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On the nonlinear impulsive Volterra-Fredholm integro differential equations

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Abstract

In this paper, we investigate existence and uniqueness of solutions of nonlinear Volterra-Fredholm impulsive integrodifferential equations. Utilizing theory of Picard operators we examine data dependence of solutions on initial conditions and on nonlinear functions involved in integrodifferential equations. Further, we extend the integral inequality for piece-wise continuous functions to mixed case and apply it to investigate the dependence of solution on initial data through ϵ-approximate solutions. It is seen that the uniqueness and dependency results got by means of integral inequity requires less restrictions on the functions involved in the equations than that required through Picard operators theory.

Keywords: Volterra-Fredholm equation; Integral inequality; Impulse condition; ϵ-approximate solution; Dependence of solutions.
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1. Introduction

Numerous evolution processes are described through the specific snapshots of time as they experience a difference in state unexpectedly. In such a cases span may be irrelevant in correlation with the length of the process. It is expected that in such cases these perturbations act instantaneously, means in the form of impulses. Different issues of the theoretical and practical importance lead us to consider the evolution of real processes with short-term perturbations. Such process are often described in the frameworks of differential and integrodifferential equations with impulse effect [1, 2].

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It is seen that [3] the differential equations with impulse conditions are commonly used to model the phenomena that cannot be modeled by the conventional initial value problems. In the perspective on its application the differential and integrodifferential equations with impulse effect have been analyzed by various scientist for existence, uniqueness, stability and different types data dependency by using various techniques [4, 5, 6, 7, 8, 9, 10] and the references cited therein.

Frigon and O'regan [11], using the fixed point approach proved existence results for impulsive initial value problem

$$\begin{aligned} &w'(\tau) = f\left(\tau, w(\tau)\right), \, 0 < \tau < b, \;\; \tau \neq \tau_k, \\ &\Delta w(\tau_k) = I_k\left(w(\tau_k)\right), \;\; k = 1, 2, \cdots, m, \, m \in \mathbb{N}, \\ &w(0) = w_0 \end{aligned}$$

and utilizing the idea of upper and lower solutions, authors have derived existence results for the boundary value problem

$$\begin{aligned} &w'(\tau) = f\left(\tau, w(\tau)\right), \, 0 < \tau < b, \;\; \tau \neq \tau_k, \\ &\Delta w(\tau_k) = I_k\left(w(\tau_k)\right), \;\; k = 1, 2, \cdots, m, \, m \in \mathbb{N}, \\ &w(0) = w(b), \end{aligned}$$

where $\Delta w(\tau_k) = w(\tau_k^+) - w(\tau_k^-)$, where $w(\tau_k^+) = \lim\limits_{\epsilon \to 0^+} w(\tau_k + \epsilon)$ and $w(\tau_k^-) = \lim\limits_{\epsilon \to 0^-} w(\tau_k + \epsilon)$.

Using Picard, weakly Picard operators theory Bielecki norms, Wang et al. [4], have examined nonlocal problem

$$\begin{aligned} &w'(\tau) = f\left(\tau, w(\tau)\right), \, \tau \in [0, b], \\ &w(0) = w_0 + g(w), \end{aligned}$$

for existence, uniqueness and data dependence . Authors have expanded the acquired outcomes at that point to a class of impulsive Cauchy problems by adapting the same strategies. Wang et al. [12], by applying the integral inequality of Gronwall type for piece-wise continuous functions investigated Ulam–Hyers stability for impulsive ordinary differential equations. Liu [3], studied the existence and uniqueness of mild and classical solutions for a nonlinear impulsive evolution equation

$$\begin{aligned} &w'(\tau) = \mathscr{A} w(\tau) + f\left(\tau, w(\tau)\right), \, 0 < \tau < b, \;\; \tau \neq \tau_k, \\ &\Delta w(\tau_k) = I_k\left(w(\tau_k)\right), \;\; k = 1, 2, \cdots, \;\; \tau_0 < \tau_1 < \cdots < b \\ &w(0) = w_0 \end{aligned}$$

in a Banach space X, where $\mathscr{A}$ is the generator of a strongly continuous semigroup.

Anguraj et al. [13], using semigroup theory and contraction mapping principle, proved the existence and uniqueness of the mild and classical solutions for the impulsive evolution mixed Volterra-Fredholm integrodifferential equation. Muresan [14] explored existence, uniqueness and data dependence of the solutions to mixed Volterra-Fredholm integrodifferential equation in Banach space by Utilizing Picard and weakly Picard operators' method and Bielecki norms.

On the other hand, Campiti [15] has explored various properties and a resolvent estimate for second-order differential operators with non-local Ventcel's boundary conditions. Park et al. [16] have inspected the Hyers-Ulam stability of the set-valued additive functional equations by utilizing the fixed point strategy.

It is noticed that in many of the works [17]-[26], differential and integral inequalities [25, 26] play central role in the investigation of different properties of solution such as uniqueness, boundedness,

stability etc. Motivated by works [4, 12, 13, 14], we will investigate the existence, uniqueness and continuous data dependence of solutions of nonlinear Volterra-Fredholm impulsive integrodifferential equations (VFIIDEs) of the form:

$$w'(\tau) = \mathscr{A}w(\tau) + G\left(\tau, w(\tau), \int_0^\tau F_1(\tau, \sigma, w(\sigma))d\sigma, \int_0^b F_2(\tau, \sigma, w(\sigma))d\sigma\right),$$
$$\tau \in I, \tau \neq \tau_k, k = 1, 2, \cdots, n, \tag{1.1}$$
$$w(0) = w_0,\ w_0 \in X \tag{1.2}$$
$$\Delta w(\tau_k) = I_k(w(\tau_k)),\ k = 1, 2, \cdots, n, \tag{1.3}$$

where $I = [0, b]$, $\mathscr{A} : X \to X$ is the infinitesimal generator of C_0-semigroup $\{\mathscr{T}(\tau)\}_{\tau \geq 0}$ in Banach space $(X, \|\cdot\|)$, $I_k : X \to X\ (k = 1, \cdots n)$ are continuous functions and G, F_1 and F_2 are the functions specified later. The impulsive moments τ_k are such that $0 \leq \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} \leq b, n \in \mathbb{N}$. Further, $\Delta w(\tau_k) = w(\tau_k^+) - w(\tau_k^-)$, where $w(\tau_k^+) = \lim_{h \to 0^+} w(\tau_k + h)$ and $w(\tau_k^-) = \lim_{h \to 0^-} w(\tau_k + h)$ are respectively the right and left limits of w at τ_k.

The dependence of solutions on initial conditions is firstly obtained via Picards' operator technique. Further, we extend the integral inequality for piece-wise continuous functions given in Theorem 2 of [27] for mixed case. The extended version of integral inequality we obtained then utilized to analyze the dependence of solution on initial data through ϵ-approximate solutions. It is seen that results we obtained via integral inequity regrading uniqueness and dependence of solution requires less restrictions on the nonlinear functions involved in the equations than that are demanded through Picard operators theory.

This paper is organized as follows. Section 2, relates with preliminaries. We will discuss existence, uniqueness and continuous data dependence in section 3. Section 4, deals with dependency of solutions via Picard theory. In section 5, we prove the variant of integral inequality for piece-wise continuous functions. In section 6, we provide the application of integral inequality we obtained to study of data dependence via ϵ-approximate solution to VFIIDEs. Paper finishes with concluding remarks.

2. Preliminaries

Definition 2.1 ([12, 28, 29]). *. Let (X, d) be a metric space. An operator $\mathcal{A} : X \to X$ is a Picard operator* (PO)*, if there exists $w^* \in X$ satisfying the following conditions:*

(a) $\mathbf{F}_{\mathcal{A}} = \{w^*\}$, *where* $\mathbf{F}_{\mathcal{A}} := \{w \in X : \mathcal{A}(w) = w\}$.
(b) *the sequence* $(\mathcal{A}^n(w_0))_{n \in \mathbb{N}}$ *converges to* w^* *for all* $w_0 \in X$.

Theorem 2.2 ([12, 28, 29]). *. Let (Y, d) be a complete metric space and $\mathcal{A}, \mathcal{B} : Y \to Y$ two operators. We suppose the following:*

(a) $\mathcal{A}$ *is a contraction with contraction constant* α *and* $\mathbf{F}_{\mathcal{A}} = \{w^*_{\mathcal{A}}\}$;
(b) $\mathcal{B}$ *has fixed point and* $w^*_{\mathcal{B}} \in \mathbf{F}_{\mathcal{B}}$;
(c) *there exists* $\rho > 0$ *such that* $d(\mathcal{A}(w), \mathcal{B}(w)) \leq \rho$ *for all* $w \in Y$.

Then

$$d(w^*_{\mathcal{A}}, w^*_{\mathcal{B}}) \leq \frac{\rho}{1 - \alpha}.$$

Lemma 2.3 ([27], Theorem 16.4). . *Let for $\tau \geq \tau_0$ the inequality*

$$u(\tau) \leq a(\tau) + \int_{\tau_0}^{\tau} g(\tau,\sigma)u(\sigma)d\sigma + \sum_{\tau_0<\tau_k<\tau} \beta_k(\tau)\, u(\tau_k),$$

hold, where $\beta_k(\tau)\,(k \in \mathbb{N})$ are nondecreasing functions for $\tau \geq \tau_0$, $a \in PC([\tau_0,\infty),\mathbb{R}_+)$ is a nondecreasing function, $u \in PC([\tau_0,\infty),\mathbb{R}_+)$, and $g(\tau,\sigma)$ is a continuous nonnegative function for $\tau,\sigma \geq \tau_0$ and nondecreasing with respect to τ for any fixed $\sigma \geq \tau_0$.
Then, for $\tau \geq \tau_0$ the following inequality is valid:

$$u(\tau) \leq a(\tau) \prod_{\tau_0<\tau_k<\tau} (1+\beta_k(\tau)) \exp\left(\int_{\tau_0}^{\tau} g(\tau,\sigma)d\sigma\right).$$

Theorem 2.4 ([27], Theorem 2). . *Let for $\tau \geq \tau_0$ the following inequality hold*

$$u(\tau) \leq a(\tau) + \int_{\tau_0}^{\tau} b(\tau,\sigma)u(\sigma)d\sigma + \int_{\tau_0}^{\tau}\left(\int_{\tau_0}^{\sigma} k(\tau,\sigma,\varsigma)u(\varsigma)d\varsigma\right)d\sigma + \sum_{\tau_0<\tau_k<\tau} \beta_k(\tau)\, u(\tau_k),$$

where $u,a \in PC([\tau_0,\infty),\mathbb{R}_+)$, a is a nondecreasing, $b(\tau,\sigma)$ and $k(\tau,\sigma,\varsigma)$ are continuous and nonnegative functions for $\tau,\sigma,\varsigma \geq \tau_0$ and are nondecreasing with respect to τ, $\beta_k(\tau)\,(k \in \mathbb{N})$ are nondecreasing for $\tau \geq \tau_0$. Then, for $\tau \geq \tau_0$, the following inequality is valid:

$$u(\tau) \leq a(\tau) \prod_{\tau_0<\tau_k<\tau} (1+\beta_k(\tau)) \exp\left(\int_{\tau_0}^{\tau} b(\tau,\sigma)d\sigma + \int_{\tau_0}^{\tau}\int_{\tau_0}^{\sigma} k(\tau,\sigma,\varsigma)d\varsigma d\sigma\right).$$

We need the following theorem from Pazy [30].

Theorem 2.5 ([30]). . *Let $\{\mathscr{T}(\tau)\}_{\tau\geq 0}$ is a C_0-semigroup. There exists constants $\omega \geq 0$ and $\mathfrak{M} \geq 1$ such that*

$$\|\mathscr{T}(\tau)\| \leq \mathfrak{M}e^{\omega\tau},\ 0 \leq \tau < \infty.$$

3. Existence and uniqueness

Consider the following space

$$\Theta = \{w : I \to X : w(\tau) \text{ is continuous at } \tau \neq \tau_k, \text{left continuous at } \tau = \tau_k,$$
$$\text{the right limit } w(\tau_k^+) \text{ exists for } k = 1,\cdots n,\ n \in \mathbb{N} \text{ and } w(0) = w_0\}.$$

Consider the following Banach space $\Theta_{\mathcal{PB}} = (\Theta, \|\cdot\|_{\mathcal{PB}})$, where

$$\|w\|_{\mathcal{PB}} = \sup_{\tau\in I}\left\{\frac{\|w(\tau)\|}{e^{\gamma\tau}}\right\}, w \in \Theta,\ \gamma > 0,$$

is the piece-wise Bielecki norm, and $\Theta_{\mathcal{PC}} = (\Theta, \|\cdot\|_{\mathcal{PC}})$, where $\|w\|_{\mathcal{PC}} = \sup_{t\in I}\{\|w(\tau)\|\}$, $w \in \Theta$ is the piece-wise Chebyshev norm.

Definition 3.1. *A function $w \in \Theta$ is called a mild solution of (1.1)-(1.3) if it satisfies the following impulsive integral equation*

$$w(\tau) = \mathscr{T}(\tau)w_0 + \int_0^{\tau} \mathscr{T}(\tau-\sigma)G\left(\sigma, w(\sigma), \int_0^{\sigma} F_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \int_0^{b} F_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)d\sigma$$
$$+ \sum_{0<\tau_k<\tau} \mathscr{T}(\tau-\tau_k)I_k\left(w(\tau_k)\right),\ \tau \in I. \tag{3.1}$$

We need the following hypothesis to obtain our main results.

(H1) Let $G : I \times X \times X \times X \to X$ be continuous function and there exist constant $L_G > 0$ such that

$$\|G(\tau, v_1, v_2, v_3) - G(\tau, w_1, w_2, w_3)\| \leq L_G \left(\sum_{i=1}^{3} \|v_i - w_i\| \right),$$

for all $\tau \in I$ and $v_i, w_i \in X$ $(i = 1, 2, 3)$.

(H2) Let F_j $(j = 1, 2) : I \times I \times X \to X$ are continuous functions and there exist constants L_{F_j} $(j = 1, 2) > 0$ such that

$$\|F_j(\tau, \sigma, v_1) - F_j(\tau, \sigma, w_1)\| \leq L_{F_j} \|v_1 - w_1\|, \ j = 1, 2;$$

for all $\tau, \sigma \in I$ and $v_1, w_1 \in X$.

(H3) There exist constant $L_{I_k} > 0$ such that $\|I_k(v) - I_k(w)\| \leq L_{I_k} \|v - w\|$; for $v, w \in X$, $(k = 1, \cdots, n)$.

Theorem 3.2. *Suppose that hypothesis* (H1)-(H3) *are holds and there exist constant* $\gamma > 0$ *such that*

$$L_{\mathcal{R}} = \frac{\mathscr{M} L_G}{\gamma} \left[\left(1 - e^{-\gamma b}\right) \left(1 + \frac{L_{F_1}}{\gamma}\right) + L_{F_2} b e^{\gamma b} \right] + \mathscr{M} e^{\gamma b} \sum_{k=1}^{n} L_{I_k} < 1.$$

Then the VFIIDE (1.1)-(1.3) *has a unique solution in* $\Theta_{\mathcal{PB}}$.

Proof .

Define the operator $\mathcal{R} : \Theta_{\mathcal{PB}} \to \Theta_{\mathcal{PB}}$, $\Theta_{\mathcal{PB}} = (\Theta, \|\cdot\|_{\mathcal{PB}})$ by

$$\begin{aligned} \mathcal{R}(w)(\tau) = \mathscr{T}(\tau) w_0 + \int_0^\tau \mathscr{T}(\tau - \sigma) G \left(\sigma, w(\sigma), \int_0^\sigma F_1(\sigma, \varsigma, w(\varsigma)) d\varsigma, \int_0^b F_2(\sigma, \varsigma, w(\varsigma)) d\varsigma \right) d\sigma \\ + \sum_{0 < \tau_k < \tau} \mathscr{T}(\tau - \tau_k) I_k \left(w(\tau_k) \right), \ \tau \in [0, b]. \end{aligned}$$

Then fixed point of the operator $\mathcal{R}$ is the solution of the problem (1.1)-(1.3). Let any $w, v \in \Theta$ and $\tau \in [0, b]$. Then

$$\begin{aligned} &\|\mathcal{R}(w)(\tau) - \mathcal{R}(v)(\tau)\| \\ &\leq \int_0^\tau \|\mathscr{T}(\tau - \sigma)\|_{B(X)} \left\| G \left(\sigma, w(\sigma), \int_0^\sigma F_1(\sigma, \varsigma, w(\varsigma)) d\varsigma, \int_0^b F_2(\sigma, \varsigma, w(\varsigma)) d\varsigma \right) \right. \\ &\quad \left. - G \left(\sigma, v(\sigma), \int_0^\sigma F_1(\sigma, \varsigma, v(\varsigma)) d\varsigma, \int_0^b F_2(\sigma, \varsigma, v(\varsigma)) d\varsigma \right) \right\| d\sigma \\ &\quad + \sum_{0 < \tau_k < \tau} \|\mathscr{T}(\tau - \tau_k)\|_{B(X)} \|I_k \left(w(\tau_k) \right) - I_k \left(v(\tau_k) \right)\|. \end{aligned} \tag{3.2}$$

By the Theorem 2.5 there exist constant $\mathscr{M} \geq 1$ such that

$$\|\mathscr{T}(\tau)\|_{B(X)} \leq \mathscr{M}, \ \tau \geq 0. \tag{3.3}$$

Using hypothesis (H1)-(H3) and the condition (3.3) to the inequality (3.2), we have

$\|\mathcal{R}(w)(\tau) - \mathcal{R}(v)(\tau)\|$

$$
\begin{aligned}
&\leq \int_0^\tau \mathscr{M}\, L_G \left[\|w(\sigma)-v(\sigma)\|\, e^{-\gamma\sigma}\right] e^{\gamma\sigma} d\sigma + \int_0^\tau \int_0^\sigma \mathscr{M}\, L_G\, L_{F_1} \left[\|w(\varsigma)-v(\varsigma)\|\, e^{-\gamma\varsigma}\right] e^{\gamma\varsigma} d\varsigma d\sigma \\
&\quad + \int_0^\tau \int_0^b \mathscr{M}\, L_G\, L_{F_2} \left[\|w(\varsigma)-v(\varsigma)\|\, e^{-\gamma\varsigma}\right] e^{\gamma\varsigma} d\varsigma d\sigma + \sum_{0<\tau_k<\tau} \mathscr{M}\, L_{I_k} \left[\|w(\tau_k)-v(\tau_k)\|\, e^{-\gamma\tau_k}\right] e^{\gamma\tau_k} \\
&\leq \int_0^\tau \mathscr{M}\, L_G \|w-v\|_{\mathcal{PB}}\, e^{\gamma\sigma} d\sigma + \int_0^\tau \int_0^\sigma \mathscr{M}\, L_G\, L_{F_1} \|w-v\|_{\mathcal{PB}}\, e^{\gamma\varsigma} d\varsigma d\sigma \\
&\quad + \int_0^\tau \int_0^b \mathscr{M}\, L_G\, L_{F_2} \|w-v\|_{\mathcal{PB}}\, e^{\gamma\varsigma} d\varsigma d\sigma + \mathscr{M} \sum_{k=1}^{n} L_{I_k} e^{\gamma\tau_k}\, L_{I_k} \|w-v\|_{\mathcal{PB}} \\
&= \left\{ \mathscr{M}\, L_G \left(\frac{e^{\gamma\tau}}{\gamma} - \frac{1}{\gamma}\right) + \mathscr{M}\, L_G L_{F_1} \left(\frac{e^{\gamma\tau}}{\gamma^2} - \frac{1}{\gamma^2} - \frac{\tau}{\gamma}\right) \right. \\
&\qquad \left. + \mathscr{M}\, L_G L_{F_2} \left(\frac{e^{\gamma b}}{\gamma} - \frac{1}{\gamma}\right)\tau + \mathscr{M} \sum_{k=1}^{n} e^{\gamma\tau_k}\, L_{I_k} \right\} \|w-v\|_{\mathcal{PB}} \\
&\leq \left\{ \mathscr{M}\, L_G \left(\frac{e^{\gamma\tau}}{\gamma} - \frac{1}{\gamma}\right) + \mathscr{M}\, L_G L_{F_1} \left(\frac{e^{\gamma\tau}}{\gamma^2} - \frac{1}{\gamma^2}\right) + \mathscr{M}\, L_G L_{F_2}\, b \frac{e^{\gamma b}}{\gamma} + \mathscr{M}\, e^{\gamma b} \sum_{k=1}^{n} L_{I_k} \right\} \|w-v\|_{\mathcal{PB}}, \tau \in I.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{R}(w)(\tau) - \mathcal{R}(v)(\tau)\|\, e^{-\gamma\tau} &\leq \left\{ \mathscr{M}\, L_G \frac{1-e^{-\gamma\tau}}{\gamma} + \mathscr{M}\, L_G L_{F_1} \frac{1-e^{-\gamma\tau}}{\gamma^2} \right. \\
&\qquad \left. + \mathscr{M}\, L_G L_{F_2} \frac{b}{\gamma} e^{\gamma b} e^{-\gamma\tau} + \mathscr{M}\, e^{\gamma b}\, e^{-\gamma\tau} \sum_{k=1}^{n} L_{I_k} \right\} \|w-v\|_{\mathcal{PB}} \\
&\leq \left\{ \mathscr{M}\, L_G \frac{1-e^{-\gamma\tau}}{\gamma} + \mathscr{M}\, L_G L_{F_1} \frac{1-e^{-\gamma\tau}}{\gamma^2} \right. \\
&\qquad \left. + \mathscr{M}\, L_G L_{F_2} \frac{b}{\gamma} e^{\gamma b} + \mathscr{M}\, e^{\gamma b} \sum_{k=1}^{n} L_{I_k} \right\} \|w-v\|_{\mathcal{PB}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathcal{R}(w) - \mathcal{R}(v)\|_{\mathcal{PB}} &= \sup_{\tau\in I} \left\{ \frac{\|\mathcal{R}(w)(\tau) - \mathcal{R}(v)(\tau)\|}{e^{\gamma\tau}} \right\} \\
&\leq \left(\frac{\mathscr{M}\, L_G}{\gamma} \left[\left(1 - e^{-\gamma b}\right) \left(1 + \frac{L_{F_1}}{\gamma}\right) + L_{F_2}\, b\, e^{\gamma b} \right] + \mathscr{M}\, e^{\gamma b} \sum_{k=1}^{n} L_{I_k} \right) \|w-v\|_{\mathcal{PB}} \\
&= L_{\mathcal{R}} \|w-v\|_{\mathcal{PB}}.
\end{aligned}
$$

Choose $\gamma > 0$ such that

$$
L_{\mathcal{R}} = \frac{\mathscr{M}\, L_G}{\gamma} \left[\left(1 - e^{-\gamma b}\right) \left(1 + \frac{L_{F_1}}{\gamma}\right) + L_{F_2}\, b\, e^{\gamma b} \right] + \mathscr{M}\, e^{\gamma b} \sum_{k=1}^{n} L_{I_k} < 1.
$$

Then $\mathcal{R}$ is contraction operator. By Banach fixed point theorem it has a fixed point $\bar{w} \in \Theta_{\mathcal{PB}}$ which is unique solution of VFIIDE (1.1)-(1.3). □

4. Dependency of solutions via PO

In this section, we analyse the dependency of solutions on the initial condition and functions in the equations by means of Picard operator theory.

Consider the following problem

$$w'(\tau) = \mathscr{A}w(\tau) + \widehat{G}\left(\tau, w(\tau), \int_0^\tau \widehat{F}_1(\tau,\sigma,w(\sigma))d\sigma, \int_0^b \widehat{F}_2(\tau,\sigma,w(\sigma))d\sigma\right),$$
$$\tau \in I, \tau \neq \tau_k, k = 1,2,\cdots,n, \tag{4.1}$$
$$w(0) = \widehat{w}_0,\ \widehat{w}_0 \in X \tag{4.2}$$
$$\Delta w(\tau_k) = \hat{I}_k(w(\tau_k)),\ k = 1,2,\cdots,n, \tag{4.3}$$

where $\widehat{G} : I \times X \times X \times X \to X$, $\widehat{F}_j\,(j = 1,2) : I \times I \times X \to X$ and $\hat{I}_k : X \to X\,(k = 1,\cdots n)$ are the continuous functions.

A function $w \in \Theta$ is called a mild solution of (4.1)-(4.3) if it satisfies the following impulsive integral equation

$$w(\tau) = \mathscr{T}(\tau)\widehat{w}_0 + \int_0^\tau \mathscr{T}(\tau-\sigma)\widehat{G}\left(\sigma, w(\sigma), \int_0^\sigma \widehat{F}_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \int_0^b \widehat{F}_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)d\sigma$$
$$+ \sum_{0<\tau_k<\tau} \mathscr{T}(\tau-\tau_k)\hat{I}_k\left(w(\tau_k)\right),\ \tau \in [0,b]. \tag{4.4}$$

Theorem 4.1. *Suppose that the following conditions are holds*

(A1) *All the conditions in Theorem 3.2 are satisfied and $w^* \in \Theta$ is the unique solution of the integral equation* (3.1).

(A2) *There exists constants $L_{\widehat{G}}, L_{\widehat{F}_1}, L_{\widehat{F}_2} > 0$ such that*

$$\left\|\widehat{G}(\tau,w_1,w_2,w_3) - \widehat{G}(\tau,v_1,v_2,v_3)\right\| \leq L_{\widehat{G}}\left(\sum_{i=1}^{3}\|w_i - v_i\|\right)$$

and

$$\left\|L_{\widehat{F}_j}(\tau,\sigma,w_1) - L_{\widehat{F}_j}(\tau,\sigma,v_1)\right\| \leq L_{L_{\widehat{F}_j}}\left(\|w_1 - v_1\|\right),\ j = 1,2;$$

for all $\tau,\sigma \in I$ and $w_i, v_i \in X\,(i = 1,2,3)$.

(A3) *There exist constant $L_{\hat{I}_k}$ such that $\left\|\hat{I}_k(w) - \hat{I}_k(v)\right\| \leq L_{\hat{I}_k}\|v - w\|$; for $v, w \in X$.*

(A4) *There exists a constants $\mu,\ \eta > 0$ such that*

$$\left\|G(\tau,u,v,w) - \widehat{G}(\tau,u,\tilde{v},\tilde{w})\right\| \leq \mu$$

and

$$\left\|I_k(w) - \hat{I}_k(w)\right\| \leq \eta;$$

for all $\tau \in I$ and $u,\ v,\ \tilde{v},\ w,\ \tilde{w} \in X$.

Then, if v^* is the solution of integral equations (4.4) *then*

$$\|w^* - v^*\|_{\mathcal{PB}} \leq \frac{\mathscr{M}\|w_0 - \hat{w}_0\| + b\,\mathscr{M}\,\mu + n\,\mathscr{M}\,\eta}{1 - L_{\mathcal{R}}}. \tag{4.5}$$

Proof .

Define the operators $\mathcal{R}, \mathcal{T} : (\Theta, \|\cdot\|_{\mathcal{PB}}) \to (\Theta, \|\cdot\|_{\mathcal{PB}})$ defined by

$$\mathcal{R}(w)(\tau) = \mathscr{T}(\tau)w_0 + \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)d\sigma + \sum_{0<\tau_k<\tau} \mathscr{T}(\tau-\tau_k)I_k\left(w(\tau_k)\right)$$

and

$$\mathcal{T}w(\tau) = \mathscr{T}(\tau)\widehat{w}_0 + \int_0^\tau \mathscr{T}(\tau-\sigma)\widehat{G}\left(\sigma, w(\sigma), \int_0^\sigma \widehat{F}_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \int_0^b \widehat{F}_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)d\sigma + \sum_{0<\tau_k<\tau} \mathscr{T}(\tau-\tau_k)\hat{I}_k\left(w(\tau_k)\right).$$

With (A1), it is already prove that $\mathcal{R}$ is a contraction. On the similar line $\mathcal{T}$ is contraction provided that

$$L_{\mathcal{T}} = \frac{\mathscr{M}\, L_{\widehat{G}}}{\gamma}\left[\left(1-e^{-\gamma\, b}\right)\left(1+\frac{L_{\widehat{F}_1}}{\gamma}\right) + L_{\widehat{F}_2}\, b\, e^{\gamma\, b}\right] + \mathscr{M}\ e^{\gamma\, b}\sum_{k=1}^{n} L_{\hat{I}_k} < 1.$$

Let $\mathbf{F}_{\mathcal{R}} = \{w^*\}$ and $\mathbf{F}_{\mathcal{T}} = \{v^*\}$. For any $w \in \Theta$. Then any $\tau \in I$, we have

$$\begin{aligned}
&\|\mathcal{R}(w)(\tau) - \mathcal{T}(w)(\tau)\| \\
&\leq \|\mathscr{T}(\tau)\|_{B(X)}\, \|w_0 - \hat{w}_0\| + \int_0^\tau \|\mathscr{T}(\tau-\sigma)\|_{B(X)} \left\| G\left(\sigma, w(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \right.\right. \\
&\qquad \left.\left. \int_0^b F_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right) - \widehat{G}\left(\sigma, w(\sigma), \int_0^\sigma \widehat{F}_1(\sigma,\varsigma,w(\varsigma))d\varsigma, \int_0^b \widehat{F}_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)\right\| \\
&\qquad + \sum_{0<\tau_k<\tau} \|\mathscr{T}(\tau-\tau_k)\|_{B(X)} \left\| I_k\left(w(\tau_k)\right) - \hat{I}_k\left(w(\tau_k)\right)\right\|.
\end{aligned}$$

In the view of assumptions (A4), we have

$$\|\mathcal{R}(w)(\tau) - \mathcal{T}(w)(\tau)\| \leq \mathscr{M}\, \|w_0 - \hat{w}_0\| + b\, \mathscr{M}\, \mu + n\, \mathscr{M}\, \eta.$$

Therefore,

$$\begin{aligned}
\|\mathcal{R}(w) - \mathcal{T}(w)\|_{\mathcal{PB}} &= \sup_{\tau\in I}\left\{\frac{\|\mathcal{R}(w)(\tau) - \mathcal{T}(w)(\tau)\|}{e^{\gamma\tau}}\right\} \\
&\leq \mathscr{M}\, \|w_0 - \hat{w}_0\| + b\, \mathscr{M}\, \mu + n\, \mathscr{M}\, \eta. \qquad (4.6)
\end{aligned}$$

Applying the Theorem 2.2, to the inequality (4.6), we obtain

$$\|w^* - v^*\|_{\mathcal{PB}} \leq \frac{\mathscr{M}\, \|w_0 - \hat{w}_0\| + b\, \mathscr{M}\, \mu + n\, \mathscr{M}\, \eta}{1 - L_{\mathcal{R}}},$$

which is desired inequality (4.5). □

5. Extended version of integral inequality for piece-wise continuous functions

In this section, firstly we extended the integral inequality given in the Theorem 2.4 to the mixed case, so as the results related to dependency of solutions on different data can be obtained for mixed Volterra-Fredholm integrodifferential equations with impulses.

Theorem 5.1. *Let $\tau \in [0,b]$, the following integral inequality hold*

$$u(\tau) \le a(\tau) + \int_0^\tau b(\tau,\sigma)u(\sigma)d\sigma + \int_0^\tau \left(\int_0^\sigma k_1(\tau,\sigma,\varsigma)u(\varsigma)d\varsigma\right)d\sigma + \int_0^\tau \left(\int_0^b k_2(\tau,\sigma,\varsigma)u(\varsigma)d\varsigma\right)d\sigma + \sum_{0<\tau_k<\tau} \beta_k(\tau)u(\tau_k) \tag{5.1}$$

where $u, a \in PC([0,b],\mathbb{R}_+)$, a is a nondecreasing, $b(\tau,\sigma)$, $k_1(\tau,\sigma,\varsigma)$ and $k_2(\tau,\sigma,\varsigma)$ are continuous and nonnegative functions for $\tau,\sigma,\varsigma \in [0,\tau]$ and are nondecreasing with respect to τ, $\beta_k(\tau)$ $(k \in \mathbb{N})$ are nondecreasing for $\tau \in [0,\tau]$. Then, for $\tau \in [0,\tau]$, the following inequality is valid:

$$u(\tau) \le a(\tau) \prod_{0<\tau_k<\tau} (1+\beta_k(\tau)) \exp\left(\int_0^\tau b(\tau,\sigma,)d\sigma + \int_0^\tau \int_0^\sigma k_1(\tau,\sigma,\varsigma)d\varsigma d\sigma + \int_0^\tau \int_0^b k_2(\tau,\sigma,\varsigma)d\varsigma d\varsigma\right). \tag{5.2}$$

Proof . Denote the right hand side of following inequality (5.1) by $\mathscr{V}(\tau)$

$$\mathscr{V}(\tau) = a(\tau) + \int_0^\tau b(\tau,\sigma)u(\sigma)d\sigma + \int_0^\tau \left(\int_0^\sigma k_1(\tau,\sigma,\varsigma)u(\varsigma)d\varsigma\right)d\sigma + \int_0^\tau \left(\int_0^b k_2(\tau,\sigma,\varsigma)u(\varsigma)d\varsigma\right)d\sigma + \sum_{0<\tau_k<\tau} \beta_k(\tau)u(\tau_k).$$

Then the function $\mathscr{V}(\tau) \in PC([0,b],\mathbb{R}_+)$ is nondecreasing, $u(\tau) \le \mathscr{V}(\tau)$ and

$$\begin{aligned}\mathscr{V}(\tau) &\le a(\tau) + \int_0^\tau b(\tau,\sigma)\mathscr{V}(\sigma)d\sigma + \int_0^\tau \left(\int_0^\sigma k_1(\tau,\sigma,\varsigma)\mathscr{V}(\varsigma)d\varsigma\right)d\sigma + \int_0^\tau \left(\int_0^b k_2(\tau,\sigma,\varsigma)\mathscr{V}(\varsigma)d\varsigma\right)d\sigma \\ &\quad + \sum_{0<\tau_k<\tau} \beta_k(\tau)\mathscr{V}(\tau_k) \\ &\le a(\tau) + \int_0^\tau b(\tau,\sigma)\mathscr{V}(\sigma)d\sigma + \int_0^\tau \left(\int_0^\sigma k_1(\tau,\sigma,\varsigma)\mathscr{V}(\sigma)d\varsigma\right)d\sigma + \int_0^\tau \left(\int_0^b k_2(\tau,\sigma,\varsigma)\mathscr{V}(\sigma)d\varsigma\right)d\sigma \\ &\quad + \sum_{0<\tau_k<\tau} \beta_k(\tau)\mathscr{V}(\tau_k) \\ &= a(\tau) + \int_0^\tau \left[f(\tau,\sigma) + \int_0^s p(\tau,\sigma,\varsigma)d\varsigma + \int_0^b q(\tau,\sigma,\varsigma)d\varsigma\right]\mathscr{V}(\sigma)d\sigma + \sum_{0<t_k<t} \beta_k(\tau)\mathscr{V}(\tau_k). \end{aligned} \tag{5.3}$$

Applying Lemma 2.3 to the inequality (5.3), with

$$\begin{aligned} u(\tau) &= \mathscr{V}(\tau), \\ a(\tau) &= a(\tau), \\ g(\tau,\sigma) &= f(\tau,\sigma) + \int_0^\sigma p(\tau,\sigma,\varsigma)d\varsigma + \int_0^b q(\tau,\sigma,\varsigma)d\varsigma, \\ \beta_k(\tau) &= \beta_k(\tau), \end{aligned}$$

we obtain

$$\mathscr{V}(\tau) \le a(\tau) \prod_{0<\tau_k<\tau} (1+\beta_k(\tau)) \exp\left(\int_0^\tau b(\tau,\sigma)d\sigma + \int_0^\tau \int_0^\sigma k_1(\tau,\sigma,\varsigma)d\varsigma\, d\sigma + \int_0^\tau \int_0^b k_2(\tau,\sigma,\varsigma)d\varsigma\, d\sigma\right). \tag{5.4}$$

From the inequality (5.4), we obtain the desired inequality (5.3). □

6. Applications of mixed version of integral inequality

In this section, we give the application of the mixed version of integral inequality to examine the continuous dependence of solutions on initial data and functions involved in equation. Further, we analyse the dependency by means of concept of ϵ-approximate solutions and utilizing mixed version of integral inequality.

Theorem 6.1. *Suppose that the hypothesis* (H1)-(H3) *and* (A4) *are satisfied. Let w and v are the mild solutions of* (1.1)-(1.2) *and* (4.1)-(4.2) *respectively. Then*

$$\begin{aligned}\|w-v\|_{\mathcal{PB}} \leq{}& (\mathscr{M}\,\|w_0-\widehat{w}_0\|+b\,\mathscr{M}\,\mu+n\,\mathscr{M}\,\eta)\\ &\prod_{k=1}^{n}(1+\mathscr{M}\,L_{I_k})\exp\left(\mathscr{M}\,L_G\,b+\mathscr{M}\,L_G\,L_{F_1}\frac{b^2}{2}+\mathscr{M}\,L_G\,L_{F_2}\,b^2\right).\end{aligned}\tag{6.1}$$

Proof .

Let w and v be the mild solution of (1.1)-(1.3) and (4.1)-(4.3) respectively. Then utilizing hypothesis (H1), (H2), (H3) and (A4), we get

$$\begin{aligned}&\|w(\tau)-v(\tau)\|\\ &\leq \|\mathscr{T}(\tau)\|_{B(X)}\,\|w_0-\widehat{w}_0\|+\int_0^\tau \|\mathscr{T}(\tau-\sigma)\|_{B(X)}\left\|G\left(\sigma,w(\sigma),\int_0^\sigma F_1(\sigma,\varsigma,w(\varsigma))d\varsigma,\right.\right.\\ &\quad\left.\left.\int_0^b F_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)-\widehat{G}\left(\sigma,v(\sigma),\int_0^\sigma \widehat{F}_1(\sigma,\varsigma,v(\varsigma))d\varsigma,\int_0^b \widehat{F}_2(\sigma,\varsigma,v(\varsigma))d\varsigma\right)\right\|d\sigma\\ &\quad+\sum_{0<\tau_k<\tau}\|\mathscr{T}(\tau-\tau_k)\|_{B(X)}\left\|I_k\left(w(\tau_k)\right)-\hat{I}_k(v(\tau_k))\right\|\\ &\leq \|\mathscr{T}(\tau)\|_{B(X)}\,\|w_0-\widehat{w}_0\|+\int_0^\tau \|\mathscr{T}(\tau-\sigma)\|_{B(X)}\left\|G\left(\sigma,w(\sigma),\int_0^\sigma F_1(\sigma,\varsigma,w(\varsigma))d\varsigma,\right.\right.\\ &\quad\left.\left.\int_0^b F_2(\sigma,\varsigma,w(\varsigma))d\varsigma\right)-G\left(\sigma,v(\sigma),\int_0^\sigma F_1(\sigma,\varsigma,v(\varsigma))d\varsigma,\int_0^b F_2(\sigma,\varsigma,v(\varsigma))d\varsigma\right)\right\|d\sigma\\ &\quad+\int_0^\tau \|\mathscr{T}(\tau-\sigma)\|_{B(X)}\left\|G\left(\sigma,v(\sigma),\int_0^\sigma F_1(\sigma,\varsigma,v(\varsigma))d\varsigma,\right.\right.\\ &\quad\left.\left.\int_0^b F_2(\sigma,\varsigma,v(\varsigma))d\varsigma\right)-\widehat{G}\left(\sigma,v(\sigma),\int_0^\sigma \widehat{F}_1(\sigma,\varsigma,v(\varsigma))d\varsigma,\int_0^b \widehat{F}_2(\sigma,\varsigma,v(\varsigma))d\varsigma\right)\right\|d\sigma\\ &\quad+\sum_{0<\tau_k<\tau}\|\mathscr{T}(\tau-\tau_k)\|_{B(X)}\,\|I_k\left(w(\tau_k)\right)-I_k(v(\tau_k))\|\\ &\quad+\sum_{0<\tau_k<\tau}\|\mathscr{T}(\tau-\tau_k)\|_{B(X)}\left\|I_k\left(v(\tau_k)\right)-\hat{I}_k(v(\tau_k))\right\|\\ &\leq \mathscr{M}\,\|w_0-\widehat{w}_0\|+\tau\mathscr{M}\,\mu+n\,\mathscr{M}\,\eta+\int_0^\tau \mathscr{M}L_G\,\|w(\sigma)-v(\sigma)\|\,d\sigma+\int_0^\tau\int_0^\sigma \mathscr{M}L_GL_{F_1}\\ &\quad\|w(\varsigma)-v(\varsigma)\|\,d\varsigma d\sigma+\int_0^\tau\int_0^b \mathscr{M}L_GL_{F_2}\,\|w(\varsigma)-v(\varsigma)\|\,d\varsigma d\sigma+\sum_{0<\tau_k<\tau}\mathscr{M}\,L_{I_k}\,\|w(\tau_k)-v(\tau_k)\|.\end{aligned}\tag{6.2}$$

Applying impulsive inequality from the Theorem 5.1 to (6.2) with

$$\begin{aligned}u(\tau)&=\|w(\tau)-v(\tau)\|,\\ a(\tau)&=\mathscr{M}\,\|w_0-\widehat{w}_0\|+\tau\mathscr{M}\,\mu+n\,\mathscr{M}\,\eta,\\ b(\tau,\sigma)&=\mathscr{M}\,L_G,\end{aligned}$$

$$k_1(\tau,\sigma,\varsigma) = \mathscr{M}\, L_G\, L_{F_1},$$
$$k_2(\tau,\sigma,\varsigma) = \mathscr{M}\, L_G\, L_{F_2},$$
$$\beta_k(\tau) = \mathscr{M} L_{I_k},$$

we obtain

$$\begin{aligned}
&\|w(\tau) - v(\tau)\| \\
&\leq (\mathscr{M}\,\|w_0 - \widehat{w}_0\| + \tau\mathscr{M}\mu + n\,\mathscr{M}\,\eta) \prod_{0<\tau_k<0} (1 + \mathscr{M}\, L_{I_k}) \exp\left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2\right) \\
&\leq (\mathscr{M}\,\|w_0 - \widehat{w}_0\| + b\,\mathscr{M}\,\mu + n\,\mathscr{M}\,\eta) \prod_{k=1}^{n} 1 + \mathscr{M}\, L_{I_k}) \exp\left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2\right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|w - v\|_{\mathcal{PC}} &= \sup_{\tau\in I}\left\{\frac{\|w(\tau) - v(\tau)\|}{e^{\gamma\tau}}\right\} \\
&\leq (\mathscr{M}\,\|w_0 - \widehat{w}_0\| + b\,\mathscr{M}\,\mu + n\,\mathscr{M}\,\eta) \times \\
&\times \prod_{k=1}^{n}(1 + \mathscr{M}\, L_{I_k}) \exp\left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2\right),
\end{aligned}$$

which is desired inequality (6.1). □

Definition 6.2. *For a given constant $\epsilon \geq 0$, a function $w \in \Theta_{\mathcal{PB}}$ satisfying the inequality*

$$\left\| w'(\tau) - \mathscr{A} w(\tau) - G\left(\tau, w(\tau), \int_0^\tau F_1(\tau,\sigma,w(\sigma))d\sigma, \int_0^b F_2(\tau,\sigma,w(\sigma))d\sigma\right)\right\| \leq \epsilon,\ \tau \in I.$$

subject to $w(0) = w_0$ and $\Delta w(\tau_k) = I_k(w(\tau_k)),\ k = 1, 2, \cdots, n$, is called a ϵ-approximate solution of the VFIIDE (1.1).

Theorem 6.3. *Assume that (H1)-(H3) holds. If $w_j(\tau)$, $(j = 1, 2)$ be ϵ_j-approximate solutions of VFIIDE (1.1) corresponding to $w^j(0) = w_0^j \in X$, $\Delta w_j(\tau_k) = I_k(w_j(\tau_k)) \in X,\ k = 1, 2, \cdots, n$ respectively. Then*

$$\begin{aligned}
\|w_1 - w_2\|_{\mathcal{PB}} \leq \{&(\epsilon_1 + \epsilon_2)\ \mathscr{M}(b + n) + \mathscr{M}\left\|w_0^1 - w_0^2\right\|\} \\
&\prod_{k=1}^{n}(1 + \mathscr{M}\, L_{I_k}) \exp\left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2\right). \qquad (6.3)
\end{aligned}$$

Proof . Let $w_j(\tau)$, $(j = 1, 2)$ be ϵ_j-approximate solutions of VFIIDE (1.1) corresponding to $w^j(0) = w_0^j \in X$, $\Delta w_j(\tau_k) = I_k(w_j(\tau_k)) \in X,\ k = 1, 2, \cdots, n$ respectively. Then we have

$$\left\| w_j'(\tau) - \mathscr{A} w_j(\tau) - G\left(\tau, w_j(\tau), \int_0^\tau F_1(\tau,\sigma,w_j(\sigma))d\sigma, \int_0^b F_2(\tau,\sigma,w_j(\sigma))d\sigma\right)\right\| \leq \epsilon_j,\ \tau \in I. \qquad (6.4)$$

Then there exist $P_{w_j} \in PC(I, X)$ and a sequence $(P_{w_j})_k$ (dependence on w_j) such that

(i) $\left\|P_{w_j}(\tau)\right\| \leq \epsilon_j,\ \tau \in I$ and $\left\|(P_{w_j})_k\right\| \leq \epsilon_j,\ k = 1, 2, \cdots, n.$

(ii) $w_j'(\tau) = \mathscr{A} w_j(\tau) + G\left(\tau, w_j(\tau), \int_0^\tau F_1(\tau,\sigma,w_j(\sigma))d\sigma, \int_0^b F_2(\tau,\sigma,w_j(\sigma))d\sigma\right) + P_{w_j}(\tau),\ \tau \in I.$

(iii) $\Delta w_j(\tau_k) = I_k(w_j(\tau_k)) + (P_{w_j})_k,\ k = 1, 2, \cdots, n.$

This gives

$$
\begin{aligned}
w_j(\tau) &= \mathscr{T}(\tau)w_0^j + \int_0^\tau \mathscr{T}(\tau-\sigma)\left[G\left(\sigma, w_j(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w_j(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_j(\varsigma))d\varsigma\right) + P_{w_j}(\sigma)\right]d\sigma \\
&\quad + \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)\left[I_k\left(w_j(\tau_k)\right) + (P_{w_j})_k\right] \\
&\Bigg\| w_j(\tau) - \mathscr{T}(\tau)w_0^j - \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w_j(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w_j(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_j(\varsigma))d\varsigma\right)d\sigma \\
&\quad \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)I_k\left(w_j(\tau_k)\right)\Bigg\| \\
&\leq \int_0^\tau \|\mathscr{T}(\tau-\sigma)\| \left\|P_{w_j}(\sigma)\right\| d\sigma + \sum_{k=1}^{n} \|\mathscr{T}(\tau-\tau_k)\| \left\|(P_{w_j})_k\right\| \\
&\leq \tau\mathscr{M}\epsilon_j + \mathscr{M}\, n\, \epsilon_j = \epsilon_j\, \mathscr{M}(\tau+n),\ j=1,2,\ \tau\in I. \qquad (6.5)
\end{aligned}
$$

Therefore from (6.5) we have

$$
\begin{aligned}
&(\epsilon_1+\epsilon_2)\ \mathscr{M}(\tau+n) \\
&\geq \Bigg\| w_1(\tau) - \mathscr{T}(\tau)w_0^1 - \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w_1(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w_1(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_1(\varsigma))d\varsigma\right)d\sigma \\
&\quad + \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)I_k\left(w_1(\tau_k)\right)\Bigg\| + \Bigg\| w_2(\tau) - \mathscr{T}(\tau)w_0^2 - \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w_2(\sigma),\right. \\
&\quad \left.\int_0^\sigma F_1(\sigma,\varsigma,w_2(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_2(\varsigma))d\varsigma\right)d\sigma + \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)I_k\left(w_2(\tau_k)\right)\Bigg\|. \qquad (6.6)
\end{aligned}
$$

As we know for any $\xi_1, \xi_2 \in X$, $\|\xi_1-\xi_2\| \leq \|\xi_1\| + \|\xi_2\|$ and $|\|\xi_1\| - \|\xi_2\|| \leq \|\xi_1-\xi_2\|$. Using this in Eq.(6.6), we get

$$
\begin{aligned}
&(\epsilon_1+\epsilon_2)\ \mathscr{M}(\tau+n) \\
&\geq \Bigg\| \Bigg\{ w_1(\tau) - \mathscr{T}(\tau)w_0^1 - \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w_1(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w_1(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_1(\varsigma))d\varsigma\right)d\sigma \\
&\quad - \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)I_k\left(w_1(\tau_k)\right)\Bigg\} - \Bigg\{ w_2(\tau) - \mathscr{T}(\tau)w_0^2 - \int_0^\tau \mathscr{T}(\tau-\sigma)G\left(\sigma, w_2(\sigma)\right. \\
&\quad \left.\int_0^\sigma F_1(\sigma,\varsigma,w_2(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_2(\varsigma))d\varsigma\right)d\sigma - \sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)I_k\left(w_2(\tau_k)\right)\Bigg\}\Bigg\| \\
&= \Bigg\| [w_1(\tau)-w_2(\tau)] - \left(\mathscr{T}(\tau)\left[w_0^1-w_0^2\right]\right) - \Bigg\{ \int_0^\tau \mathscr{T}(\tau-\sigma)\Bigg[G\left(\sigma, w_1(\sigma), \int_0^\sigma F_1(\sigma,\varsigma,w_1(\varsigma))d\varsigma,\right. \\
&\quad \left.\int_0^b F_2(\sigma,\varsigma,w_1(\varsigma))d\sigma\right) - G\left(\sigma, w_2(\sigma)\int_0^\sigma F_1(\sigma,\varsigma,w_2(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_2(\varsigma))d\varsigma\right)\Bigg]d\sigma\Bigg\} \\
&\quad - \left\{\sum_{k=1}^{n} \mathscr{T}(\tau-\tau_k)\left[I_k\left(w_1(\tau_k)\right) - I_k\left(w_2(\tau_k)\right)\right]\right\}\Bigg\| \\
&\geq \|w_1(\tau)-w_2(\tau)\| - \left\|\mathscr{T}(\tau)\left[w_0^1-w_0^2\right]\right\| - \Bigg\| \int_0^\tau \mathscr{T}(\tau-\sigma)\Bigg[G\left(\sigma, w_1(\sigma)\int_0^\sigma F_1(\sigma,\varsigma,w_1(\varsigma))d\varsigma\right. \\
&\quad \left.\int_0^b F_2(\sigma,\varsigma,w_1(\varsigma))d\varsigma\right) - G\left(\sigma, w_2(\sigma)\int_0^\sigma F_1(\sigma,\varsigma,w_2(\varsigma))d\varsigma, \int_0^b F_2(\sigma,\varsigma,w_2(\varsigma))d\varsigma\right)\Bigg]d\sigma\Bigg\|
\end{aligned}
$$

$$- \left\| \sum_{k=1}^{n} \mathscr{T}(\tau - \tau_k) \left[I_k \left(w_1(\tau_k) \right) - I_k \left(w_2(\tau_k) \right) \right] \right\|. \tag{6.7}$$

In Eq.(6.7) can be written as

$$\begin{aligned} &\|w_1(\tau) - w_2(\tau)\| \\ &\leq (\epsilon_1 + \epsilon_2)\, \mathscr{M}(\tau + n) + \left\| \mathscr{T}(\tau) \left[w_0^1 - w_0^2 \right] \right\| + \left\| \int_0^\tau \mathscr{T}(\tau - \sigma) \left[G \left(\sigma, w_1(\sigma) \int_0^\sigma, F_1(\sigma, \sigma, w_1(\varsigma)) d\varsigma \right. \right. \right. \\ &\quad \left. \left. \left. \int_0^b F_2(\sigma, \varsigma, w_1(\varsigma)) d\varsigma \right) - G \left(\sigma, w_2(\sigma) \int_0^\sigma F_1(\sigma, \varsigma, w_2(\varsigma)) d\varsigma, \int_0^b F_2(\sigma, \varsigma, w_2(\varsigma)) d\varsigma \right) \right] d\sigma \right\| \\ &\quad + \left\| \sum_{k=1}^{n} \mathscr{T}(\tau - \tau_k) \left[I_k \left(w_1(\tau_k) \right) - I_k \left(w_2(\tau_k) \right) \right] \right\|. \end{aligned} \tag{6.8}$$

Using hypotheses (H1)–(H3) and let $\mathbf{B}(\tau) = \|w_1(\tau) - w_2(\tau)\|$ in (6.8) we get,

$$\begin{aligned} \mathbf{B}(\tau) &\leq (\epsilon_1 + \epsilon_2)\, \mathscr{M}(\tau + n) + \mathscr{M} \left\| w_0^1 - w_0^2 \right\| + \int_0^\tau \mathscr{M}\, L_G \mathbf{B}(\sigma) d\sigma + \int_0^\tau \int_0^\sigma \mathscr{M}\, L_G\, L_{F_1} \mathbf{B}(\varsigma) d\varsigma d\sigma \\ &\quad \int_0^\tau \int_0^b \mathscr{M}\, L_G\, L_{F_2} \mathbf{B}(\varsigma) d\varsigma d\sigma + \sum_{k=1}^{n} \mathscr{M}\, L_{I_k} \mathbf{B}(\tau_k). \end{aligned} \tag{6.9}$$

Applying inequality from the Theorem 5.1 to (6.9) with

$$\begin{aligned} u(\tau) &= \mathbf{B}(\tau), \\ a(\tau) &= (\epsilon_1 + \epsilon_2)\, \mathscr{M}(\tau + n) + \mathscr{M} \left\| w_0^1 - w_0^2 \right\| \\ b(\tau, \sigma) &= \mathscr{M}\, L_G, \\ k_1(\tau, \sigma, \varsigma) &= \mathscr{M}\, L_G\, L_{F_1}, \\ k_2(\tau, \sigma, \varsigma) &= \mathscr{M}\, L_G\, L_{F_2}, \\ \beta_k(\tau) &= \mathscr{M} L_{I_k}, \end{aligned}$$

we get

$$\begin{aligned} \mathbf{B}(\tau) &\leq \left\{ (\epsilon_1 + \epsilon_2)\, \mathscr{M}(\tau + n) + \mathscr{M} \left\| w_0^1 - w_0^2 \right\| \right\} \\ &\qquad \prod_{0 < \tau_k < 0} (1 + \mathscr{M}\, L_{I_k}) \exp \left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2 \right) \\ &\leq \left\{ (\epsilon_1 + \epsilon_2)\, \mathscr{M}(b + n) + \mathscr{M} \left\| w_0^1 - w_0^2 \right\| \right\} \\ &\qquad \prod_{k=1}^{n} (1 + \mathscr{M}\, L_{I_k}) \exp \left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2 \right) \end{aligned}$$

But $\mathbf{B}(\tau) = \|w_1(\tau) - w_2(\tau)\|$ we have

$$\begin{aligned} \|w_1 - w_2\|_{\mathcal{PC}} &= \sup_{\tau \in I} \left\{ \frac{\|w(\tau) - v(\tau)\|}{e^{\gamma \tau}} \right\} \\ &\leq \left\{ (\epsilon_1 + \epsilon_2)\, \mathscr{M}(b + n) + \mathscr{M} \left\| w_0^1 - w_0^2 \right\| \right\} \\ &\qquad \prod_{k=1}^{n} (1 + \mathscr{M}\, L_{I_k}) \exp \left(\mathscr{M}\, L_G\, b + \mathscr{M}\, L_G\, L_{F_1} \frac{b^2}{2} + \mathscr{M}\, L_G\, L_{F_2}\, b^2 \right). \end{aligned}$$

Which gives the required inequality (6.3). □

Remark 6.4.
- (i) *Continuous dependence of solutions of* (1.1) *on initial conditions obtained by putting* $\epsilon_1 = \epsilon_2 = 0$ *in inequality* (6.3).
- (ii) *Uniqueness of the solution of problem* (1.1)-(1.2) *obtained by putting* $\epsilon_1 = \epsilon_2 = 0$ *and* $w_0^1 = w_0^2$ *in inequality* (6.3).

7. Concluding Remarks

Existence and uniqueness of solution of the Volterra-Fredholm impulsive integrodifferential equations (VFIIDEs) have been successfully achieve, through Banach's fixed point theorem. We favourably achieve an interesting extension that is a mixed version of integral inequality for piece-wise continuous functions. Further, continuous data dependence of solutions on initial condition and functions involved in right hand side of VFIIDEs obtained by two techniques first via Picard operator theory and secondly via mixed version of integral inequality for piece-wise continuous functions.

In view of obtaining continuous data dependence via PO, the Eq.(4.5) is holds when

$$L_{\mathcal{R}} = \frac{\mathscr{M}\, L_G}{\gamma}\left[\left(1-e^{-\gamma\, b}\right)\left(1+\frac{L_{F_1}}{\gamma}\right)+L_{F_2}\, b\, e^{\gamma\, b}\right]+\mathscr{M}\, e^{\gamma\, b}\sum_{k=1}^{n} L_{I_k} < 1.$$

This restriction have removed when we obtained same results by mixed version of integral inequality.

One can extend similar types of impulsive integral inequalities in fractional case that can be applied to analyze various qualitative properties of fractional integrodifferential equations with impulse condition.

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