(t,s,x(\theta_2(t))ds, x(\upsilon_2(t)))\right), t \in [0,1], \tag{3.5}
$$

has at least one solution in the Banach space $E = C(I_a)$.

Proof . It can be verified that if $q(t) = f(t,x_1,x_2) = g(t,x_1,x_2) = 0, F(t,x_1,x_2,x_3) = F(t,x_1,x_3)$ and $G(t,x_1,x_2,x_3) = G(t,x_1x_3,x_2)$, then Eq. (1.1) is reduces to the Eq. (3.5) for a=1.

It is easy to cheek that (H2) is concluded by (K1). Now we show that (H3) is also holds. Suppose that $\|x\| \leq r_0, r_0 > 0$ and setting $M_1 = \alpha_1 + \beta_1 r_0^{p_1}, M_2 = \alpha_2 + \beta_2 r_0^{p_2}$, then we have

$$\begin{aligned}
|x(t)| &= \left| F\Big(t, \int_0^{\varphi(t)} u(t,s,x(\theta_1(t))ds, x(\tau_1(t))\Big) \times G\Big(t, x(\upsilon_1(t)) \int_0^1 v(t,s,x(\theta_2(t))ds, x(\upsilon_2(t)))\Big) \right| \\
&\leq \Big(|F\big(t, \int_0^{\varphi(t)} u(t,s,x(\theta_1(t))ds, x(\tau_1(t))\big) - F(t,0,x(\tau_1(t)))| + |F(t,0,x(\tau_1(t))| \Big) \\
&\times \Big(|G\big(t, x(\upsilon_1(t)) \int_0^1 v(t,s,x(\theta_2(t))ds, x(\upsilon_2(t)))\big) - G(t,0,x(\upsilon_2(t)))| + |G(t,0,x(\upsilon_2(t))| \Big) \\
&\leq \big(k \int_0^{\varphi(t)} |v(t,s,x(\gamma_1(s)))|ds + m_1 + k'|x(\alpha(t))|\big) \\
&\times \big(k \int_0^1 |u(t,s,x(\gamma_1(s)))|ds + m_2 + k'|x(\beta(t))|\big) \\
&\leq \big(k(\alpha_1+\beta_1)\|x(t)\|^{p_1} + m_1 + k'\|x\|\big)\big(k\|x\|(\alpha_2+\beta_2)\|x(t)\|^{p_2} + m_2 + k'\|x\|\big)
\end{aligned}$$

Hence, r_0 in (H3) is real number that satisfies

$$\big(k(\alpha_1+\beta_1)r_0^{p_1} + m_1 + k'r_0\big)\big(kr_0(\alpha_2+\beta_2)r_0^{p_2} + m_2 + k'r_0\big) \leq r_0$$

Similar argument as in the first paragraph of the proof of [32, Theorem 3.1] shows that this inequality has a solution in $(0,1)$. The proof is complete. ☐

Remark 3.4. *Like the similar argument as the above two corollaries, one can easily prove that Theorem 2 of [25], Theorem 5 of [31], Theorem 3 of [16],Theorem 3 of [24], Theorem 3.1 of [10], Theorem 3 of [34], Theorem 3 of [23], Theorem 3.2 of [26] and Theorem 2 of [8] can be obtained from Theorem 3.1.*

4. Applications

In this section, we give some examples of classical integral and functional equations considered in the applied problems of nonlinear analysis which are particular cases of equation (1.1).

- If $q(t) = g(t,x_1,x_2) = 0, f(t,x_1,x_2) = f_1(t,x_1), \alpha_1(t) = \varphi(t) = t, F(t,x_1,x_2,x_3) = p(t,x_1,x_3)$, $G(t,x_1,x_2,x_3) = q(t,x_1,x_3)$, then equation (1.1) is in the following form which was studied in [16].

$$x(t) = \Big(f_1(t,x(t)) + p\big(t, x(\tau_1(t)), \int_0^t u(t,s,x(\theta_1(s)))ds\big)\Big) \times \Big(q\big(t, x(\upsilon_1(t)), \int_0^a v(t,s,x(\theta_2(s)))ds\big)\Big).$$

- For $q(t) = f(t,x_1,x_2) = g(t,x_1,x_2) = 0, \theta_1(s) = \theta_2(s) = s, \varphi(t) = t, F(t,x_1,x_2,x_3) = p(t,x_1,x_3)$, $G(t,x_1,x_2,x_3) = q(t,x_1,x_3)$, we obtain the following nonlinear functional-integral equation studied in [23, 8].

$$x(t) = \Big(p\big(t, x(\tau_1(t)), \int_0^t u(t,s,x(s)ds\big)\Big) \times \Big(q\big(t, x(\upsilon_1(t)), \int_0^a v(t,s,x(s))ds\big)\Big).$$

- $q(t) = f(t,x_1,x_2) = g(t,x_1,x_2) = 0, a = 1, F(t,x_1,x_2,x_3) = p(t,x_1,x_3), G(t,x_1,x_2,x_3) = q(t,x_1,x_3)$, then we get the following functional-integral equation studied in [32].

$$x(t) = \Big(p\big(t, x(\tau_1(t)), \int_0^{\varphi(t)} u(t,s,x(\theta_1(s)))ds\big)\Big) \times \Big(q\big(t, x(\upsilon_1(t)), \int_0^1 v(t,s,x(\theta_2(s)))ds\big)\Big).$$

- If $q(t) = f(t, x_1, x_2) = g(t, x_1, x_2) = 0, \varphi(t) = t, F(t, x_1, x_2, x_3) = p(t, x_1, x_3)$, $G(t, x_1, x_2, x_3) = q(t, x_1, x_3)$, then equation (1.1) has the following form as in the paper [10].

$$x(t) = \Big(p\big(t, x(\tau_1(t)), \int_0^t u(t, s, x(\theta_1(s)))ds\big)\Big) \times \Big(q\big(t, x(\upsilon_1(t)), \int_0^a v(t, s, x(\theta_2(s)))ds\big)\Big).$$

- If $q(t) = g(t, x_1, x_2) = 0, f(t, x_1, x_2) = f_1(t, x_1), \alpha_1(t) = \varphi(t) = \theta_1(t) = t, F(t, x_1, x_2, x_3) = p(t, x_1, x_3)$,
$G(t, x_1, x_2, x_3) = 1$, then equation (1.1) has the following form as in the paper [25].

$$x(t) = f_1(t, x(t)) + p\big(t, x(\tau_1(t)), \int_0^t u(t, s, x(s))ds\big).$$

- If $q(t) = g(t, x_1, x_2) = 0, f(t, x_1, x_2) = f_1(t, x_1), \varphi(t) = t, F(t, x_1, x_2, x_3) = p(t, x_1)x_3$, $G(t, x_1, x_2, x_3) = 1$, then equation (1.1) has the following form as in the paper [31].

$$x(t) = f_1(t, x(\alpha_1(t))) + p\big(t, x(\tau_1(t))\big) \int_0^{\varphi(t)} u(t, s, x(\theta_1(s))ds.$$

- Moreover, if $q(t) = f(t, x_1, x_2) = 0, g(t, x_1, x_2) = 1, \upsilon_1(t) = \theta_2(t) = t, F(t, x_1, x_2, x_3) = 1$, $G(t, x_1, x_2, x_3) = 1 + x_1x_3$, and $v(t, s, x) = \frac{t\phi(s)x}{t+s}$, then equation (1.1) has the following form

$$x(t) = 1 + x(t) \int_0^a \frac{t}{t+s}\phi(s)x(s)ds.$$

  The above equation is the famous quadratic integral equation of Chandrasekhar type [11] which is applied in the theories of radiative transfer, neutron transport and kinetic energy of gases (see [11, 3, 19, 27, 20]).

Now, we present some examples of functional integral equations to illustrate the usefulness of our results and consequently, see the existence of its solutions by using Theorem 3.1.

Example 4.1. *Consider the following nonlinear Volterra integral equation*

$$\begin{aligned} x(t) = &\Big(\frac{1}{3}te^{-t} + \frac{tsin(x(\sqrt{t}))}{3(1+t)} + \frac{1}{3(e^t + 3\sin(|x(t^3)|))} \int_0^{t^3} (s\cos(tx(\sqrt{s})) + \frac{3}{2}t\ln(1 + x(\sqrt{s}))ds\Big) \\ &\times \Big(\frac{1}{2(e^{t^2} + |\cos(|x(t^2)|))} \int_0^1 [(\frac{t}{1+t+s})sin(\frac{x(1-s)}{1+x(s-1)}) + \frac{x(s-1)}{2}]ds\Big), \quad t \in [0,1] \end{aligned} \tag{4.1}$$

Eq. (4.1) is a special case of Eq. (1.1). *Here* $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, F, G : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \alpha_1, \tau_1, \beta_3, \theta, \theta_1, \theta_2 : [0, 1] \to [0, 1], u, v : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ *and comparing* (4.1) *with eq.* (1.1), *we obtain*

$$\alpha_1(t) = \theta_1(t) = \sqrt{t}, \tau_1 = \varphi = t^3, a = 1, \upsilon_1(t) = t^2, \theta_2(t) = 1 - t, \qquad \text{for all} \quad t \in [0, 1],$$

$$q(t)=\frac{1}{3}te^{-t}, f(t,x_1,x_2)=\frac{t}{3(1+t)}\sin(x_1), g=0,$$
$$F(t,x_1,x_2,z)=\frac{z}{3(e^t+3\sin(x_1))},\quad z=\int_0^{t^3}(s\cos(tx(\sqrt{s}))+\frac{3}{2}t\ln(1+x(\sqrt{s}))ds,$$
$$G(t,x_1,x_2,w)=\frac{w}{2(e^{t^2}+|\cos(x_1)|},\quad w=\int_0^1[(\frac{t}{1+t+s})sin(\frac{x(1-s)}{1+x(s-1)})+\frac{x(s-1)}{2}]ds,$$
$$u(t,s,\theta_1(s))=(s\cos(tx(\sqrt{s}))+\frac{3}{2}t\ln(1+x(\sqrt{s})),\quad |u(t,s,x)|\le 1+\frac{3}{2}|x|$$
$$v(t,s,\theta_2(s))=(\frac{t}{1+t+s})sin(\frac{x(1-s)}{1+x(s-1)})+\frac{x(s-1)}{2},\quad |v(t,s,x)|\le\frac{1}{2}+\frac{1}{2}|x|$$

Now, we examine the solution in $C[0,1]$. It is easy to prove that these functions satisfy the assumptions (H1) and (H2). We show that (H3) also holds. Suppose that $\|x\|\le r_0, r_0>0$, then we have

$$|x(t)|=|\left(\frac{1}{3}te^{-t}+\frac{tsin(x(\sqrt{t}))}{3(1+t)}+\frac{1}{3(e^t+3\sin(|x(t^3)|))}\int_0^{t^3}(s\cos(tx(\sqrt{s}))+\frac{3}{2}t\ln(1+x(\sqrt{s}))ds\right)$$
$$\times\left(\frac{1}{2(e^{t^2}+|\cos(|x(t^2)|))}\int_0^1[(\frac{t}{1+t+s})sin(\frac{x(1-s)}{1+x(s-1)})+\frac{x(s-1)}{2}]ds\right)|\le r_0,$$

for all $t\in I_a$. Hence (H3) holds if,

$$(\frac{1}{2}r_0+1)(\frac{1}{2}+\frac{1}{4}r_0)\le r_0.$$

This shows that $r_0=2$. Hence, from Theorem 3.1 equation (4.1) has at least one solution in Banach space $C[0,1]$.

Example 4.2. *Consider the following nonlinear integral equation*

$$x(t)=\left(\frac{t^2}{6+6t^2}\ln(1+|x(t^3)|+\frac{t}{4}\int_0^t(t\sin(x(\sqrt{s}))+\arctan(\frac{|x(\sqrt{s}))|}{1+|x(\sqrt{s})|})ds\right)$$
$$\times\left(\frac{1}{4}\cos(x(1-t))+\frac{1}{3}\int_0^1[e^{-3t^2}(e^t+t\cos(s)+\sin(\frac{x(s)}{1+x(s)}]ds\right),\qquad t\in[0,1]. \tag{4.2}$$

Here,

$$\alpha_1(t)=t^3,\theta_1(t)=\sqrt{t},\varphi=\theta_1(t)=t,a=1,\beta_1(t)=1-t,\qquad \textit{for all}\quad t\in[0,1],$$

$$q(t)=0, f(t,x_1,x_2)=\frac{t^2}{6+6t^2}\ln(1+|x_1|), g(t,x_1,x_2)=\frac{1}{4}\cos(x_1),$$
$$F(t,x_1,x_2,z)=\frac{tz}{4},\quad z=\int_0^t(t\sin(x(\sqrt{s}))+\arctan(\frac{|x(\sqrt{s})|}{1+|x(\sqrt{s})|})ds,$$
$$G(t,x_1,x_2,w)=\frac{tw}{3},\quad w=\int_0^1[e^{-3t^2}(e^t+t\cos(s)+\sin(\frac{x(s)}{1+x(s)}]ds,$$
$$u(t,s,\theta_1(s))=t\sin(x(\sqrt{s}))+\arctan(\frac{|x(\sqrt{s})|}{1+|x(\sqrt{s}|}),\quad |u(t,s,x)|\le 1+|x|$$
$$v(t,s,\theta_2(s))=e^{-3t^2}(e^t+t\cos(s)+\sin(\frac{x(s)}{1+x(s)}),\quad |v(t,s,x)|\le e+2$$

for all $t \in [0,1]$.
Now, we can see that these functions satisfy the assumptions (H1) and (H2). We check that (H3) also holds. Suppose that $\|x\| \leq r_0, r_0 > 0$, *then we have*

$$|x(t)| = |\left(\frac{t^2}{6+6t^2}\ln(1+|x(t^3)|+\frac{t}{4}\int_0^t (t\sin(x(\sqrt{s}))+\arctan(\frac{|x(\sqrt{s})|}{1+|x(\sqrt{s})|})ds\right)$$
$$\times\left(\frac{1}{4}\cos(x(1-t))+\frac{1}{3}\int_0^1 [e^{-3t^2}(e^t+t\cos(s)+\sin(\frac{x(s)}{1+x(s)}]ds\right)| \leq r_0,$$

for all $t \in [0,1]$. *Hence (H3) holds if,*

$$(\frac{1}{6}+\frac{1}{4}(1+r_0))(\frac{1}{4}+\frac{1}{3}(e+2)) \leq r_0.$$

. *Hence, (H3) holds if* $r_0 \geq 1.8946$. *Hence, from Theorem 3.1 equation* (4.3) *has at least one solution in Banach space* $[0,1]$.

Example 4.3. *Consider the following nonlinear integral equation*

$$x(t) = \left(\frac{e^t}{2+t}\sin(x(t))+\frac{1}{2+t^2}\int_0^{\sqrt{t}}\frac{(\sqrt{1+|x(\sqrt{s})|}+ts)(1+cos(s))}{4+s^2}ds\right)$$
$$\times\left(e^{-t}+\frac{1}{5+t^3}\int_0^1\frac{(\sin(\sqrt{t}))(\sqrt{1+|x(\sqrt{s})|})}{1+s+ln(1+s)}ds\right), \qquad t \in [0,1]. \tag{4.3}$$

Here,

$$\alpha_1(t) = t, \theta_1(t) = \sqrt{t}, \varphi = \theta_2(t) = \sqrt{t}, a = 1, \qquad \textit{for all} \quad t \in [0,1],$$

$$q(t) = 0, f(t,x_1,x_2) = \frac{e^t}{2+t}\sin(x_1), g(t,x_1,x_2) = e^{-t},$$
$$F(t,x_1,x_2,z) = \frac{z}{2+t^2}, \quad z = \int_0^{\sqrt{t}}\frac{(\sqrt{1+|x(\sqrt{s})|}+ts)(1+cos(s))}{4+s^2}ds,$$
$$G(t,x_1,x_2,w) = \frac{w}{5+t^3}, \quad w = \int_0^1\frac{(\sin(\sqrt{t}))(\sqrt{1+|x(\sqrt{s})|})}{1+s+ln(1+s)}ds,$$
$$u(t,s,\theta_1(s)) = \frac{(\sqrt{1+|x(\sqrt{s})|}+ts)(1+cos(s))}{4+s^2}, \quad |u(t,s,x)| \leq \frac{1}{2}\sqrt{1+|x|}$$
$$v(t,s,\theta_2(s)) = \frac{(\sin(\sqrt{t}))(\sqrt{1+|x(\sqrt{s})|})}{1+s+ln(1+s)}, \quad |v(t,s,x)| \leq \sqrt{1+|x|}$$

for all $t \in [0,1]$.
Now, we can see that these functions satisfy the assumptions (H1) and (H2). We check that (H3) also holds. Suppose that $\|x\| \leq r_0, r_0 > 0$, *then we have*

$$|x(t)| = |\left(\frac{e^t}{2+t}\sin(x(t))+\frac{1}{2+t^2}\int_0^{\sqrt{t}}\frac{(\sqrt{1+|x(\sqrt{s})|}+ts)(1+cos(s))}{4+s^2}ds\right)$$
$$\times\left(e^{-t}+\frac{1}{5+t^3}\int_0^1\frac{(\sin(\sqrt{t}))(\sqrt{1+|x(\sqrt{s})|})}{1+s+ln(1+s)}ds\right)ds| \leq r_0,$$

for all $t \in I_a$. Hence (H3) holds if,

$$(\frac{1}{2}+\frac{1}{4}\sqrt{1+r_0})(1+\frac{1}{5}\sqrt{1+r_0}) \leq r_0.$$

This shows that $r_0 = 1.1147$. Hence, from Theorem 3.1 equation (4.3) has at least one solution in Banach space $[0,1]$. Since there is no constants $\alpha_1, \beta_1, \alpha_2$ and β_2 satisfying the inequalities (Sublinear condition)

$$|u(t,s,x)| \leq \alpha_1 + \beta_1|x|,$$
$$|v(t,s,x)| \leq \alpha_2 + \beta_2|x|$$

for all $t, s \in I_a$ and $x \in \mathbb{R}$, the results in [25], [27], [10] and [26] are inapplicable to the integral equation (4.3).

5. Conclusion and Perspective

In this paper, we have discussed about the existence of the solutions of nonlinear functional-integral equations in Banach algebra by using a strategy which is different from other authors approach [2, 13, 25, 31, 27, 32, 33, 16, 4, 24, 14, 15, 10, 34, 23]. The advantage of Theorem 2.6 among the others (Darbo and Schauder fixed point theorems) lies in that in applying the theorem, one does not need to verify the involved operator maps a closed convex subset onto itself. Also in future, the researchers can acheive solvability of infinite systems of the Eq. (1.1) and the existence of solution of implicit fractional integral equations or implicit fractional differential equations using Petryshyn's Fixed point theorem with numerical methods in different function spaces. Further, condition (2.4) deals with the eigenvalue of nonlinear operator Γ which the author hope that this can be constitute to further study in this field of research.

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