Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 467–484
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.21181.2240

A numerical scheme for solving variable order Caputo–Prabhakar fractional integro–differential equation

B. Bagherzadeh Tavasani[a], A.H. Refahi Sheikhani[a,*], H. Aminikhah[b]

[a]*Department of Applied Mathematics, Faculty of Mathematical Sciences, Lahijan Branch, Islamic Azad University, Lahijan, Iran.*
[b]*Department of Applied Mathematics and Computer Science, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht 41938, Iran*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we apply the Chebyshev polynomials for the numerical solution of variable-order fractional integro–differential equations with initial conditions. Moreover, a class of variable-order fractional integro–differential equations with a fractional derivative of Caputo–Prabhakar sense is considered. The main aim of the Chebyshev polynomials is to derive four kinds of operational matrices of these polynomials. With such operational matrices, an equation is transformed into the products of several dependent matrices, which can also be viewed as the system of linear equations after dispersing the variables. Finally, numerical examples have been presented to demonstrate the accuracy of the proposed method, and the results have been compared with the exact solution.

Keywords: Variable order fractional; Prabhakar fractional derivative; Chebyshev polynomials; Numerical method; Operational matrices.
2010 MSC: Primary 26A33, 34A08; Secondary 65N35.

1. Introduction

Fractional differential equations have profound physical background and rich theory and are particularly noticeable in recent years. They are equations containing fractional derivative or fractional

*Corresponding author
Email addresses: Bgh.Bagharzadeh@gmail.com (B. Bagherzadeh Tavasani), ah_refahi@liau.ac.ir (A.H. Refahi Sheikhani), hossein.aminikhah@gmail.com (H. Aminikhah)

Received: August 2020 *Accepted:* December 2020

integrals, which have received great interest across disciplines such as physics, biology, and chemistry. More specifically, they are widely used in dynamical systems with chaotic dynamical behavior, quasi-chaotic dynamical systems, the dynamics of complex material or porous media, and random walks with memory [12]. In this paper, we investigate approximate solutions of the following fractional integro–differential equation using a numerical method based on shifted Chebyshev polynomials:

$$ {}^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big] + \frac{\partial z(x,t)}{\partial t} = r(x,t) - \int_0^t z(x,Y).k(x,Y)dY - \int_0^t z(x,Y)dY, \tag{1.1} $$

$$ z(x,0) = w(x),\ x \in [0,1],\ z(0,t) = v(t),\ t \in [0,1], \tag{1.2} $$

where in above symbol ${}^{CP}\mathfrak{D}_{\mu(t)}$ is named Caputo-Prabhakar fractional derivative of order $\mu(t) \in (0,1]$ and $\mu(t)$ is a continuous function. The history of this type of derivative is considered in this article goes back to the reference [30, 6] that it as an extension of Riemann-Liouville and Caputo derivatives is expressed and this type of the ${}^{CP}\mathfrak{D}_{\mu(t)}$ on function $f(t)$ for $m = 1$ is defined by:

$$ {}^{CP}\mathfrak{D}_{\mu(t)} f(t) = {}^{IP}\mathfrak{E}^{-\gamma}_{1-\mu(t)} \frac{d}{dt} f(t), \tag{1.3} $$

where ${}^{IP}\mathfrak{E}^{-\gamma}_{m-\mu(t)}$ is the Prabhakar fractional integral of order $1-\mu(t)$ and it is defined by:

$$ {}^{IP}\mathfrak{E}^{-\gamma}_{1-\mu(t)} f(t) = \int_0^t (t-\varrho)^{-\mu(t)} E^{-\gamma}_{\rho,\mu(t)}(\omega(t-\varrho)^{\rho}) f(\varrho) d\varrho, \tag{1.4} $$

and in the relation (1.4), $E^{\gamma}_{\rho,\mu(t)}(\omega t^{\rho})$ is as a generalization of one-parameter Mittag-Leffler and two-parameter Mittag-Leffler functions and it called Prabhakar generalized Mittag function which is given by[6]:

$$ E^{\gamma}_{\rho,\mu(t)}(z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n + \mu(t))} z^n,\ \mu(t), \gamma, \rho \in \mathbb{C},\ \Re(\rho) > 0. \tag{1.5} $$

Also in equation (1.1), the functions $z(x,t)$, $r(x,t)$, $w(x,t)$, $k(x,t)$ according to time casual functions are considered that $r(x,t)$, $w(x,t)$, $k(x,t)$ are determined and $z(x,t)$ is indeterminated. Due of the abundant application of the Prabhakar generalized Mittag function in fractional calculus a reason was to choose this kind of the Caputo-Prabhakar fractional derivative of order $\mu(t)$. Applications of the three-parameter Mittag-Leffler function can be found in mathematical fields as physics and stochastic processes, electromagnetic, viscosity, various materials, and different media[27, 16, 18, 15, 20, 31, 33]. Recently, the Prabhakar fractional derivative with three-parameter Mittag-Leffler function kernel has attracted increasing attention in the real-world problems, with a growing number of applications in sciences. For example, in Garra et al. [6], Kilbas et al. [17] and Prabhakar [30], authors developed the fractional Riemann–Liouville (or Caputo) derivative and integral to the Prabhakar fractional derivative and integral containing the three-parameter Mittag-Leffler function in their kernels. This form of fractional integral and derivative can suitably explain anomalous relaxation of Havriliak-Negami models in the scope of dielectric materials [11, 9, 19, 8, 28], the corresponding applications in the time-evolution of polarization processes [6, 8, 13], the fractional Poisson process [6], the fractional Maxwell model in linear viscoelasticity [10], the generalized reaction-diffusion equations [1].

Getting approximate solutions to the equation (1.1) which is called fractional integro–differential equation of variable order is not easy, so in this article, a numerical method for finding the numerical solutions of this type of equation is presented. Some authors have been paid to solve integro–differential equations involving fractional derivatives using numerical methods. For example, in [26] was studied a numerical algorithm base on the variational iteration, the numerical method base on Adomian decomposition algoritm[5, 14], the generalized differential transform algorithm[24], the wavelet algorithm[3], the finite difference algorithm[38], a numerical algorithm base on the collocation method[39] and implicit RBF Meshless method for obtaining solutions of two-dimensional fractional cable equation of variable order [23] and other methods [32, 20, 21, 22, 2, 25] must be used. In this paper, we expressing a fractional integro–differential in terms of a generalized derivative of order $\mu(t)$ and using a numerical method based on matrix operator that this operator is made of the shifted Chebyshev polynomials to solve the equation.

For this aim, the following paper structure is composed as follows: in section 2 we introduce some lemmas which are applied in the next section. In section 3, first, we introduce a Chebyshev polynomial of degree n and then using the Chebyshev polynomials to make shifted Chebyshev polynomials and in this section, we get the approximation function to find the solutions of the proposed equation. In section 4, applied the approximate function in section 3 to obtain numerical solutions of the integro–differential equation (1.1). In section 5, we show two examples for the performance and accuracy of the proposed method in this paper.

2. Some properties of Caputo–Prabhakar fractional derivative

This section describes Lemmas which are used for the next section.

Lemma 2.1. *[4]. Let $\nu(t) \in (0, 1)$ and $k > 0$. Then*

$$\mathbb{I}_t^{\nu(t)} t^k = \frac{t^{k+\nu(t)}\Gamma(k+1)}{\Gamma(k+1+\nu(t))}, \tag{2.1}$$

where $\mathbb{I}_t^{\nu(t)}$ is the Riemann–Liouville fractional integral of order $\nu(t)$ which is defined in [4].

Lemma 2.2. *Let $\rho, \gamma, \mu(t), \varsigma, \omega \in \mathbb{C}$. Then for any $\Re(\rho), \Re(\mu(t)), \Re(\varsigma) > 0$ the following relation is hold:*

$$\int_0^t (t-u)^{\mu(t)-1} E^{\gamma}_{\rho,\mu(t)}(\omega(t-u)^{\rho}) u^{\varsigma-1} du = \Gamma(\varsigma) t^{\mu(t)+\varsigma-1} E^{\gamma}_{\rho,\mu(t)+\varsigma}(\omega t^{\rho}). \tag{2.2}$$

Proof . The use of (1.5), we obtain:

$$\begin{aligned}\int_0^t (t-\tau)^{\mu(t)-1} & E^{\gamma}_{\rho,\mu(t)}\Big(\omega(t-\tau)^{\rho}\Big)\tau^{\varsigma-1} d\tau \\ &= \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t))} \int_0^t (t-\tau)^{\rho k+\mu(t)-1} \tau^{\varsigma-1} d\tau. \end{aligned} \tag{2.3}$$

Now, employing $\int_0^t (t-\tau)^{\rho k+\mu(t)-1}\tau^{\varsigma-1}d\tau = \Gamma(\rho k+\mu(t))\Big(\mathbb{I}_t^{\rho k+\mu(t)}t^{\varsigma-1}\Big)$, we have:

$$\begin{aligned}
&\int_0^t (t-\tau)^{\mu(t)-1}E^{\gamma}_{\rho,\mu(t)}\Big(\omega(t-\tau)^{\rho}\Big)\tau^{\varsigma-1}d\tau \\
&= \frac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t))}\Big(\Gamma(\rho k+\mu(t))\Big(\mathbb{I}_t^{\rho k+\mu(t)}t^{\varsigma-1}\Big)\Big) \\
&= \frac{\Gamma(\varsigma)}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t)+\varsigma)}t^{\varsigma+\rho k+\mu(t)-1} \\
&= \Gamma(\varsigma)t^{\varsigma+\mu(t)-1}\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\omega^k t^{\rho k}}{k!\Gamma(\rho k+\mu(t)+\varsigma)} \\
&= \Gamma(\varsigma)t^{\varsigma+\mu(t)-1}E^{\gamma}_{\rho,\mu(t)+\varsigma}(\omega t^{\rho}). \qquad (2.4)
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *For any $\Re(\rho), \Re(\mu(t)) > 0$ the following relation is hold:*

$$^{CP}\mathfrak{D}_{\mu(t)}\Big(x^{\zeta-1}\Big)(t) = \Gamma(\zeta)t^{\zeta-\mu(t)-1}E^{-\gamma}_{\rho,\zeta-\mu(t)}(\omega t^{\rho}), \zeta > 1. \qquad (2.5)$$

Proof . Using (1.3) and (1.4), we get

$$\begin{aligned}
^{CP}\mathfrak{D}_{\mu(t)}\Big(x^{\zeta-1}\Big)(t) &= {}^{IP}\mathfrak{E}^{-\gamma}_{1-\mu(t)}\frac{d}{dt}(t^{\zeta-1}) \\
&= (\zeta-1)\int_0^t (t-\varrho)^{-\mu(t)}E^{-\gamma}_{\rho,\mu(t)}(\omega(t-\varrho)^{\rho})\varrho^{\zeta-2}d\varrho, \qquad (2.6)
\end{aligned}$$

with the help of Lemma 2.2, we obtain:

$$\begin{aligned}
^{CP}\mathfrak{D}_{\mu(t)}\Big(x^{\zeta-1}\Big)(t) &= (\zeta-1)\int_0^t (t-\varrho)^{-\mu(t)}E^{-\gamma}_{\rho,\mu(t)}(\omega(t-\varrho)^{\rho})\varrho^{\zeta-2}d\varrho \\
&= \Gamma(\zeta)t^{\zeta-\mu(t)-1}E^{-\gamma}_{\rho,\zeta-\mu(t)}(\omega t^{\rho}). \qquad (2.7)
\end{aligned}$$

Therefore the proof is completed. □

3. Properties of Chebyshev orthogonal polynomial and shifted Chebyshev orthogonal polynomial

A Chebyshev polynomials of degree n in the interval $x \in [-1,1]$ that with the symbol $T_n(x)$ is shown, in the form of a recursive sequence is defined as follows[34]:

$$\begin{aligned}
&T_{n+1}(x)+T_{n-1}(x) = 2xT_n(x),\ n = 1,2,3,\cdots, \\
&T_0(x) = 1,\ T_1(x) = x. \qquad (3.1)
\end{aligned}$$

The Chebyshev polynomial can be represented as a finite series as follows:

$$T_n(x) = n\sum_{i=0}^{[\frac{n}{2}]}(-1)^i 2^{n-2i-1}\frac{\binom{n-i}{i}}{(n-i)}x^{n-2i}. \qquad (3.2)$$

The orthogonal condition for this Chebyshev polynomial respect to a weight function $\Sigma_1(x) = \frac{1}{\sqrt{1-x^2}}$ is given by:

$$\int_{-1}^{1} T_i(x)T_j(x)\Sigma_1(x)dx = \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}}dx = \begin{cases} \pi, & i=j=0, \\ \frac{\pi}{2}, & i=j\neq 0, \\ 0 & j\neq i, \end{cases} \tag{3.3}$$

Now we change the variable $x \in [-1,1]$ in the Chebyshev polynomial to $x = 2t-1, t \in [0,1]$ that the Chebyshev polynomial of degree n, $T_n(x)$ changes to the shifted Chebyshev polynomial of degree n as $T_n(2t-1) = T_n^*(t)$. The recursive sequence of this shifted Chebyshev polynomial of degree can be defined as follows:

$$\begin{aligned} &T^*_{n+1}(t)+T^*_{n-1}(t) = 2(2t-1)T^*_n(t),\ n = 1,2,3,\dots, \\ &T^*_0(t) = 1,\ T^*_1(t) = 2t-1. \end{aligned} \tag{3.4}$$

Here, this polynomial is introduced in the relation(3.4) has a series representation as follows:

$$T^*_n(t) = n\sum_{k=0}^{n}(-1)^{n-k}2^{2k}\frac{\begin{pmatrix} n+k \\ 2k \end{pmatrix}}{(n+k)}t^k. \tag{3.5}$$

The orthogonality condition for $T^*_n(t)$ respect to a weight function $\Sigma_2(x) = \frac{1}{\sqrt{1-x^2}}$ is given by:

$$\int_{0}^{1} T^*_i(x)T^*_j(x)\Sigma_2(x)dx = \int_{0}^{1} \frac{T^*_i(x)T^*_j(x)dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & i=j=0, \\ \frac{\pi}{2}, & i=j\neq 0, \\ 0 & j\neq i. \end{cases} \tag{3.6}$$

Here we introduce a vector function $\Upsilon(t)$ as follows:

$$\Upsilon(t) = \Big[T^*_0(t), T^*_1(t), \dots, T^*_n(t)\Big]^T, \tag{3.7}$$

where $T^*_i(t), 0 \leq i \leq n$ are the shifted Chebyshev polynomials of degree n and we can display the $\Upsilon(t)$ as follows:

$$\Upsilon(t) = \Pi\mathbf{T}_n(t), \tag{3.8}$$

where Π is defined by:

$$\Pi = \Big[\Big(a_{i,j}\Big)\Big] = \begin{cases} 0, & j > i, \\ (i-1)(-1)^{i-j}\frac{2^{2(j-1)}(i+j-3)!}{(2(j-1))!(i-j)!} & j \leq i, \end{cases} \tag{3.9}$$

where $i = 1,\dots,n+1, j = 1,\dots,n+1$ and $\mathbf{T}_n(t)$ is defined by:

$$\mathbf{T}^T_n(t) = \Big[1, t, \dots, t^n\Big]. \tag{3.10}$$

Since Π is invertible then we can rewrite the matric representation (3.8) as:

$$\mathbf{T}_n(t) = \Pi^{-1}\Upsilon(t). \tag{3.11}$$

Using orthogonal conditions for $T^*_n(t)$ respect to the weight function $\Sigma_2(x)$ which is stated in relation(3.6), we can be expanded any arbitrary function $z(x,t)$ in terms of the shifted Chebyshev polynomials as follows:

$$z(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z_{i,j} T^*_i(x) T^*_j(t), x \in L^2[0,1], t \in L^2[0,1], \tag{3.12}$$

where the coefficient $z_{i,j}$ for $i = 1,\ldots,n+1, j = 1,\ldots,n+1$ can be calculated. For calculate $z_{i,j}$, we multiply the two sides of the relation (3.12) in $\Sigma_2(x)T^*_{k_1}(x)\Sigma_2(t)T^*_{k_2}(t), k_1 = 1,\ldots,n+1, k_2 = 1,\ldots,n+1$, we have:

$$\Sigma_2(x)T^*_{k_1}(x)\Sigma_2(t)T^*_{k_2}(t)z(x,t) = \Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z_{i,j} T^*_i(x) T^*_j(t)\Big)\Sigma_2(x)T^*_{k_1}(x)\Sigma_2(t)T^*_{k_2}(t), \tag{3.13}$$

by integrating both sides of the equation (3.13), we obtain

$$\int_0^1\int_0^1 \Sigma_2(x)T^*_i(x)\Sigma_2(t)T^*_i(t)z(x,t)dxdt = z_{i,i}\langle T^*_i(t), T^*_i(t)\rangle_{\Sigma_2(t)} \times \langle T^*_i(x), T^*_i(x)\rangle_{\Sigma_2(x)},$$

$$z_{i,i} = \frac{\int_0^1\int_0^1 \Sigma_2(x)T^*_i(x)\Sigma_2(t)T^*_i(t)z(x,t)dxdt}{\langle T^*_i(t), T^*_i(t)\rangle_{\Sigma_2(t)} \times \langle T^*_i(x), T^*_i(x)\rangle_{\Sigma_2(x)}}. \tag{3.14}$$

Considering the first $(n+1)$ sentence of the infinite series (3.12), we can approximate the function $z(x,t)$ as:

$$z(x,t) \cong z_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} z_{i,j} T^*_i(x) T^*_j(t) = \underbrace{[1, x, \ldots, x^n]_{1\times(n+1)}}_{\Upsilon^T(x)}$$

$$\times \underbrace{\begin{bmatrix} z_{0,0} & z_{0,1} & \cdots & z_{0,n} \\ z_{1,0} & z_{1,1} & \cdots & z_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n,0} & z_{n,1} & \cdots & z_{n,n} \end{bmatrix}}_{\mathbb{Z}} \times \underbrace{\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}_{(n+1)\times 1}}_{\Upsilon(t)} = (\Pi \mathbf{T}_n(x))^T \mathbb{Z} (\Pi \mathbf{T}_n(t)). \tag{3.15}$$

Theorem 3.1. *Let* $\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} |z_{i,j}|^2 < \infty$ *and the following relations are hold:*

$$\begin{aligned} \mathfrak{L}_1 : {}^{CP}\mathfrak{D}_{\mu(t)}\Big[z_n(x,t).w(x,t)\Big] &+ \frac{\partial z_n(x,t)}{\partial t} \\ &- r(x,t) + \int_0^t z_n(x,Y).k(x,Y)dY + \int_0^t z_n(x,Y)dY, \\ \mathfrak{L}_2 : {}^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big] &+ \frac{\partial z(x,t)}{\partial t} \\ &- r(x,t) + \int_0^t z(x,Y).k(x,Y)dY + \int_0^t z(x,Y)dY, \end{aligned} \tag{3.16}$$

where $z(x,t)$ *is the exact solution of the equation* (1.1) *and* $z_n(x,t)$ *is the numerical solution of the equation* (1.1)*. Then we have:*

$$|\mathfrak{L}_1 - \mathfrak{L}_2| \underbrace{\rightarrow}_{n\rightarrow\infty} 0. \tag{3.17}$$

Proof . We want to show that the following relation holds:

$$\lim_{n\to\infty}\mathfrak{L}_1=\mathfrak{L}_2\Rightarrow|\mathfrak{L}_1-\mathfrak{L}_2|\underbrace{\to}_{n\to\infty}0. \tag{3.18}$$

From definitions $\mathfrak{L}_1,\mathfrak{L}_2$ we get:

$$\begin{aligned}\mathfrak{L}_1-\mathfrak{L}_2 =& {}^{CP}\mathfrak{D}_{\mu(t)}\Big[(z_n(x,t)-z(x,t)).w(x,t)\Big]+\frac{\partial}{\partial t}(z_n(x,t)-z(x,t))\\ &+\int_0^t(z_n(x,Y)-z(x,Y)).k(x,Y)dY+\int_0^t(z_n(x,Y)-z(x,Y))dY, \end{aligned}\tag{3.19}$$

$$\begin{aligned}|\mathfrak{L}_1-\mathfrak{L}_2| \leq& |{}^{CP}\mathfrak{D}_{\mu(t)}\Big[(z_n(x,t)-z(x,t)).w(x,t)\Big]|+|\frac{\partial}{\partial t}(z_n(x,t)-z(x,t))|\\ &+|\int_0^t(z_n(x,Y)-z(x,Y)).k(x,Y)dY|+|\int_0^t(z_n(x,Y)-z(x,Y))dY|,. \end{aligned}\tag{3.20}$$

To proof Eq.(3.20), we show the following relation is hold:

$$|z_n(x,t)-z(x,t)|\to 0,\text{as}\, n\to\infty. \tag{3.21}$$

For this aim, we have:

$$\begin{aligned}|z_n(x,t)-z(x,t)| &= |\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}z_{i,j}T_i^*(x)T_j^*(t)-\sum_{i=0}^{n}\sum_{j=0}^{n}z_{i,j}T_i^*(x)T_j^*(t)|\\ &= |\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}z_{i,j}T_i^*(x)T_j^*(t)|, \end{aligned}\tag{3.22}$$

using the Cauchy–Schwarz inequality for equation (3.22), we obtain

$$\begin{aligned}|z_n(x,t)-z(x,t)| &\leq \Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|z_{i,j}|^2\Big)^{\frac{1}{2}}\times\Big(\sum_{i=n+1}^{\infty}|T_i^*(x)|^2\Big)^{\frac{1}{2}}\times\Big(\sum_{j=n+1}^{\infty}|T_j^*(t)|^2\Big)^{\frac{1}{2}}\\ &\leq \Big(\sum_{i=n+1}^{\infty}|T_i^*(x)|^2\Big)^{\frac{1}{2}}\times\Big(\sum_{j=n+1}^{\infty}|T_j^*(t)|^2\Big)^{\frac{1}{2}},\ \text{since}\ \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}|z_{i,j}|^2<\infty. \end{aligned}$$

Then we have

$$\underbrace{|z_n(x,t)-z(x,t)|\to 0}_{n\to\infty}, \tag{3.23}$$

since $\underbrace{\Big(\sum_{j=n+1}^{\infty}|T_j^*(t)|^2\Big)^{\frac{1}{2}}\to 0}_{n\to\infty,\, t\in(0,1)}$ and $\underbrace{\Big(\sum_{j=n+1}^{\infty}|T_i^*(x)|^2\Big)^{\frac{1}{2}}\to 0}_{n\to\infty,\, x\in(0,1)}$.

So from the equation (3.23), for the equation (3.20) is used and we conclude

$$\mathfrak{L}_1-\mathfrak{L}_2\to 0,\ \text{as}\, n\to\infty. \tag{3.24}$$

The proof is completed. □

4. Numerical approximation by the operational matrix

In this section we obtain the numerical solutions of the proposed equation presented in Eqs. (1.1) and (1.2).

4.1. Calculation of operators $\int_0^t z(x,Y).k(x,Y)dY, \int_0^t z(x,Y)dY$

Assume the function $k(x,t)$ as the function $z(x,t)$ can be approximated as follows:

$$\begin{aligned} k(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j}T_i^*(x)T_j^*(t) = (\Pi \mathbf{T}_n(t))^T \mathbb{K}(\Pi \mathbf{T}_n(x)) \\ &= \Upsilon^T(t)\mathbb{K}\Upsilon(x), x \in L^2[0,1], t \in L^2[0,1], \end{aligned} \tag{4.1}$$

where $\mathbb{K} = [k_{i,j}]$. So, using Eqs. (3.15),(4.1), we obtain

$$\begin{aligned} &\int_0^t z(x,Y).k(x,Y)dY = \int_0^t \Big(\Upsilon^T(x)\mathbb{Z}\Upsilon(Y))\Big)\Big(\Upsilon^T(Y)\mathbb{K}\Upsilon(x))\Big)dY \\ &= \Upsilon^T(x)\mathbb{Z}\Big(\int_0^t \Upsilon(Y)\Upsilon^T(Y)dY\Big)\mathbb{K}\Upsilon(x) \\ &= \Upsilon^T(x)\mathbb{Z}\Big(\int_0^t \begin{bmatrix} 1 & Y & Y^2 & \dots & Y^n \\ Y & Y^2 & \ddots & \ddots & Y^{n+1} \\ Y^2 & Y^3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ Y^n & Y^{n+1} & \dots & Y^{2n-1} & Y^{2n} \end{bmatrix} dY\Big)\mathbb{K}\Upsilon(x) \\ &= \Upsilon^T(x)\mathbb{Z}\underbrace{\begin{bmatrix} \int_0^t 1dY & \int_0^t YdY & \int_0^t Y^2dY & \dots & \int_0^t Y^ndY \\ \int_0^t YdY & \int_0^t Y^2dY & \ddots & \ddots & \int_0^t Y^{n+1}dY \\ \int_0^t Y^2dY & \int_0^t Y^3dY & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \int_0^t Y^ndY & \int_0^t Y^{n+1}dY & \dots & \int_0^t Y^{2n-1}dY & \int_0^t Y^{2n}dY \end{bmatrix}}_{\mathfrak{H}}\mathbb{K}\Upsilon(x). \end{aligned} \tag{4.2}$$

Also, with a similar process for $\int_0^t z(x,Y)dY$, we have

$$\begin{aligned} \int_0^t z(x,Y)dY &= \int_0^t \Upsilon^T(x)\mathbb{Z}\Upsilon(Y)dY = \Upsilon^T(x)\mathbb{Z}\int_0^t \Upsilon(Y)dY \\ &= (\Pi\mathbf{T}_n(x))^T\mathbb{Z}\Pi \begin{bmatrix} \int_0^t 1dY \\ \int_0^t YdY \\ \int_0^t Y^2dY \\ \vdots \\ \int_0^t Y^ndY \end{bmatrix}. \end{aligned} \tag{4.3}$$

4.2. Calculation of operators ${}^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big], \frac{\partial z(x,t)}{\partial t}$

Operator calculation ${}^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big]$ in the form of a theorem is stated as follows:

Theorem 4.1. *Let $0 < \mu(t) \leq 1$ and $z(x,t)$, $w(x,t) \in L^2[0,1]$. Then the operational matrix of Caputo-Prabhakar fractional derivative of variable order $\mu(t)$ for multiplication the functions $z(x,t).w(x,t)$ can be expressed in the following from:*

$$^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big] = \Upsilon(x)\mathbb{Z}\Pi\boldsymbol{M}\Pi^T\mathbb{W}\Phi(x), \tag{4.4}$$

that $\boldsymbol{M}$ has a matric representation as follows:

$$\boldsymbol{M} = \begin{bmatrix} 0 & \Gamma(2)t^{1-\mu(t)}E^{-\gamma}_{\rho,2-\mu}(\omega t^\rho) & \dots & \Gamma(n+1)t^{n-\mu(t)}E^{-\gamma}_{\rho,n-\mu+1}(\omega t^\rho) \\ \Gamma(2)t^{1-\mu(t)}E^{-\gamma}_{\rho,2-\mu}(\omega t^\rho) & \Gamma(3)t^{2-\mu(t)}E^{-\gamma}_{\rho,3-\mu}(\omega t^\rho) & \dots & \Gamma(n+22)t^{n+1-\mu(t)}E^{-\gamma}_{\rho,n+2-\mu}(\omega t^\rho) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(n+1)t^{n-\mu(t)}E^{-\gamma}_{\rho,n+1-\mu}(\omega t^\rho) & \Gamma(n+2)t^{n+1-\mu(t)}E^{-\gamma}_{\rho,n+2-\mu}(\omega t^\rho) & \dots & \Gamma(2n+1)t^{2n-\mu(t)}E^{-\gamma}_{\rho,2n+1-\mu}(\omega t^\rho) \end{bmatrix} \tag{4.5}$$

where, the function $\mathbb{Z}$ is unknown and the function $\mathbb{W}$ is known.

Proof . Let $w(x,t) = \Upsilon^T(t)\mathbb{W}\Upsilon(x)$ be as an approximation of the function $w(x,t)$. Then we have:

$$\begin{aligned} &^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big] = {}^{CP}\mathfrak{D}_{\mu(t)}\Big[\Upsilon^T(x)\mathbb{Z}\Phi(t).\Upsilon^T(t)\mathbb{W}\Upsilon(x)\Big] \\ &= \Upsilon^T(x)\mathbb{Z}^{CP}\mathfrak{D}_{\mu(t)}\Big[\Upsilon(t)\Upsilon^T(t)\Big]\mathbb{W}\Upsilon(x) = \Upsilon^T(x)\mathbb{Z}^{CP}\mathfrak{D}_{\mu(t)}\Big[\Pi\mathbf{T}^*_n(t)\big(\Pi\mathbf{T}^*_n(t)\big)^T\Big] \\ &\times\Pi^T\mathbb{W}\Upsilon(x) = \Upsilon^T(x)\mathbb{Z}\Pi^{CP}\mathfrak{D}_{\mu(t)}\Big[\mathbf{T}^*_n(t)\big(\mathbf{T}^*_n(t)\big)^T\Big]\Pi^T\mathbb{W}\Upsilon(x) \\ &= \Upsilon^T(x)\mathbb{Z}\Pi^{CP}\mathfrak{D}_{\mu(t)}\Big(\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}.\begin{pmatrix} 1 & t & \dots & t^n \end{pmatrix}\Big)\Pi^T\mathbb{W}\Upsilon(x) \\ &= \Upsilon^T(x)\mathbb{Z}\Pi^{CP}\mathfrak{D}_{\mu(t)}\Big(\begin{pmatrix} 1 & t & \dots & t^n \\ t & t^2 & \dots & t^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t^n & t^{2n} & \dots & t^{2n} \end{pmatrix}\Big)\Upsilon^T\mathbb{W}\Upsilon(x). \end{aligned} \tag{4.6}$$

Using the Lemma 2.3, we obtain

$$\begin{aligned} ^{CP}\mathfrak{D}_{\mu(t)}\Big[z(x,t).w(x,t)\Big] &= \Upsilon^T(x)\mathbb{Z}\Pi \\ &\times\begin{bmatrix} 0 & \Gamma(2)t^{1-\mu(t)}E^{-\gamma}_{\rho,2-\mu}(\omega t^\rho) & \dots & \Gamma(n+1)t^{n-\mu(t)}E^{-\gamma}_{\rho,n-\mu+1}(\omega t^\rho) \\ \Gamma(2)t^{1-\mu(t)}E^{-\gamma}_{\rho,2-\mu}(\omega t^\rho) & \Gamma(3)t^{2-\mu(t)}E^{-\gamma}_{\rho,3-\mu}(\omega t^\rho) & \dots & \Gamma(n+22)t^{n+1-\mu(t)}E^{-\gamma}_{\rho,n+2-\mu}(\omega t^\rho) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(n+1)t^{n-\mu(t)}E^{-\gamma}_{\rho,n+1-\mu}(\omega t^\rho) & \Gamma(n+2)t^{n+1-\mu(t)}E^{-\gamma}_{\rho,n+2-\mu}(\omega t^\rho) & \dots & \Gamma(2n+1)t^{2n-\mu(t)}E^{-\gamma}_{\rho,2n+1-\mu}(\omega t^\rho) \end{bmatrix}\times\Pi^T\mathbb{W}\Upsilon(x) \\ &= \Upsilon^T(x)\mathbb{Z}\Pi\mathbf{M}\Pi^T\mathbb{W}\Upsilon(x). \end{aligned} \tag{4.7}$$

The relation (4.5) is obtained. □ To calculation the operator $\frac{\partial z(x,t)}{\partial t}$ we have:

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial(\Pi\mathbf{T}_n(x))^T\mathbb{Z}(\Pi\mathbf{T}_n(t))}{\partial t}$$
$$= (\Pi\mathbf{T}_n(x))^T\mathbb{Z}(\Pi\mathbf{T}'_n(t)) = (\Pi\mathbf{T}_n(x))^T\mathbb{Z}\Pi\begin{bmatrix}0\\1\\\vdots\\nt^{n-1}\end{bmatrix}. \tag{4.8}$$

To obtain the numerical solution of the equations (1.1) and (1.2), we Substitute Eqs. (4.2), (4.3), (4.7) and (4.8) into the equation (1.1) and the result is obtained.

5. Numerical Examples

In the following section, three numerical examples are showed that their demonstrate the performance and accuracy of the proposed method.

Example 5.1. *We consider the equations* (1.1) *and* (1.2) *with* $k(x,t) = (x+t), w(x,t) = (x+t+1), z(0,t) = t^2, z(x,0) = x^2$ *and*

$$\mu(t) = \frac{t}{3},$$
$$r(x,t) = 2t + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^3x}{3} + tx^2 + \frac{t^2x^2}{3} + tx^3 - \frac{3t^{1-\frac{t}{3}}\Big[6t(9+8t) - 6(-9+t)tx\Big]}{(-9+t)(-6+t)(-3+t)\Gamma 1 - \frac{t}{3}}, \tag{5.1}$$

where for this example analytical solution is $z(x,t) = x^2 + t^2$. *Let the maximum error in this paper as* $\| E \| = \max_{1\le i\le n} |u_n(M_i) - u(M_i)|$ *is defined. Applying the proposed method on this example, taking* $n = 2$, *dispersing* $x_i = \frac{k_i}{3} - \frac{1}{6}$, $x_j = \frac{k_j}{3} - \frac{1}{6}$, $(k_i, k_j = 1, 2, 3)$. *For other values* n, x_i, x_j *are defined as:*

$$x_i = \frac{k_i}{n+1} - \frac{1}{2n+2},\ x_j = \frac{k_j}{n+1} - \frac{1}{2n+2},\ (k_i, k_j = 1, 2, 3, \dots, n+1). \tag{5.2}$$

The numerical solution and the exact solution with $n = 2$ *for Example 5.1 are showed in Figure 1 also, plots of approximate solution and its absolute error for* $n = 2, 3$ *are shown in Figs. 2,3, 4. The absolute error between the exact solution and the numerical solution is showed in Table1 also, the absolute error between the exact solution and the numerical solution when* $n = 3$ *is displayed in Table 2.*

Example 5.2. *For this example, we study the fractional integro-differential equation of variable order* $\mu(t) = \sin(\frac{t}{3})$ *with* $k(x,t) = (x+t), w(x,t) = xt, z(0,t) = (1+t)^2, z(x,0) = (1+x)^2$ *and*

$$r(x,t) = 2(1+x+t) + t + \frac{3t^2}{2} + t^3 + \frac{t^4}{4} + 3tx + tx^2 + 3t^2x + t^3x + \frac{3t^2x^2}{2} + tx^3 - \frac{3t^{1-\sin(\frac{t}{3})}x\Big[6(1+x+t)^2 + (1+x)\sin t(-3(5+4t+5x) + (1+x)\sin t)\Big]}{(-9+\sin t)(-6+\sin t)(-3+\sin t)\Gamma 1 - \sin(\frac{t}{3})}. \tag{5.3}$$

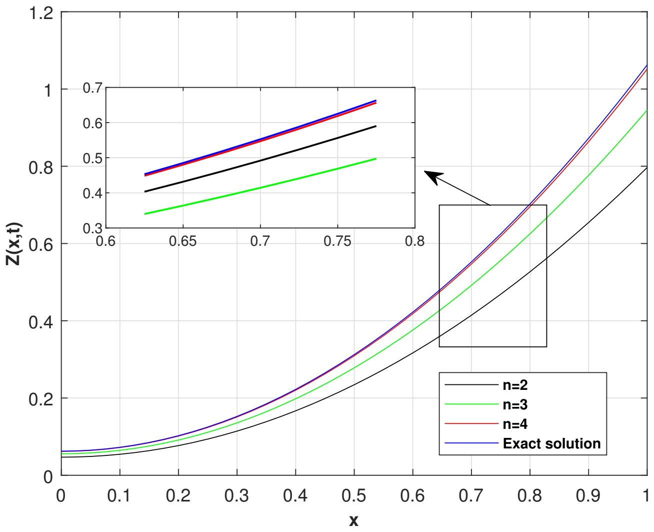

Figure 1: The numerical solution and the exact solution with $n = 2, 3, 4$, $\mu(t) = \frac{t}{3}$ for Example 5.1 at $t = 0.25$ and $\rho = \omega = \gamma = 1$.

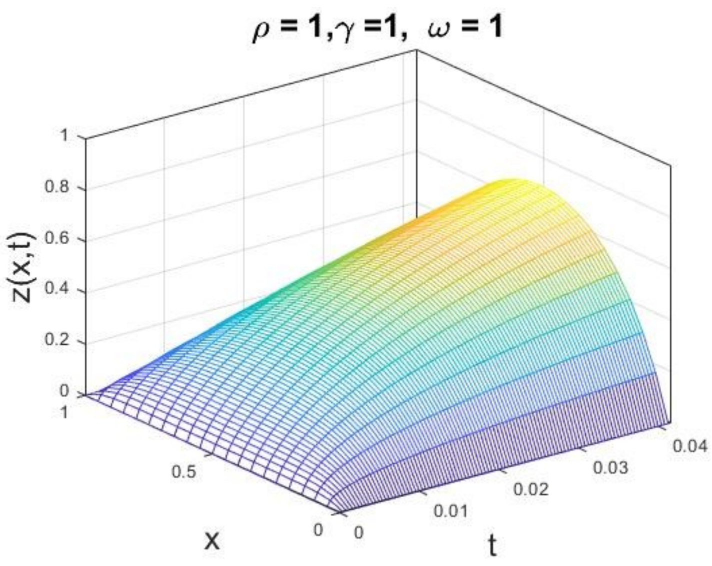

Figure 2: The graph of the approximate solution when $n = 2$, $\mu(t) = \frac{t}{3}$.

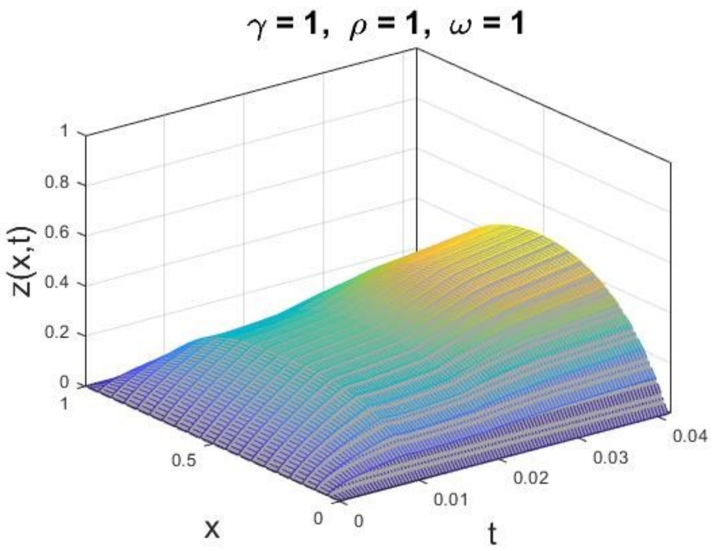

Figure 3: The graph of the approximate solution when $n = 3$, $\mu(t) = \frac{t}{3}$.

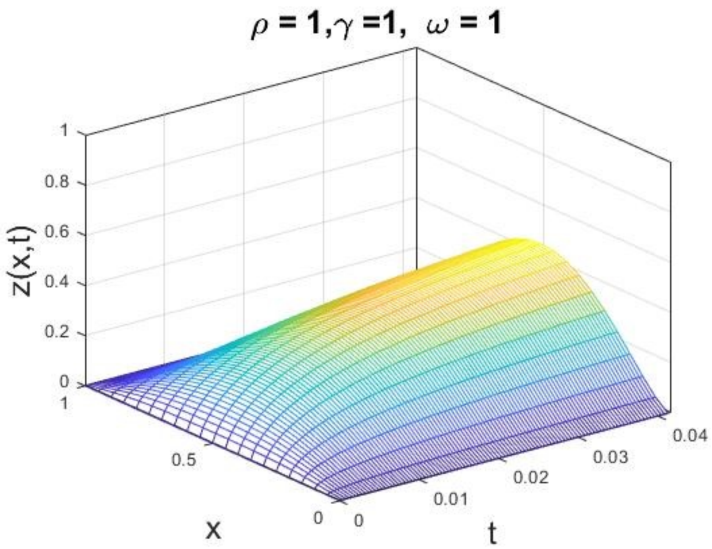

Figure 4: The graph of the approximate solution when $n = 4$, $\mu(t) = \frac{t}{3}$.

Table 1: The absolute error the numerical solution and the exact solution when $n = 2$, $\mu(t) = \frac{t}{3}$

	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$x = 0.0$	0	0	0	0	0
$x = 0.1$	$1.5421e-004$	$3.3854e-004$	$4.1031e-004$	$1.8787e-004$	$1.3624e-004$
$x = 0.2$	$2.1482e-004$	$6.3007e-004$	$8.4527e-004$	$5.9813e-004$	$2.1503e-004$
$x = 0.3$	$3.0023e-004$	$1.1501e-004$	$1.0426e-004$	$7.8044e-004$	$3.2674e-004$
$x = 0.4$	$4.6589e-004$	$1.6078e-004$	$1.0934e-004$	$7.2928e-004$	$4.2013e-004$
$x = 0.5$	$5.2218e-004$	$1.3728e-004$	$1.6714e-004$	$8.5494e-004$	$4.2354e-004$
$x = 0.6$	$5.1048e-004$	$1.2018e-004$	$1.2223e-004$	$7.1452e-004$	$3.2264e-004$
$x = 0.7$	$4.0076e-004$	$1.2054e-004$	$1.2032e-004$	$6.2901e-004$	$2.2054e-004$
$x = 0.8$	$3.0602e-004$	$1.2140e-004$	$1.2454e-004$	$6.0143e-004$	$1.1043e-004$
$x = 0.9$	$2.1009e-004$	$5.2237e-004$	$7.2118e-004$	$4.2063e-004$	$1.0178e-004$
$x = 1$	$1.5308e-004$	$2.2549e-004$	$4.2054e-004$	$2.0183e-004$	$1.2078e-004$

Table 2: The absolute error the numerical solution and the exact solution when $n = 3$, $\mu(t) = \frac{t}{3}$

	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$x = 0.0$	0	0	0	0	0
$x = 0.1$	$2.0787e-004$	$3.2546e-004$	$5.2691e-004$	$5.8754e-004$	$5.2162e-004$
$x = 0.2$	$2.2662e-004$	$3.3782e-004$	$6.2004e-004$	$7.8834e-004$	$3.5003e-004$
$x = 0.3$	$2.2055e-004$	$4.1003e-004$	$6.2615e-004$	$9.8654e-004$	$8.2054e-004$
$x = 0.4$	$4.1152e-004$	$3.6376e-004$	$3.2040e-004$	$9.2953e-004$	$8.2006e-004$
$x = 0.5$	$4.0272e-004$	$4.6331e-004$	$3.26782e-004$	$9.1534e-004$	$6.1014e-004$
$x = 0.6$	$5.2232e-004$	$2.2004e-004$	$4.0706e-004$	$8.1041e-004$	$6.2432e-004$
$x = 0.7$	$5.2139e-004$	$3.2104e-004$	$4.2504e-004$	$7.2901e-004$	$3.1084e-004$
$x = 0.8$	$6.0642e-004$	$2.1704e-004$	$2.2014e-004$	$7.0083e-004$	$3.0413e-004$
$x = 0.9$	$6.1008e-004$	$7.2013e-004$	$2.2623e-004$	$5.2003e-004$	$4.01370e-004$
$x = 1$	$7.7001e-004$	$3.2022e-004$	$4.2294e-004$	$3.0014e-004$	$4.2089e-004$

where analytical solution is given by $z(x,t) = (1 + x + t)^2$. We consider a similar process as Example 5.1 for this example and it is solve that here we obtain the matrix $\mathbb{Z}$ as follows:

$$\mathbb{Z} = \begin{bmatrix} 1 & \frac{5}{2} & \frac{8}{3} \\ \frac{5}{2} & 4.00765 & 5.08665 \\ \frac{8}{3} & 7.003462 & 8.006243 \end{bmatrix}. \tag{5.4}$$

The numerical solution and the exact solution with $n = 2$ for example 5.2 are displayed in Fig. 5. Also, the approximate solution and its absolute error for $n = 2, 3$ are shown in Figs.6,7, 8. The absolute error between the exact solution and the numerical solution is displayed in Table 3 also, the absolute error between the exact solution and the numerical solution when $n = 3$ is displayed in Table4.

6. Conclusion

In this paper, we presented a numerical method based on shifted Chebyshev polynomials for finding the solution of the fractional integro–differential equation of variable order with Caputo-Prabhakar fractional derivative of order $\mu(t)$. We are used the proposed method to reduces the

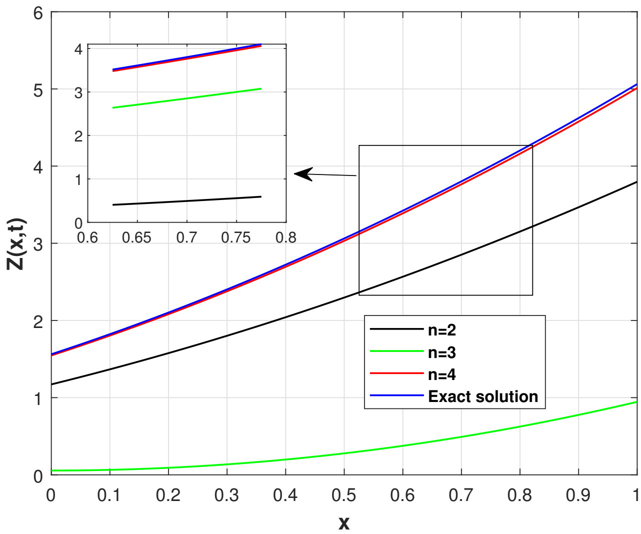

Figure 5: The numerical solution and the exact solution with $n = 2, 3, 4$, $\mu(t) = \sin(\frac{t}{3})$ for Example 5.2 at $t = 0.25$ and $\rho = \omega = \gamma = 1$.

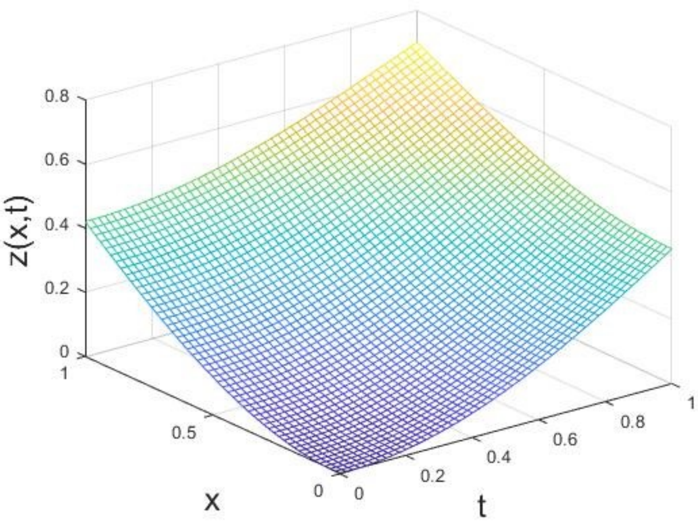

Figure 6: The graph of the approximate solution when $n = 2$, $\mu(t) = \sin(\frac{t}{3})$.

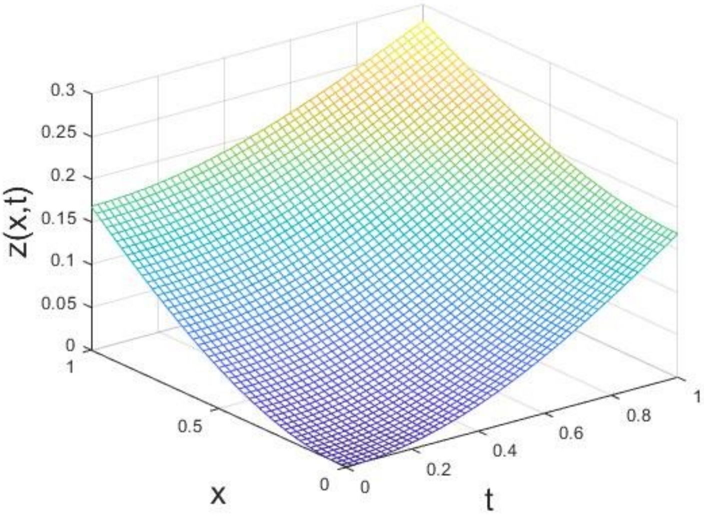

Figure 7: The graph of the approximate solution when $n = 3$, $\mu(t) = \sin(\frac{t}{3})$.

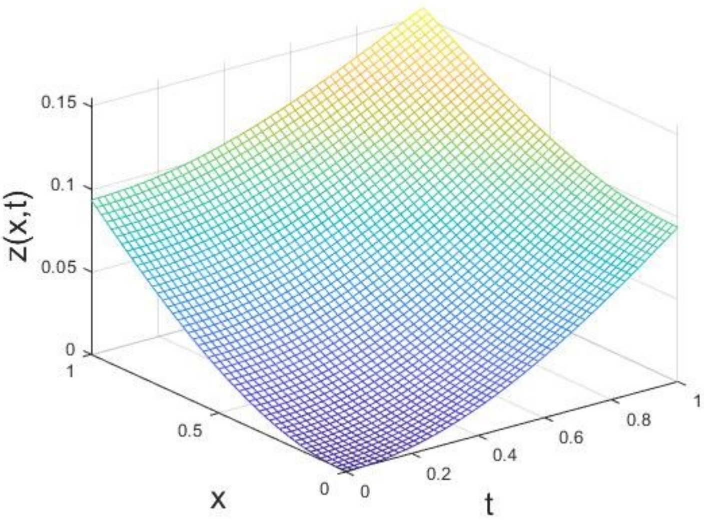

Figure 8: The graph of the approximate solution when $n = 4$, $\mu(t) = \sin(\frac{t}{3})$.

Table 3: The absolute error the numerical solution and the exact solution when $n = 2$, $\mu(t) = \sin(\frac{t}{3})$

	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$x = 0.0$	0	0	0	0	0
$x = 0.1$	$2.2634e-004$	$4.2634e-004$	$5.2034e-004$	$2.8754e-004$	$1.2634e-004$
$x = 0.2$	$3.2726e-004$	$7.3408e-004$	$9.2334e-004$	$6.9834e-004$	$2.5603e-004$
$x = 0.3$	$4.2879e-004$	$2.1523e-004$	$2.2131e-004$	$8.8754e-004$	$3.2614e-004$
$x = 0.4$	$5.1572e-004$	$2.6542e-004$	$2.2634e-004$	$8.2953e-004$	$4.2004e-004$
$x = 0.5$	$6.3392e-004$	$2.6531e-004$	$2.2634e-004$	$9.2634e-004$	$4.2014e-004$
$x = 0.6$	$6.5432e-004$	$2.2034e-004$	$2.2234e-004$	$8.1732e-004$	$3.2264e-004$
$x = 0.7$	$5.2609e-004$	$2.2364e-004$	$2.2034e-004$	$7.2981e-004$	$2.2084e-004$
$x = 0.8$	$4.0652e-004$	$2.2764e-004$	$2.2214e-004$	$7.0043e-004$	$1.0043e-004$
$x = 0.9$	$3.1078e-004$	$7.2214e-004$	$8.2278e-004$	$5.2903e-004$	$1.0078e-004$
$x = 1$	$2.7043e-004$	$3.2541e-004$	$5.2064e-004$	$3.0023e-004$	$1.2089e-004$

Table 4: The absolute error the numerical solution and the exact solution when $n = 3$, $\mu(t) = \sin(\frac{t}{3})$

	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$x = 0.0$	0	0	0	0	0
$x = 0.1$	$1.0437e-004$	$2.2634e-004$	$4.2634e-004$	$4.8754e-004$	$4.2634e-004$
$x = 0.2$	$1.2701e-004$	$2.3128e-004$	$5.2634e-004$	$6.8834e-004$	$2.5603e-004$
$x = 0.3$	$1.2049e-004$	$3.1343e-004$	$5.2634e-004$	$8.8654e-004$	$7.2614e-004$
$x = 0.4$	$3.1322e-004$	$2.6562e-004$	$2.2044e-004$	$8.2953e-004$	$7.2036e-004$
$x = 0.5$	$3.0292e-004$	$3.6781e-004$	$2.26364e-004$	$8.1534e-004$	$5.2014e-004$
$x = 0.6$	$4.2332e-004$	$1.2024e-004$	$3.0756e-004$	$7.1721e-004$	$5.2674e-004$
$x = 0.7$	$4.2039e-004$	$2.2304e-004$	$3.2634e-004$	$6.2981e-004$	$2.2084e-004$
$x = 0.8$	$5.0612e-004$	$1.2704e-004$	$1.2024e-004$	$6.0783e-004$	$2.0453e-004$
$x = 0.9$	$5.1018e-004$	$6.2014e-004$	$1.2653e-004$	$4.2903e-004$	$3.02370e-004$
$x = 1$	$6.7041e-004$	$2.2021e-004$	$3.2294e-004$	$2.0015e-004$	$3.2019e-004$

equation to a set of algebraic relations. The numerical results and absolute errors are presented. It has been shown that the obtained results are in excellent agreement with the exact solution.

Acknowledgments

We are very grateful to anonymous referees for their careful reading and valuable comments which led to the improvement of this paper.

References

[1] R. Agarwal, S. Jain, R.P. Agarwal, *Analytic solution of generalized space time fractional reaction–diffusion equation*, Fract Differ Calc, 7(2017), 169-84.

[2] H. Aminikhah, A.R. Sheikhani, H. Rezazadeh, *Exact solutions for the fractional differential equations by using the first integral method*, Nonlinear Engineering, 4(1)(2015), 15-22.

[3] Y. Chen, M. Yi, C. Yu, *Error analysis for numerical solution of fractional differential equation by Haar wavelets method*, Journal of Computational Science, 3(5)(2012), 367-373.

[4] E.H. Doha, M.A. Abdelkawy, A.Z.M. Amin, A. M. Lopes, *On spectral methods for solving variable-order fractional integro–differential equations*, Computational and Applied Mathematics, 37(3)(2018), 3937-3950.

[5] I.L. El-Kalla, *Convergence of the Adomian method applied to a class of nonlinear integral equations*, Applied Mathematics Letters, 21(4)(2008), 372–376.

[6] R. Garra, R. Gorenflo, F. Polito, Ž. Tomovski, *Hilfer-Prabhakar derivatives and some applications*, Applied mathematics and computation, 242(2014), 576-589.

[7] R. Garra, R. Garrappa, The Prabhakar or three parameters Mittag-Leffler function: Theory and application, Communications in Nonlinear Science and Numerical Simulation, 56(2018), 314-329.

[8] R. Garrappa, *Grünwald–Letnikov operators for fractional relaxation in Havriliak-Negamimodels*, Commun Nonlinear Sci Numer Simul 38(2016), 178–191.

[9] R. Garra, R. Garrappa, *The Prabhakar or three parameters Mittag–Leffler function: Theory and application*, Communications in Nonlinear Science and Numerical Simulation, 56(2018), 314-329.

[10] A. Giusti, I. Colombaro, *Prabhakar-like fractional viscoelasticity*, Communications in Nonlinear Science and Numerical Simulation, 56(2018), 138-143.

[11] A. Giusti, I. Colombaro, R. Garra, R. Garrappa, F. Polito, M. Popolizio, F. Mainardi, *A practical guide to Prabhakar fractional calculus*, Fractional Calculus and Applied Analysis, 23(1)(2020), 9-54.

[12] B. Guo, X. Pu, F. Huang, *Fractional Partial Differential Equations and Their Numerical Solutions*, World Scientific Publishing, Singapore, 2015.

[13] R.K. Gupta, B.S. Shaktawat, D. Kumar, *Certain relation of generalized fractional calculus associated with the generalized Mittag-Leffler function.* J. Raj. Acad. Phy. Sci, 15(3)(2016), 117-126.

[14] M.M. Hosseini, *Adomian decomposition method for solution of nonlinear differential algebraic equations*, Applied mathematics and computation, 181(2)(2006), 1737-1744.

[15] M. Ichise, Y. Nagayanagi, T. Kojima, *An analog simulation of non-integer order transfer functions for analysis of electrode processes*, Journal of Electroanalytical Chemistry and Interfacial Electrochemistry, 33(2)(1971), 253-265.

[16] S. Khubalkar, A. Junghare, M. Aware, S. Das, *Unique fractional calculus engineering laboratory for learning and research*, International Journal of Electrical Engineering Education, 57(1)(2020), 3-33.

[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations* , Elsevier Science Limited, 2006.

[18] J. Lai, S. Mao, J. Qiu, H. Fan, Q. Zhang, Z. Hu, J. Chen, *Investigation progress and applications of fractional derivative model in geotechnical engineering*, Mathematical Problems in Engineering, 2016.

[19] F. Mainardi, R. Garrappa, *On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics*, Journal of Computational Physics, 293(2015), 70-80.

[20] M. Mashoof, A.H.R. Sheikhani, *Numerical Solution of Fractional Control System by Haar-wavelet Operational Matrix Method*, Int. J. Industrial Mathematics, 8(2016), 289-298.

[21] M. Mashoof, A.H.R. Sheikhani, H.S. Najafi, *Stability Analysis of Distributed-Order Hilfer-Prabhakar Systems Based on Inertia Theory*, Mathematical Notes, 104(1-2)(2018), 74-85.

[22] M. Mashoof, A.H.R. Sheikhani, H.S. Naja, *Stability analysis of distributed order Hilfer-Prabhakar differential equations*, Hacettepe Journal of Mathematics and Statistics, 47(2)(2018), 299-315.

[23] A. Mohebbi, M. Saffarian, *Implicit RBF Meshless Method for the Solution of Two-dimensional Variable Order Fractional Cable Equation*, Journal of Applied and Computational Mechanics, 6(2)(2020), 235-247.

[24] S. Momani, Z. Odibat, V.S. Erturk, *Generalized differential transform method for solving a space and time fractional diffusion wave equation*, Physics Letters A, 370(5-6)(2007), 379-387.

[25] H.S. Najafi, S.A. Edalatpanah, A.R.H. Sheikhani, *Convergence analysis of modified iterative methods to solve linear systems*, Mediterranean journal of mathematics, 11(3)(2014), 1019-1032.

[26] Z.M. Odibat, *A study on the convergence of variational iteration method*, Mathematical and Computer Modelling, 51(9-10)(2010), 1181-1192.

[27] M.D. Ortigueira, *Fractional calculus for scientists and engineers*, Springer Science & Business Media, 2011.

[28] S.C. Pandey, *The Lorenzo–Hartley's function for fractional calculus and its applications pertaining to fractional order modelling of anomalous relaxation in dielectrics*, Computational and Applied Mathematics, 37(3)(2018), 2648-2666.

[29] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[30] T.R. Prabhakar, *A singular integral equation with a generalized Mittag Leffler function in the kernel*, 1971.

[31] F. Shariffar, A.H.R. Sheikhani, *A New Two-stage Iterative Method for Linear Systems and Its Application in Solving Poissons Equation*, International Journal of Industrial Mathematics, 11(4)(2019), 283–291.

[32] F. Shariffar, A.H.R. Sheikhani, H.S. Najafi, *An efficient chebyshev semi-iterative method for the solution of large systems*, University Politehnica of Bucharest Scientific Bulletin-Series A Applied Mathematics and Physics. 80(4)(2018), 239-252.

[33] A.H.R. Sheikhani, M. Mashoof, *A Collocation Method for Solving Fractional Order Linear System*, Journal of the Indonesian Mathematical Society, 23(1)(2017), 27-42.

[34] M.A. Snyder, *Chebyshev methods in numerical approximation* , Prentice-Hall, 1966.

[35] H.M. Srivastava, R.K. Saxena, T.K. Pogany, R. Saxena, *Integral transforms and special functions*, Applied Mathematics and Computation, 22(7)(2011), 487-506.

[36] K. Sun, M. Zhu, *Numerical algorithm to solve a class of variable order fractional integral-differential equation based on Chebyshev polynomials*, Mathematical Problems in Engineering, 2015.

[37] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Communications in Nonlinear Science and Numerical Simulation, 64(2018), 213-231.

[38] Y. Xu, V. Ertürk, *A finite difference technique for solving variable order fractional integro-differential equations*, Bulletin of the Iranian Mathematical Society, 40(3)(2014), 699-712

[39] M. Zayernouri, G.E. Karniadakis, *Fractional spectral collocation methods for linear and nonlinear variable order FPDEs*, Journal of Computational Physics, 293(2015), 312-338.