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Some common fixed point results of multivalued mappings on fuzzy metric space

Mohit Kumar[a,*], Ritu Arora[b], Ajay Kumar[c]

[a] *Department of Mathematics, School of Computing, University of Engineering and technology Roorkee, Haridwar, Uttarakhand-247667, India*
[b] *Department of Mathematics, Kanya Gurukula Campus, Gurukula Kangri Vishwavidyalaya, Haridwar, Uttarakhand-249404, India*
[c] *Department of Mathematics, Shaheed Srimati Hansa Dhanai Government Degree College, Agrora (Dharmandal), Tehri Garhwal, Uttarakhand -249127, India*

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Abstract

The aim of this paper is to establish new fixed point theorems for single-valued and multivalued maps which satisfy $\alpha - \psi$-contraction conditions in the complete fuzzy metric space. In this paper, we extend the results of Hussain et al. and Samet et al. Some comparative examples are also given which demonstrate the superiority of our results from the exiting results in the literature.

Keywords: Fuzzy metric space, fixed point, $\alpha - \psi-$contractive mapping, α_*-admissible, Hausdorff fuzzy metric space
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1. Introduction

The concept of fuzzy set was introduced in 1965 by Zadeh [10] and fuzzy metric space was initially introduced in 1975 by Kramosil and Michalek [7]. In fact, in 1988, Grabiec [11] introduced Banach contraction (in the sense of Kramosil and Michalek [7]) into fuzzy metric space and extended fixed point theorems of Banach and Edelstein. George and Veeramani [1] modified the fuzzy metric space given by Kramosil and Michalek [7]. In 2002, Gregori and Sapena [18] introduced the notion of fuzzy contractive mapping and proved certain fixed point theorems in various classes of complete fuzzy metric spaces(in the sense of George and Veeramani [1], Kramosil and Michalek [7] and Grabiec

*Corresponding author
Email addresses: mkdgkv@gmail.com (Mohit Kumar), ritu.arora29@gmail.com (Ritu Arora), ajaygraph.gkv.math@gmail.com (Ajay Kumar)

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[11]).
Samet et al. [2] introduced contraction mapping and admissible mapping and established various fixed point theorems in complete metric spaces. Arora and Kumar [14] extended the results of Samet et al. [2] in fuzzy metric space. Phiangsungnoen et al. [17] introduced the fuzzy fixed point in Hausdorff fuzzy metric space and established certain fixed point theorems for fuzzy mapping in Hausdorff fuzzy metric space. We extend the results of Hussain et al. [13] by using the results of Arora and Kumar [14] in fuzzy metric space.The main objective of this paper is to derive the fixed point theorem for multivalued contractive mapping in complete fuzzy metric spaces. We also give some examples for supporting our results.

2. Preliminaries

Definition 2.1. *(Schweizer and Sklar [16]) A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *satisfies the following conditions*

[B.1] $*$ *is commutative and associative*

[B.2] $*$ *is continuous*

[B.3] $a * 1 = a \quad \forall\, a \in [0,1]$

[B.4] $a * b \le c * d$ *whenever* $a \le c, b \le d$ *and* $a, b, c, d \in [0,1]$.

Definition 2.2. *(A. George and P. Veeramani [1]) The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary non –empty set,* $*$ *is an continuous t-norm and* M *is a fuzzy metric in* $X^2 \times [0,\infty] \to [0,1]$*,satisfying the following conditions:* $\forall x, y, z \in X$ *and* $t, s > 0$.

[FM.1] $M(x, y, 0) = 0$

[FM.2] $M(x, y, t) = 1 \quad \forall\, t > 0$ *if and only if* $x = y$

[FM.3] $M(x, y, t) = M(y, x, t)$

[FM.4] $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$

[FM.5] $M(x, y, \cdot) : [0,\infty) \to [0,1]$*, is left continuous.*

[FM.6] $\lim_{n\to\infty} M(x, y, t) = 1$.

Definition 2.3. *(A. George and P. Veeramani [1]) Let* $(X, M, *)$ *be a fuzzy metric space and let a sequence* $x_n \in X$ *is said to be converge to* $x \in X$ *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$.

Definition 2.4. *(A. George and P. Veeramani [1]) A sequence* $x_n \in X$ *is called Cauchy sequence if* $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$*, for each* $t > 0$ *and* $p = 1, 2, 3, \ldots$

Definition 2.5. *(A. George and P. Veeramani [1]) A fuzzy metric space* $(X, M, *)$ *is said to be complete if every Cauchy sequence in* X *is convergent in* X.
A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

Definition 2.6. *(A. George and P. Veeramani [1]) A self mapping $T : X \to X$ is called fuzzy contractive mapping if $M(Tx, Ty, t) > M(x, y, t)$ for each $x \neq y \in X$ and $t > 0$.*

Let ψ the family of functions $\psi : [0, \infty) \to [0, 1]$ such that $\sum_{n=1}^{\infty} \psi^n(t) = 1$ for each $t > 0$, where ψ^n is the n^{th} itereation of ψ.

Lemma 2.7. *For every fuction $\psi : [0, \infty) \to [0, 1]$ the following hold: if ψ is decrease, then for each $t > 0, \lim_{n=1} \psi^n(t) = 1$ implies $\psi(t) > t$.*

Lemma 2.8. *(Nawab Hussain, Jamshaid Ahmad and Akbar Azam [13]) If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n \in N}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in (0, +\infty)$*;*

(ii) $\psi(t) < t$ *for all* $t > 0$*;*

(iii) $\psi(t) = 0$ *if and only if* $t > 0$*.*

Definition 2.9. *(B. Samet, C. Vetro and P. Vetro [2]) Let (X, d) be a fuzzy metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi-$contractive mapping if there exists two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 2.10. *(R. Arora and M. Kumar [14]) Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi-$contractive mapping if there exists two functions $\alpha : X \times X \times [0, \infty) \to [0, 1]$ and $\psi \in \Psi$ such that*

$$\alpha(x, y, t)M(Tx, Ty, t) \geq \psi(M(x, y, t)) \tag{2.1}$$

for all $x, y \in X$.

Remark 2.11. *If $T : X \to X$ satisfied the Banach contraction principle, then T is an $\alpha-\psi-$contractive mapping, where $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1]$.*

Definition 2.12. *(B. Samet, C. Vetro and P. Vetro [2]) Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$, we say that T is $\alpha-$admissible if $x, y \in X, \quad \alpha(x, y,) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1$.*

Definition 2.13. *(R. Arora and M. Kumar [14]) Let $T : X \to X$ and $\alpha : X \times X \times [0, \infty) \to [0, 1]$, we say that T is $\alpha-$admissible if $x, y \in X, \quad \alpha(x, y, t) \leq 1 \quad \Rightarrow \quad \alpha(Tx, Ty, t) \leq 1$.*

Definition 2.14. *(J. Rodriguez-Lopez and S. Romaguera [9]) Let $(X, M, *)$ be a fuzzy metric space. The Hausdroff fuzzy metric $H_M : (K_M(X))^2 \times (0, \infty)$ is defined by*

$$H_M(A, B, t) = \min(\inf_{x \in A}(\sup_{y \in B} M(x, y, t)), \inf_{y \in B}(\sup_{x \in A} M(x, y, t)))$$

for all $A, B \in (K_M(X)$ and $t > 0$ and $K_M(X)$ denotes the set of its non-empty compact subsets.

Definition 2.15. *(J. Hasanzade Asl, S. Rezapour and N. Shahzad [8]) Let* (X, d) *be a metric space,* $T : X \to 2^X$ *be a closed-valued multifunction,* $\psi \in \Psi$ *and* $\alpha : X \times X \to [0, \infty)$ *be a function. In this case, we say that* T *is a* $\alpha_* - \psi-$*contractive multifunction whenever* $\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ *for* $x, y \in X$*, where* H *is the Hausdroff generalized metric,* $\alpha_*(A, B) = \inf\{(\alpha(a, b) : a \in A, b \in B)\}$ *and* 2^X *denote the family of all non-empty subsets of* X*. Also, we say that* T *is* α_*-*admissible whenever* $\alpha(x, y) \geq 1$ *implies* $\alpha_*(Tx, Ty) \geq 1$.

Definition 2.16. *(N. Hussain, J. Ahmad and A. Azam [13]) Let* $T : X \to 2^X$ *be a multifunction,* $\alpha, \eta : X \times X \to R$ *be two functions where* η *is bounded. We say that* T *is* α_*-*admissible mapping with respect to* η *if* $\alpha(x, y) \geq \eta(x, y)$ *implies* $\alpha_*(x, y) \geq \eta_*(x, y), x, y \in X$ *where* $\alpha_*(A, B) = \inf_{x \in A, y \in B} \alpha(x, y)$ *and* $\eta_*(A, B) = \sup_{x \in A, Y \in B} \eta(x, y)$*. If* $\eta(x, y) = 1$ *for all* $x, y \in X$*, then this definition reduces to definition 2.15. If in the case* $\alpha(x, y) = 1$ *for all* $x, y \in X$*,* T *is called* η_*-*sub-admissible mapping.*

Definition 2.17. *(George and Veeramani [1]) Let* $(X, M, *)$ *be a fuzzy metric space. The open ball* $B(x, r, t)$ *and closed ball* $B[x, r, t]$ *with centre* $x \in X$ *and radius* $r, 0 < r < 1, t > 0$ *respectively, are defined as follows:*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$
$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

Lemma 2.18. *([15]) Let* A *and* B *be non-empty closed and bounded subsets of a metric space* (X, d) *and* $0 < h \in R$*. Then for every* $b \in B$*, there exists* $a \in A$ *such that* $d(x, b) \leq H(A, B) + h$*.*

Lemma 2.19. *([12]) Let* (X, d) *be a metric space and* B *be non-empty closed subset of* X *and* $q > 1$*. Then, for each* $x \in X$ *with* $d(x, B) > 0$ *and* $q > 1$*, there exists* $b \in B$*such that* $d(x, b) < qd(x, B)$*.*

3. Main Results

The following result regarding the existence of the fixed point of the mapping satisfying an $\alpha - \psi-$contractive condition on the closed ball, is very useful in that it requires the contractiveness of the mapping only on the closed ball instead of the whole space.

Definition 3.1. *Let* $(X, M, *)$ *be a fuzzy metric space,* $T : X \to 2^X$ *be closed-valued multifunction,* $\psi \in \Psi$ *and* $\alpha : X \times X \times [0, \infty) \to [0, 1]$ *be a function. In this case, we say that* T *is a* $\alpha_* - \psi-$*contractive multifunction whenever* $\alpha_*(Tx, Ty, t)H_M(Tx, Ty, t) \geq \psi(M(x, y, t))$ *for all* $x, y \in X$*, where* H_M *is the Hausdroff fuzzy metric,* $\alpha_*(x, y, t) = inf\{\alpha(a, b, t) : a \in A, b \in B\}$ *and* 2^X *denote the family of all non-empty subsets of* X*. Also we say that* T *is* α_*-*admissible whenever* $\alpha(x, y, t) \leq 1$ *implies* $\alpha_*(Tx, Ty, t) \leq 1$.

Definition 3.2. *Let* $T : X \to 2^X$ *be a multifunction,* $\alpha, \eta : X \times X \times [0, \infty) \to [0, 1]$ *be two functions where* η *is bounded. We say that* T *is* α_*-*admissible mapping with respect to* η *if* $\alpha(x, y, t) \leq \eta(x, y, t)$ *implies* $\alpha_*(x, y, t) \leq \eta_*(x, y, t)$*,* $x, y \in X$*, where* $\alpha_*(A, B, t) = \inf_{x \in A, y \in B} \alpha(x, y, t)$ *and* $\eta_*(A, B, t) = \sup_{x \in A, y \in B} \eta(x, y, t)$*. if* $\eta(x, y, t) = 1$ *for all* $x, y \in X$*, then this definition reduces to definition 3.1. In the case* $\alpha(x, y, t) = 1$ *for all* $x, y \in X$*,* T *is called* η_*-*sub-admissible mapping.*

Theorem 3.3. *Let $(M, X, *)$ be a complete fuzzy metric space and $T : X \to 2^X$ be a α_*-admissible and closed-valued multifunction on X. Assume that $\psi \in \Psi$,*

$$\alpha_*(Tx, Ty, t)H_M(Tx, Ty, t) \geq \psi(M(x, y, t)) \tag{3.1}$$

for all $x, y \in \overline{B(x_0, r, t)}$ and $x_0 \in X$, there exists $x_1 \in Tx_0$ such that

$$\sum_{i=0}^{n} \psi^i(M(x_0, x_1, t)) \geq 1 - r \tag{3.2}$$

for all $n \in N$ and $0 < r < 1, t > 0$. Also suppose that the following assertions hold:

(i) *$\alpha(x_0, x_1, t) \leq 1$ for $x_0 \in X$ and $x_1 \in Tx_0$;*

(ii) *for a sequence $\{x_n\}$ in $\overline{B(x_0, r, t)}$ converging to $x \in \overline{B(x_0, r, t)}$ and $\alpha(x_n, x_{n+1}, t) \leq 1$ for all $n \in N$, we have $\alpha(x_n, x, t) \leq 1$ for all $n \in N$. Then T has a fixed point.*

Proof . Since $\alpha(x_0, x_1, t) \leq 1$ and T is α_*-admissible, so $\alpha_*(x_0, x_1, t) \leq 1$. From (3.2) we get

$$M(x_0, x_1, t) \geq \sum_{i=0}^{n} \psi^i(M(x_0, x_1, t)) \geq 1 - r.$$

If follows that $x_1 \in \overline{B(x_0, r, t)}$. If $x_0 = x_1$, then

$$\alpha_*(Tx_0, Tx_1, t)H_M(Tx_0, Tx_1, t) \geq \psi(M((x_0, x_1, t))) = 0$$

implies that $Tx_0 = Tx_1$ and we have finished. Assume that $x_0 \neq x_1$. By Lemmas 2.8 and 2.18, we take $x_2 \in Tx_1$ and $h > 0$ as $h = \psi^2(M(x_0, x_1, t))$. then

$$\begin{aligned} M(x_1, x_2, t) &\geq H_M(Tx_0, Tx_1, t) + h \\ &\geq \psi(M(x_0, x_1, t) + \psi^2(M(x_0, x_1, t)) \\ &= \sum_{i=1}^{2} \psi^2(M(x_0, x_1, t). \end{aligned}$$

Note that $x_2 \in \overline{B(x_0, r, t)}$, since

$$\begin{aligned} M(x_0, x_2, t_1 + t_2) &\geq M(x_0, x_1, t_1) + M(x_1, x_2, t_2) \\ &\geq M(x_0, x_1, t_1) + \psi(M(x_0, x_1, t_2)) + \psi^2(M(x_0, x_1, t_2)) \\ &= \sum_{i=1}^{2} \psi^2(M(x_0, x_1, t) \geq 1 - r. \end{aligned}$$

By repeating this process, we can construct a sequence $\{x_n\}$ of points in $\overline{B(x_0, r, t)}$ such that $x_{n+1} \in Tx_n, x_n \neq x_{n+1}, \alpha(x_n, x_{n+1}, t) \leq 1$ with

$$M(x_n, x_{n+1}, t) \geq \sum_{i=1}^{n+1} \psi^i(M(x_0, x_1, t)). \tag{3.3}$$

Now for each $n \in N, p > 0$,

$$M(x_n, x_{n+p}, t) \geq \sum_{k=n}^{n+p-1} M(x_k, x_{k+1}, t) \geq \sum_{k=n}^{n+p} \psi^k(M(x_0, x_1, t)) \tag{3.4}$$

$M(x_n, x_{n+1}, t) \geq \psi(M(x_0, x_1, \frac{t}{p})) * \psi^2(M(x_0, x_1, \frac{t}{p})) * \cdots * \psi^{n+p}(M(x_0, x_1, \frac{t}{p}))$.
Thus we prove that $\{x_n\}$ is a Cauchy sequence. Since $\overline{B(x_0, r, t)}$ is closed. So there exists $x^* \in \overline{B(x_0, r, t)}$ such that $x_n \to x^*$ as $n \to \infty$. Now we prove that $x^* \in Tx^*$. Since $\alpha(x_n, x^*, t) \leq 1$ for all n and T is α_*-admissible with respect to η, so $\alpha_*(Tx_n, Tx^*, t) \leq 1$ for all n. Then

$$M(x^*, Tx^*, t) \geq \alpha_*(Tx_n, Tx^*, \frac{t}{2})H_M(Tx_n, Tx^*, \frac{t}{2}) * M(x_n, x^*, \frac{t}{2}) \quad \geq \psi(M(x_n, x^*, \frac{t}{2})) * M(x_n, x^*, \frac{t}{2}) \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we get $M(x^*, Tx^*, t) = 1$. Thus $x^* \in Tx^*$. □

Example 1 Let $X = [0, 1]$with the standard fuzzy metric, define $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and for all $t > 0$. Define the multivalued mapping $T : X \to 2^X$

$$Tx = \begin{cases} [0, \frac{1}{x+3}], & if \quad x \in [0, 1], \\ 0, & otherwise. \end{cases}$$

Consider $x_0 = \frac{1}{3}$ and $x_1 = \frac{1}{4}, r = \frac{1}{12}$ then $\overline{B(x_0, r, t)} = [0, 1]$ and

$$\alpha(x, y, t) = \begin{cases} 1, & if \quad x \in [0, 1], \\ 0, & otherwise. \end{cases}$$

Clearly T is $\alpha - psi-$contractive mapping with $\psi(t) = \frac{1}{t+3}$. Now $M(x_0, x_1, t) = \frac{1}{4}$,

$$\sum_{i=1}^{n} \psi^i(M(x_0, x_1, t)) \geq \frac{1}{4} \sum_{i=1}^{n} \frac{1}{3^n} \geq \frac{1}{3} \geq 1 - r.$$

We prove that all the condition of our theorem 3.3 is satisfied only for $x, y \in \overline{B(x_0, r, t)}$. We suppose that $x \leq y$. The contractive condition of theorem is trivial for the case when $x = y$. So we suppose that $x < y$. Then

$$\alpha_*(Tx, Ty, t)H_M(Tx, Ty, t) = \frac{1}{2}(\frac{1}{1 + |x - y|}) = \psi(M(x_0, x_1, t))$$

put $x_0 = \frac{1}{3}$ and $x_1 = 1$. Then $\alpha(x_0, x_1, t) \leq 1$, then T has a fixed point.

Theorem 3.4. *Let $(M, X, *)$ be a complete fuzzy metric space and $T : X \to 2^X$ be a α_*-admissible and closed-valued multifunction on X. Assume that $\psi \in \Psi$,*

$$\alpha_*(Tx, Ty, t)H_M(Tx, Ty, t) \geq \psi(\max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \frac{M(x, Tx, t)M(y, Ty, t)}{1 + M(x, y, t)}\}) \tag{3.6}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

(i) *There exists* $x_0 \in X$ *and* $x_1 \in Tx_0$ *with* $\alpha(x_0, x_1, t) \leq 1$*;*

(ii) *for a sequence* $\{x_n\}$ *in* X *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}, t) \leq 1$ *for all* $n \in N$*, we have* $\alpha(x_n, x, t) \leq 1$ *for all* $n \in N$*. Then* T *has a fixed point.*

Proof . Since $\alpha(x_0, x_1, t) \leq 1$ and T is α_*-admissible, so $\alpha_*(x_0, x_1, t) \leq 1$. If $x_0 = x_1$ then we have nothing to prove. Let $x_0 \neq x_1$. If $x_1 \in Tx_1$ then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$ then from (3.6), we get

$$\begin{aligned} M(x_1, Tx_1, t) &\geq \alpha_*(Tx_0, Tx_1, t)H_M(Tx_0, Tx_1, t) \\ &\geq \psi(\max\{M(x_0, x_1, t), M(x_0, Tx_0, t), M(x_1, Tx_1, t), \frac{M(x_0, Tx_0, t)M(x_1, Tx_1, t)}{1 + M(x_0, x_1, t)}\}) \\ &\geq \psi(\max\{M(x_0, x_1, t), M(x_0, x_1, t), M(x_1, Tx_1, t), \frac{M(x_0, x_1, t)M(x_1, Tx_1, t)}{1 + M(x_0, x_1, t)}\}) \\ &= \psi(\max\{M(x_0, x_1, t), M(x_1, Tx_1, t)\}). \end{aligned}$$

If $\max\{M(x_0, x_1, t), M(x_1, Tx_1, t)\} = M(x_1, Tx_1, t)$, then $M(x_1, Tx_1, t) \geq \psi(M(x_1, Tx_1, t))$. Since $\psi(t) \geq t$ for all $t > 0$. Then we get a contradiction. Hence we obtain $\max\{M(x_1, Tx_1, t), M(x_0, x_1, t)\} = M(x_0, x_1, t)$. So $M(x_1, Tx_1, t) \geq \psi(M(x_0, x_1, t))$. Let $q < 1$, then from Lemma 2.19 we take $x_2 \in Tx_2$ such that

$$M(x_1, x_2, t) \geq q(M(x_1, Tx_1, t) \geq q\psi(M(x_0, x_1, t))). \tag{3.7}$$

It is clear that $x_1 \neq x_2$. Put $q_1 = \frac{\psi(q\psi(M(x_0,x_1,t))))}{\psi(M(x_1,x_2,t))}$. Then $q_1 < 1$ and $\alpha(x_1, x_2, t) \leq 1$. Since T is α_*-admissible, so $\alpha_*(x_1, x_2, t) \leq 1$. If $x_2 \in Tx_2$ then x_2 is a fixed point of T. Assume that $x_2 \neq Tx_2$. Then from (3.6), we get

$$\begin{aligned} M(x_2, Tx_2, t) &\geq \alpha_*(Tx_1, Tx_2, t)H_M(Tx_1, Tx_2, t) \\ &\geq \psi(\max\{M(x_1, x_2, t), M(x_1, Tx_1, t), M(x_2, Tx_2, t), \frac{M(x_1, Tx_1, t)M(x_2, Tx_2, t)}{1 + M(x_1, x_2, t)}\}) \\ &\geq \psi(\max\{M(x_1, x_2, t), M(x_1, x_2, t), M(x_2, Tx_2, t), \frac{M(x_1, x_2, t)M(x_2, Tx_2, t)}{1 + M(x_1, x_2, t)}\}) \\ &= \psi(\max\{M(x_1, x_2, t), M(x_2, Tx_2, t)\}). \end{aligned}$$

If $\max\{M(x_2, Tx_2, t), M(x_1, x_2, t)\} = M(x_2, Tx_2, t)$, we get contradiction to the fact $M(x_2, Tx_2, t) \geq \psi(M(x_2, Tx_2, t))$. Hence we obtain $\max\{M(x_2, Tx_2, t), M(x_1, x_2, t)\} = M(x_1, x_2, t)$. so $M(x_2, Tx_2, t) \geq \psi(M(x_1, x_2, t))$. Since $q_1 < 1$, so by Lemma 2.19, we can find $x_3 \in Tx_2$ such that $M(x_2, x_3, t) \geq q_1 M(x_2, Tx_2, t) \geq q_1\psi(M(x_1, x_2, t))$,

$$M(x_2, x_3, t) \geq q_1\psi(M(x_1, x_2, t)) = \psi(q\psi(M(x_0, x_1, t))) \tag{3.8}$$

It is clear that $x_2 \neq x_3$. Put $q_2 = \frac{\psi^2(q\psi M(x_0,x_1,t))}{\psi M(x_2,x_3,t)}$. Then $q_2 < 1$ and $\alpha(x_2, x_3, t) \leq 1$. Since T is α_*-admissible, so $\alpha_*(x_2, x_3, t) \leq 1$. If $x_3 \in Tx_3$, then x_3 is a fixed point of T. Assume that $x_3 \neq Tx_3$. Then from (3.6), we have

$$\begin{aligned} M(x_3, Tx_3, t) &\geq \alpha_*(Tx_2, Tx_3, t)H_M(Tx_2, Tx_3, t) \\ &\geq \psi(\max\{M(x_2, x_3, t), M(x_2, Tx_2, t), M(x_3, Tx_3, t), \frac{M(x_2, Tx_2, t)M(x_3, Tx_3, t)}{1 + M(x_2, x_3, t)}\}) \\ &\geq \psi(\max\{M(x_2, x_3, t), M(x_2, x_3, t), M(x_3, Tx_3, t), \frac{M(x_2, x_3, t)M(x_3, Tx_3, t)}{1 + M(x_2, x_3, t)}\}) \\ &= \psi(\max\{M(x_2, x_3, t), M(x_3, Tx_3, t)\}). \end{aligned}$$

If $\max\{M(x_3,Tx_3,t),M(x_2,x_3,t)\}=M(x_3,Tx_3,t)$. Then we get contradiction. So $\max\{M(x_3,Tx_3,t),M(x_2,x_3,t)\}=M(x_2,x_3,t)$. Thus$M(x_3,Tx_3,t)\geq\psi((x_2,x_3,t))$. Since $q_2<1$, so by Lemma 2.19 we can find $x_4\in Tx_3$ such that

$$M(x_3,x_4,t)\geq q_2M(x_3,Tx_3,t)\geq q_2\psi(M(x_2,x_3,t))=\psi^2(q\psi(M(x_0,x_1,t))) \tag{3.9}$$

continuing in this way, we can generate a sequence $\{x_n\}$ in X such that $x_n\in Tx_{n-1}$ and $x_n\neq x_{n-1}$ and

$$M(x_n,x_{n+1},t)\geq\psi^{n-1}(q\psi(M(x_0,x_1,t))) \tag{3.10}$$

for all n. Now, for each $n\in N$ and $p>0$, we have

$$M(x_n,x_{n+p},t)\geq\psi(M(x_0,x_1,\frac{t}{p}))*\psi^2(M(x_0,x_1,\frac{t}{p}))*\cdots\psi^{n+p}(M(x_0,x_1,\frac{t}{p})).$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. We now show that $x^*\in Tx^*$. Since $\alpha(x_n,x^*,t)\leq 1$ for all n and T is α_*-admissible, so $\alpha_*(x_n,x^*,t)\leq 1$ for all n. Then

$$\begin{aligned}M(x^*,Tx^*,t)&\geq\alpha_*(Tx_n,Tx^*,t)H_M(Tx_n,Tx^*,t)*M(x_n,x^*,t)\\&\geq\psi(\max\{M(x_n,x^*,t),M(x_n,Tx_n,t),M(x^*,Tx^*,t),\frac{M(x_n,Tx_n,t)M(x^*,Tx^*,t)}{1+M(x_n,x^*,t)}\})\\&*M(x_n,x^*,t)\\&\geq\psi(\max\{M(x_n,x^*,t),M(x_n,x_{n+1},t),M(x^*,Tx^*,t),\frac{M(x_n,x_{n+1},t)M(x^*,Tx^*,t)}{1+M(x_n,x^*,t)}\})\\&*M(x_n,x^*,t)\end{aligned}$$

and taking the limit as $n\to\infty$, we get $M(x^*,Tx^*,t)=1$. Thus $x^*\in Tx^*$. □

Example 2 Let $X=[0,1]$with the standard fuzzy metric, define $a*b=ab$ for all $a,b\in[0,1]$ and $M(x,y,t)=\frac{t}{t+|x-y|}$, for all $x,y\in X$ and for all $t>0$. Define the multivalued mapping $T:X\to 2^X$

$$Tx=\begin{cases}[0,\frac{1}{x^2+2}], & if \quad x\in[0,1],\\ 0, & otherwise.\end{cases}$$

for all $x\in X$ and

$$\alpha(x,y,t)=\begin{cases}1, & if \quad x\in[0,1],\\ 0, & otherwise.\end{cases}$$

Then $\alpha(x,y,t)\leq 1\Rightarrow\alpha_*(Tx,Ty,t)=\inf\{\alpha(a,b,t):a\in Tx,b\in Ty\}\leq 1$. Then clearly T is α_*-admissible. Now for x,y and $x<y$, it is easy to check that

$$\alpha_*(Tx,Ty,t)H(Tx,Ty,t)\geq\psi(\max\{M(x,y,t),M(x,Tx,t),M(y,Ty,t),\frac{M(x,Tx,t)M(y,Ty,t)}{1+M(x,y,t)}\})$$

where $\psi(t)=\frac{1}{t^2+2}$ for all $t\geq 0$. Put $x_0=\frac{1}{2}$ and $x_1=1$. Then $\alpha(x_0,x_1,t)=\frac{1}{2}<1$. Then T has fixed point.

Theorem 3.5. *Let $(M, X, *)$ be a complete fuzzy metric space, $\alpha : X \times X \times [0, \infty]) \to [0, 1]$ be a mapping, $\psi \in \Psi$ and T be a self-mapping on X such that*

$$\alpha(x, y, t)M(Tx, Ty, t) = \begin{cases} \psi(\max\{\frac{M(x,Tx,t)M(y,Ty,t)}{M(x,y,t)}, M(x, y, t)\}), & for \quad x \neq y \\ 0, & for \quad x = y \end{cases} \tag{3.11}$$

for all $x, y \in X$. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, Tx_0, t) \leq 1$. If T is continuous. Then T has a fixed point.

Proof . Let $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \leq 1$, and define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x^* = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all n. Since T is $\alpha-$admissible, so it is easy to check that $\alpha(x_n, x_{n+1}, t) \leq 1$ for all natural numbers n. Thus for each natural numbers n, we have

$$\begin{aligned} M(x_{n+1}, x_n, t) &= M(Tx_n, Tx_{n-1}, t) \geq \alpha(x_n, x_{n-1}, t)M(Tx_n, Tx_{n-1}, t) \\ &\geq \psi(\max\{\frac{M(x_n, Tx_n, t)M(x_{n-1}, Tx_{n-1}, t)}{M(x_n, x_{n-1}, t)}, M(x_n, x_{n-1}, t)\}) \\ &\geq \psi(\max\{\frac{M(x_n, x_{n+1}, t)M(x_{n-1}, x_n, t)}{M(x_n, x_{n-1}, t)}, M(x_n, x_{n-1}, t)\}) \\ &\geq \psi(\max\{M(x_n, x_{n+1}, t), M(x_n, x_{n-1}, t)\}). \end{aligned}$$

If $\max\{M(x_n, x_{n+1}, t), M(x_n, x_{n-1}, t)\} = M(x_n, x_{n+1}, t)$, then $M(x_n, x_{n+1}, t) \geq \psi(M(x_n, x_{n+1}, t))$. This is a contradiction. So, we get $M(x_n, x_{n+1}, t) \geq \psi(M(x_n, x_{n-1}, t))$. Since ψ is decreasing, se we have

$$M(x_{n+1}, x_n, t) \geq \psi(M(x_n, x_{n-1}, t)), \psi^2(M(x_{n-1}, x_{n-2}, t)) * \cdots * \psi^n(M(x_0, x_1, t)) \tag{3.12}$$

for all n. It is easy to check that $\{x_n\}$ is a Cauchy sequence. Since X is complete, so there exists $x^* \in X$ such that $x_n \to x^*$. Further the continuity T implies that

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = x^*. \tag{3.13}$$

Therefore x^* is a fixed point of T in X. Now, there exists another point $u \neq x^*$ in X such that

$$\begin{aligned} M(x^*, u, t) &= M(Tx^*, Tu, t) \geq \alpha(x^*, u, t)M(Tx^*, Tu, t) \\ &\geq \psi(\max\{\frac{M(x^*, Tx^*, t)M(u, Tu, t)}{M(x^*, u, t)}, M(x^*, u, t)\}) \\ &\geq \psi(\max\{0, M(x^*, u, t)\}) = \psi(M(x^*, u, t)). \end{aligned}$$

This is a contradiction. Hence x^* is a fixed point of T in X. □

Example 3 Let $X = [0, 1]$with the standard fuzzy metric, define $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and for all $t > 0$. Define $T : X \to X$ by $Tx = \frac{1}{x+4}, x \in [0, 1]$. Also define the mapping $\psi : [0, \infty) \to [0, 1]$ by $\psi(t) = \frac{1}{t+4}$ and

$$\alpha(x, y, t) = \begin{cases} 1, & if \quad x, y \in [0, 1], \\ 0, & otherwise. \end{cases}$$

By calculation one can easily show that

$$\alpha(x, y, t)M(Tx, Ty, t) \geq \psi(\max\{\frac{M(x, Tx, t)My, Ty, t}{M(x, y, t)}, M(x, y, t)\})$$

for all $x, y \in X$ and T has a fixed point.

Conclusion

In this paper, we have established the results of Hussain et al. [13] in fuzzy metric space and proved fixed point theorems for multivalued contractive mappings in complete fuzzy metric space.

References

[1] A. George and P. Veeramani, *On some result in fixed point theorems in fuzzy metric spaces*, Fuzzy Sets Syst. 64 (1994) 395–399, .

[2] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for* $\alpha-\psi-$*contractive type mappings*, Nonlinear Anal. 75 (2012) 2154–2165.

[3] D. Gopal and C. Vetro, *Some new fixed point theorems in fuzzy metric spaces*, Iran. J. Fuzzy Syst. 11(3) (2014) 95–107.

[4] D. Gopal, M. Imded, C. Vetro and M. Hasan, *Fixed point theory for cyclic weak* $\phi-$*contraction in fuzzy metric spaces*, J. Nonlinear Anal. Appl. Article ID jnaa- pages,doi: 10.5899/2012/jnaa-0110, 2012 (2012).

[5] D. Mihet, *A Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets Syst. 144 (3) (2004) 431–439.

[6] D. Mihet, *On fuzzy contractive mappings in fuzzy metric spaces*, Fuzzy Sets Syst. 158 (8) (2007) 915–921.

[7] I. Kramosil and J. Michalek, *Fuzzy metric and Statistical metric spaces*, Ky-Bernetica 11 (1975) 336–344.

[8] J. Hasanzade Asl, S. Rezapour and N. Shahzad, *On fixed points of* $\alpha-\psi-$*contractive multifunctions*, Fixed Point Theory Appl. (2012) 2012 : 212.

[9] J. Rodriguez-Lopez and S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets Syst. 147 (2004) 273–283.

[10] L.A. Zadeh, *Fuzzy sets Information and Control*, Fuzzy Sets and Syst. 8 (1965) 338–353.

[11] M. Grabiec, *Fixed point in fuzzy metric space*, Fuzzy Sets Syst. 27 (1988) 385–389.

[12] M.U. Ali and T. Kamran, *On* $(\alpha^{\star}-\psi-)$*contractive multi-valued mappings*, Fixed Point Theory Appl. 2013 (2013) Article ID 137 .

[13] N. Hussain, J. Ahmad and A. Azam, *Generalized fixed point theorems for multi-valued* $\alpha-\psi-$ *contractive mappings*, J. Inequal. Appl. 2014 (2014) 348.

[14] R. Arora and M. Kumar, *Unique fixed point theorems for* $\alpha-\psi-$*contractive type mappings in fuzzy metric space*, Cogent. Math. Statis. 3(1) (2016) 1–8.

[15] S.B.Jr. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969) 475–478.

[16] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10 (1960) 385–389.

[17] S. Phiangsungnoen, W. Sintunavarat and P. Kumam, *Fuzzy fixed point theorems in Hausdorff fuzzy metric spaces*, J. Inequal. Appl. 2014 (2014) 201.

[18] V. Gregori and A. Sapena, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets Syst. 125(2) (2002) 245–253.

[19] V.L. Lazar, *Fixed point theory for multivalued* $\phi-$*contractions*, Fixed Point Theory Appl. 2011 (2011) 50.