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A numerical scheme for solving nonlinear parabolic partial differential equations with piecewise constant arguments

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Abstract

This article deals with the nonlinear parabolic equation with piecewise continuous arguments (EPCA). This study, therefore, with the aid of the θ -methods, aims at presenting a numerical solution scheme for solving such types of equations which has applications in certain ecological studies. Moreover, the convergence and stability of our proposed numerical method are investigated. Finally, to support and confirm our theoretical results, some numerical examples are also presented.

Keywords: (Partial differential equation with piecewise constant arguments (EPCA), θ-methods, Convergence, Trust-region-dogleg method)
2010 MSC: 65M06; 65M12; 65A05

1. Introduction

It seems that the strong interest in partial differential equations with piecewise constant arguments (EPCA) is motivated by the fact that it describes a hybrid dynamical system a combination of continuous and discrete. These types of equations have the structure of continuous dynamical systems within intervals of unit length. In fact, the equations of EPCA are a combination of both differential and difference equations. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [27]. As we studied in the literature, there are some theoretical and

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numerical works that have been published, see [9, 23] and [5, 6, 8, 18, 21, 22, 24, 25, 26] respectively. This book and many other articles have mentioned to applications of EPCA equations in biology, mechanics, electronics, and a mathematical model for the dynamics of gas absorption. There are also some authors who have considered the initial-boundary problem for linear delay partial differential equations of the parabolic type and give a sufficient condition for the stability of the solution of this initial-boundary problem [1, 3, 4, 5, 6, 7, 8, 20, 25, 26].

Ashyralyev and Agirseven, in 2018, presented first and second order accuracy difference schemes for the solution of one dimensional nonlinear hyperbolic equation with time delay [10]. Poorkarimi et al, in the nineties, are investigated the existence and uniqueness of a bounded solution for a nonlinear parabolic and hyperbolic partial differential equations with piecewise continuous time delay [17, 18, 19].

In recent years, Bereketoglu and Lafci investigated the behavior of the solutions of a PDE with a piecewise constant argument [11]. Also, Büyükkahraman and Bereketoglu have a study On a partial differential equation with piecewise constant mixed arguments [12]. In addition, Esmailzadeh et al in [14, 15], have used the finite difference technique for solving hyperbolic partial differential equations with piecewise constant arguments and variable coefficients and diffusion-convection equation, respectively. In both papers, the stability condition of the numerical method is investigated, and also with the aid of the figures and the tables of errors are compared the numerical and analytical solutions to demonstrate the validity of the proposed scheme.

However, few authors have worked on nonlinear EPCA. So providing numerical methods for this category of equations still requires a lot of work. Hence in this article, we will use finite difference methods to investigate the numerical solution for treating a wider class of nonlinear EPCA equations. This type of equation may be considered as a generalization of Fisher's equation which has applications in certain ecological studies.

Consider the following initial-boundary value problem

$$u_t\left(x,t\right) = a^2 u_{xx}\left(x,t\right) + C\left(x,t,u\left(x,t\right),u\left(x,[t]\right)\right), \quad a<x<b, \quad t>0. \tag{1.1}$$

with the initial condition

$$u\left(x,0\right) = \varphi\left(x\right), \quad a<x<b, \tag{1.2}$$

and boundary conditions

$$u\left(a,t\right) = g_a\left(t\right), \quad u\left(b,t\right) = g_b\left(t\right), \quad t \geq 0, \tag{1.3}$$

in the domain $\Omega = (a,b) \times (0,\infty)$, where $[t]$ denotes the greatest-integer function. The analytic solution of Eq (1.1) on the interval $0 \leq t < 1$ is as follows [17]:

$$\begin{aligned} u\left(x,t\right) = &\frac{1}{\sqrt{4a^2\pi t}}\int_a^b e^{-\frac{(x-\xi)^2}{4a^2 t}}\varphi(\xi)d\xi \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left(\frac{1}{\sqrt{4a^2\pi t}}\int_a^b e^{-\frac{(x-\xi)^2}{4a^2 t}} C(\xi,\tau,u(\xi,\tau),\varphi(\xi))d\xi\right)d\tau. \end{aligned} \tag{1.4}$$

As you can see, the analytical solution has a lot of computational complexity, and this has interested us to provide a suitable approximation for this type of equation using θ-methods.

Definition 1.1. *[17] A function $u(x,t)$ is called a solution of the initial-boundary problem* (1.1)–(1.3) *if it satisfies the following conditions:*

i. $u(x,t)$ *is continuous in* $\Omega = (a,b) \times (0,\infty)$.

ii. u_t *and* u_{xx} *exist and are continuous in* Ω *with the possible exception of the points* $(x,[t])$ *where one-sided derivatives exist.*

iii. $u(x,t)$ *satisfies* $u_t(x,t) = a^2 u_{xx}(x,t) + C(x,t,u(x,t),u(x,[t]))$ *in* Ω *with the possible exception of the points* $(x,[t])$, *and condition* $u(x,0) = \varphi(x)$, $a < x < b$.

The conditions of existence and uniqueness of the analytical solution of the problem (1.1)–(1.3) are as follow

Theorem 1.2. *[17] Assume the following hypotheses:*

i. The function $\varphi(x)$ *is twice continuously differentiable and bounded on* $\mathbb{R}$.

ii. The function $C(x,t,u,v) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and bounded on* Ω, *and satisfies the Lipschitz condition* $|C(x,t,u,w) - C(x,t,v,y)| \le L|u-v|$, *where* L *is a positive constant and* $u, v \in (-\infty,\infty)$.

Then there exists a unique solution to problem (1.1)–(1.3) *which is bounded in* Ω, *Eq (1.4).*

2. Numerical solution

In this section, an efficient finite difference technique for the problem (1.1)–(1.3) is introduced. Based on the good experience we had of using the θ-methods method to the EPCA linear equations in [14] and [15] we were interested in testing the efficiency of this method for nonlinear equations as well. The numerical scheme will be presented in a finite domain $a \le x \le b$ and $0 \le t \le T$.

Let Δx and Δt be step-sizes of spatial and temporal directions which satisfy $\Delta x = \frac{b-a}{M}$ and $\Delta t = \frac{1}{N}$, where $M, N \ge 1$ are positive integers. Denote the spatial and temporal nodes as $x_m = a + m\Delta x$, $m = 0,1,\dots,M$ and $t_n = n\Delta t$, $n = 0,1,\dots,NT$ respectively, and U_m^n as an approximation to $u(x_m,t_n)$. Using Taylor expansion we have

$$(U_t)_m^n \approx \frac{U_m^{n+1} - U_m^n}{\Delta t}, \tag{2.1}$$

$$(U_{xx})_m^n \approx \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{(\Delta x)^2}. \tag{2.2}$$

By applying the θ-methods and using the approximations (2.1) and (2.2) we have

$$\begin{aligned}
&-\theta a^2 U_{m-1}^{n+1} + (\alpha + 2\theta a^2) U_m^{n+1} - \theta a^2 U_{m+1}^{n+1} \\
&= (1-\theta) a^2 U_{m-1}^n + (\alpha - 2(1-\theta)a^2) U_m^n + (1-\theta) a^2 U_{m+1}^n \\
&\quad + (1-\theta)(\Delta x)^2 C(x_m, t_n, U_m^n, U^h(x_m,[t_n])) \\
&\quad + \theta(\Delta x)^2 C(x_m, t_{n+1}, U_m^{n+1}, U^h(x_m,[t_{n+1}])),
\end{aligned} \tag{2.3}$$

where $m = 1,2,\dots,M-1$, $n = 0,1,\dots,NT-1$, $\alpha = \frac{(\Delta x)^2}{\Delta t}$ and, $U^h(x_m,[t_n])$ is an approximation to $u^h(x_m,[t_n])$.

If we denote $n = kN + l$, $k = 0, 1, \dots, T-1$ and $l = 0, 1, \dots, N-1$, then both $U^h(x_m, [t_n])$ and $U^h(x_m, [t_{n+1}])$ can be defined as U_m^{kN}. Actually U_m^{kN} is the solution at discontinuous points. So, the Eq (2.3) can be written as

$$\begin{aligned} &-\theta a^2 U_{m-1}^{kN+l+1} + (\alpha + 2\theta a^2) U_m^{kN+l+1} - \theta a^2 U_{m+1}^{kN+l+1} \\ &= (1-\theta) a^2 U_{m-1}^{kN+l} + (\alpha - 2(1-\theta)a^2) U_m^{kN+l} + (1-\theta) a^2 U_{m+1}^{kN+l} \\ &\quad + (1-\theta)(\Delta x)^2 C(x_m, t_{kN+l}, U_m^{kN+l}, U_m^{kN}) + \theta(\Delta x)^2 C(x_m, t_{kN+l+1}, U_m^{kN+l+1}, U_m^{kN}). \end{aligned} \tag{2.4}$$

To solve this nonlinear system, the trust-region-dogleg method has been used [13]. For this purpose we have used the fsolve MATLAB function.

3. Error analysis

3.1. Convergence

In this section, numerical convergence is discussed. Suppose that $u(x_m, t_n)$ and U_m^n be the exact and the approximate solution at the $(m,n)^{th}$ grid point, respectively. To prove convergence of the presented scheme, we need to examine the discretization error behavior, which is defined as follows:

$$e_m^n = u(x_m, t_n) - U_m^n. \tag{3.1}$$

Lemma 3.1. *Suppose $u(x,t)$ is a sufficiently smooth function and $0 \le \theta \le 1$. Then*

$$u_t(x, t_{n+\theta}) = (1-\theta) u_t(x, t_n) + \theta u_t(x, t_{n+1}) - (1-\theta)\theta (\Delta t)^2 u_{ttt}(x, t_{n+\theta}) + O((\Delta t)^3), \tag{3.2}$$

$$u_t(x, t_{n+\theta}) = \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} - (1-2\theta)(\Delta t) u_{tt}(x, t_{n+\theta}) + O((\Delta t)^2), \tag{3.3}$$

$$\begin{aligned} u_{xx}(x_m, t_n) &= \frac{u(x_{m+1}, t_n) - 2u(x_m, t_n) + u(x_{m-1}, t_n)}{(\Delta x)^2} \\ &\quad - \frac{(\Delta x)^2}{12} u_{xxxx}(x_m, t_n) + O((\Delta x)^4). \end{aligned} \tag{3.4}$$

Using the Taylor extension, the above lemma can be easily confirmed.

Theorem 3.2. *Let $u(x,t)$ be a sufficiently smooth function and $0 \le \theta \le 1$, then*

$$\begin{aligned} &-\theta a^2 e_{m+1}^{n+1} + (\alpha + 2\theta a^2) e_m^{n+1} - \theta a^2 e_{m-1}^{n+1} - \theta(\Delta x)^2 L_1^n e_m^{n+1} \\ &= (1-\theta) a^2 e_{m+1}^n + (\alpha - 2(1-\theta)a^2) e_m^n + (1-\theta) a^2 e_{m-1}^n + (1-\theta)(\Delta x)^2 L_2^n e_m^n \\ &\quad + (1-2\theta)(\Delta x)^2 (\Delta t) u_{tt}(x_m, t_{n+\theta}) + (\Delta x)^2 O((\Delta t)^2) + O((\Delta x)^4), \end{aligned} \tag{3.5}$$

where $L_1^n, L_2^n \in \mathbb{R}$ are the Lipschitz Constants.

Proof . Using Eq (1.1) and substituting Eq (3.3) in Eq (3.2), yields

$$\begin{aligned} \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} &= (1-\theta)\left(a^2 u_{xx}(x, t_n) + C(x, t_n, u(x, t_n), u(x, [t_n]))\right) \\ &\quad + \theta\left(a^2 u_{xx}(x, t_{n+1}) + C(x, t_{n+1}, u(x, t_{n+1}), u(x, [t_{n+1}]))\right) \\ &\quad + (1-2\theta)(\Delta t) u_{tt}(x, t_{n+\theta}) + O((\Delta t)^2). \end{aligned} \tag{3.6}$$

In Eq (3.6), first put $x = x_m$ and then by using Eq (3.4) and rewriting Eq (3.6), we have

$$\begin{aligned}
&-\theta a^2 u(x_{m+1}, t_{n+1}) + (\alpha + 2\theta a^2)(x_m, t_{n+1}) - \theta a^2 u(x_{m-1}, t_{n+1}) \\
&-\theta(\Delta x)^2 C\left(x_m, t_{n+1}, u\left(x_m, t_{n+1}\right), u\left(x_m, [t_{n+1}]\right)\right) \\
&= (1-\theta)a^2 u(x_{m+1}, t_n) + (\alpha - 2(1-\theta)a^2)u(x_m, t_n) + (1-\theta)a^2 u(x_{m-1}, t_n) \\
&\quad + (1-\theta)(\Delta x)^2 C\left(x_m, t_n, u\left(x_m, t_n\right), u\left(x_m, [t_n]\right)\right) \\
&\quad + (1-2\theta)(\Delta x)^2(\Delta t)u_{tt}(x_m, t_{n+\theta}) + (\Delta x)^2 O((\Delta t)^2) + O((\Delta x)^4). \qquad (3.7)
\end{aligned}$$

From the subtraction of Eq (3.7) and Eq (2.3), we get

$$\begin{aligned}
&-\theta a^2 e_{m+1}^{n+1} + (\alpha + 2\theta a^2)e_m^{n+1} - \theta a^2 e_{m-1}^{n+1} \\
&-\theta(\Delta x)^2\left[C\left(x_m, t_{n+1}, u\left(x_m, t_{n+1}\right), u\left(x_m, [t_{n+1}]\right)\right) - C(x_m, t_{n+1}, U_m^{n+1}, U^h\left(x_m, [t_{n+1}]\right))\right] \\
&= (1-\theta)a^2 e_{m+1}^n + (\alpha - 2(1-\theta)a^2)e_m^n + (1-\theta)a^2 e_{m-1}^n \\
&\quad + (1-\theta)(\Delta x)^2\left[C\left(x_m, t_n, u\left(x_m, t_n\right), u\left(x_m, [t_n]\right)\right) - C(x_m, t_n, U_m^n, U^h\left(x_m, [t_n]\right))\right] \\
&\quad + (1-2\theta)(\Delta x)^2(\Delta t)u_{tt}(x_m, t_{n+\theta}) + (\Delta x)^2 O((\Delta t)^2) + O((\Delta x)^4). \qquad (3.8)
\end{aligned}$$

According to the second condition of the Theorem 1.2, there are two finite real numbers L_1^n and L_2^n such that

$$C\left(x, t, u(x_m, t_{n+1}), w\right) - C\left(x, t, U_m^{n+1}, y\right) = L_1^n(u(x_m, t_{n+1}) - U_m^{n+1}) = L_1^n e_m^{n+1}, \qquad (3.9)$$

$$C\left(x, t, u(x_m, t_n), w\right) - C\left(x, t, U_m^n, y\right) = L_2^n(u(x_m, t_n) - U_m^n) = L_2^n e_m^n. \qquad (3.10)$$

By substituting Eq (3.9) and (3.10) in Eq (3.8), we have

$$\begin{aligned}
&-\theta a^2 e_{m+1}^{n+1} + (\alpha + 2\theta a^2)e_m^{n+1} - \theta a^2 e_{m-1}^{n+1} - \theta(\Delta x)^2 L_1^n e_m^{n+1} \\
&= (1-\theta)a^2 e_{m+1}^n + (\alpha - 2(1-\theta)a^2)e_m^n + (1-\theta)a^2 e_{m-1}^n + (1-\theta)(\Delta x)^2 L_2^n e_m^n \\
&\quad + (1-2\theta)(\Delta x)^2(\Delta t)u_{tt}(x_m, t_{n+\theta}) + (\Delta x)^2 O((\Delta t)^2) + O((\Delta x)^4).
\end{aligned}$$

□

Now everything is ready to prove the convergence theorem. In this theorem, we prove that by decreasing Δt and Δx, the maximum absolute error tends to zero, and this is exactly the concept of convergence.

Theorem 3.3. *Let* $E^n = \max_{0 \le m \le M} |e_m^n|$,

$$\Psi^n e_m^{n+1} = -\theta a^2 e_{m+1}^{n+1} + (\alpha + 2\theta a^2)e_m^{n+1} - \theta a^2 e_{m-1}^{n+1} - \theta(\Delta x)^2 L_1^n e_m^{n+1},$$

and $(\Delta x)^2 < \Delta t$, *that is* α *tended to zero as* Δx *and* Δt *tending to zero. Then* $E^n \to 0$ *as* $\Delta x \to 0$ *and* $\Delta t \to 0$.

Proof . Induction is used to prove the theorem. From Eq (3.5) ,we have

$$\begin{aligned}
\left|\Psi^n e_m^{n+1}\right| \le{}& (1-\theta)a^2\left|e_{m+1}^n\right| + \left|\alpha - 2(1-\theta)a^2\right||e_m^n| + (1-\theta)a^2\left|e_{m-1}^n\right| + (1-\theta)(\Delta x)^2|L_2^n||e_m^n| \\
&+ |1-2\theta|(\Delta x)^2(\Delta t)|u_{tt}(x_m, t_{n+\theta})| + (\Delta x)^2\left|O((\Delta t)^2) + O((\Delta x)^2)\right|.
\end{aligned}$$

That is

$$\begin{aligned}
\max_{0 \le m \le M}\left|\Psi^n e_m^{n+1}\right| \le K_n E^n + (\Delta x)^2\Bigg(&\Delta t\,|1-2\theta| \max_{0 \le m \le M}|u_{tt}(x_m, t_{n+\theta})| \\
&+ \left|O((\Delta t)^2) + O((\Delta x)^2)\right|\Bigg), \qquad (3.11)
\end{aligned}$$

where $K_n = 2(1-\theta)a^2 + |\alpha - 2(1-\theta)a^2| + (1-\theta)(\Delta x)^2 \left|L_2^n\right|$.

Note that the initial discretization error is zero because the initial values used with the finite difference formula are those given for the partial differential equation. Therefore by knowing $E^0 = 0$, for $n = 0$ in Eq (3.11), yields

$$\max_{0\le m\le M} \left|\Psi^0 e_m^1\right| \le |1-2\theta|\,(\Delta x)^2(\Delta t) \max_{0\le m\le M} |u_{tt}(x_m,t_\theta)| + (\Delta x)^2 \left|O((\Delta t)^2)+O((\Delta x)^2)\right|.$$

Let $\Delta x \to 0$ and $\Delta t \to 0$. So, $\max\limits_{0\le m\le M} |\Psi^0 e_m^1| \to 0$, that is $E^1 \to 0$.

Now, suppose that E^n tends to zero as $\Delta x \to 0$, $\Delta t \to 0$. Then due to the recursive Eq (3.11) and boundness of K_n we have $\max\limits_{0\le m\le M} |\Psi^n e_m^{n+1}| \to 0$.

Hence, $E^{n+1} \to 0$ as $\Delta x \to 0$ and $\Delta t \to 0$. Thus the proof is complete.
□

3.2. Stability

In this subsection, we show that the numerical scheme (2.3) is stable. In the sense that by increasing the time steps in the numerical scheme (2.3), the errors resulting from subtracting the exact solution U and its calculated values remain limited.

Suppose that $\tilde{U}_m^n$ be the approximation of exact solution U_m^n for numerical scheme (2.3).

Theorem 3.4. *Let $U^n = U(x,t_n)$, $\varepsilon^n = U^n - \tilde{U}^n$, $\varepsilon_m^n = U_m^n - \tilde{U}_m^n$ and α tended to zero as Δx and Δt tending to zero. Then ε^n is bounded as $\Delta x \to 0$.*

Proof . According to the proof of Theorem 3.2 we have

$$\begin{aligned} &- \theta a^2 \left(\varepsilon_{m-1}^{n+1} + \varepsilon_{m+1}^{n+1}\right) + (\alpha + 2\theta a^2 - \theta(\Delta x)^2 L_1^n)\varepsilon_m^{n+1} \\ &= (1-\theta)a^2 \left(\varepsilon_{m-1}^{n} + \varepsilon_{m+1}^{n}\right) + (\alpha - 2(1-\theta)a^2 + (1-\theta)(\Delta x)^2 L_2^n)\varepsilon_m^{n}, \end{aligned} \tag{3.12}$$

Using Taylor expansion in (3.12) around (x_m,t) we have

$$\begin{aligned} &- \theta a^2 \left(2\varepsilon_m^{n+1} + (\Delta x)^2 (\varepsilon_{xx})_m^{n+1} + \frac{(\Delta x)^4}{12} (\varepsilon_{xxxx})_m^{n+1} + \cdots\right) \\ &\quad + (\alpha + 2\theta a^2 - \theta(\Delta x)^2 L_1^n)\varepsilon_m^{n+1} \\ &= (1-\theta)a^2 \left(2\varepsilon_m^{n} + (\Delta x)^2 (\varepsilon_{xx})_m^{n} + \frac{(\Delta x)^4}{12} (\varepsilon_{xxxx})_m^{n} + \cdots\right) \\ &\quad + (\alpha - 2(1-\theta)a^2 + (1-\theta)(\Delta x)^2 L_2^n)\varepsilon_m^{n}, \end{aligned} \tag{3.13}$$

or

$$\begin{aligned} &(\alpha - \theta(\Delta x)^2 L_1^n)\varepsilon_m^{n+1} - \theta a^2 (\Delta x)^2 \left((\varepsilon_{xx})_m^{n+1} + \frac{(\Delta x)^2}{12} (\varepsilon_{xxxx})_m^{n+1} + \cdots\right) \\ &= (\alpha + (1-\theta)(\Delta x)^2 L_2^n)\varepsilon_m^{n} + (1-\theta)a^2 (\Delta x)^2 \left((\varepsilon_{xx})_m^{n} + \frac{(\Delta x)^2}{12} (\varepsilon_{xxxx})_m^{n} + \cdots\right), \end{aligned} \tag{3.14}$$

By tending Δx to zeros we can eliminate the terms contain derivatives and the results is as follows

$$\varepsilon_m^{n+1} = W_n \varepsilon_m^n, \tag{3.15}$$

where

$$W_n = \frac{\alpha + (1-\theta)(\Delta x)^2 L_2^n}{\alpha - \theta(\Delta x)^2 L_1^n}. \tag{3.16}$$

From (3.15) one can conclude that

$$\max_{0\le m\le M} \left|\varepsilon_m^{n+1}\right| \le (W)^{n+1} \max_{0\le m\le M} \left|\varepsilon_m^0\right|, \tag{3.17}$$

where $W = \max\limits_{0\le j\le n+1} W_j$. From (3.16) we know that $\lim\limits_{\Delta x\to 0} W_n = 1$ and hence from (3.17) we have

$$\max_{0\le m\le M} \left|\varepsilon_m^{n+1}\right| \le \max_{0\le m\le M} \left|\varepsilon_m^0\right|, \tag{3.18}$$

and the error is bounded. □

4. Numerical Experiments

In this section, the efficiency of the proposed method has been evaluated using various examples. We have tried to show this with the plots and table of errors. To estimate the error, we use the following norms

$$E_{N,M} = \max_{m,k,l} \left|u(x_m, t_{kN+l}) - U_m^{kN+l}\right|, \quad 0\le m\le M,\ 0\le k\le L-1,\ 0\le l\le N,$$
$$L_\infty(t) = \max_m |u(x_m,t) - U(x_m,t)|, \quad 0\le m\le M.$$

In all numerical examples, we assume that $a = 1$.

It is notable that, we perform all of the computations by MATLAB® R2019a software (V9.6.0.1072779, 64-bit (win64), License Number: 968398, MathWorks Inc., Natick, MA) running on a Sony VAIO Laptop (Intel® Core™ i5-2410M Processor 2.30 GHz with Turbo Boost up to 2.90 GHz, 8 GB of RAM, 64-bit) PC.

Example 4.1. *In Eq (1.1), we assume that*

$$C\left(x,t,u\left(x,t\right),u\left(x,[t]\right)\right) = u(x,t)u(x,[t]) + \left(a^2-1\right)\exp(-t)\sin(x) \\ - \exp(-\left(t+[t]\right))\sin^2(x),$$

with the exact solution $u(x,t) = \exp(-t)\sin(x)$, where $-\pi \le x \le \pi$. Therefore Eq (1.1) can be written as

$$\begin{cases} u_t\left(x,t\right) = a^2 u_{xx}\left(x,t\right) + u(x,t)u(x,[t]) + (a^2-1)\exp(-t)\sin(x) \\ \qquad\qquad - \exp(-\left(t+[t]\right))\sin^2(x), \quad -\pi < x < \pi, \quad 0 < t \le 20, \\ u\left(x,0\right) = \sin(x), \quad -\pi \le x \le \pi, \\ u(-\pi,t) = u(\pi,t) = 0, \quad 0 \le t \le 20. \end{cases} \tag{4.1}$$

The finite difference schemes (2.4) *is used to solve the Eq* (4.1)*. The exact and numerical solutions on all mesh grids are plotted in Figure 1 using $N = 100$ and $M = 50$. In Figure 2 the absolute errors in all mesh grids and the logarithm of absolute errors in integer time levels are shown using $N = 100$ and $M = 50$. In Table 1, the $E_{N,M}$ error norms are reported for different values of N, M and θ. Also, the $L_\infty(t)$ error norms for different values of N and t are tabulated in Table 2 using $M = 50$ and $\theta = 0.5$.*

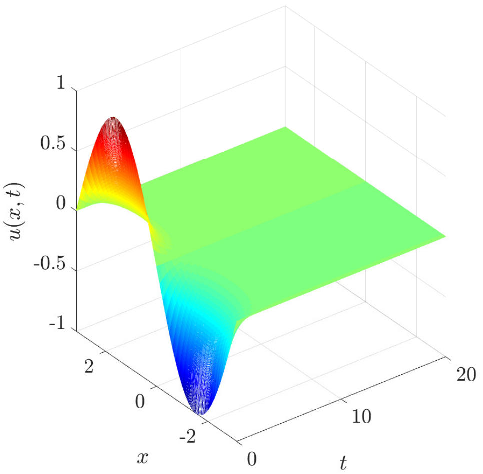

Figure 1: The exact solutions (left) and the numerical solutions (right) on all mesh grids using $N = 100$ and $M = 50$ for Example 4.1.

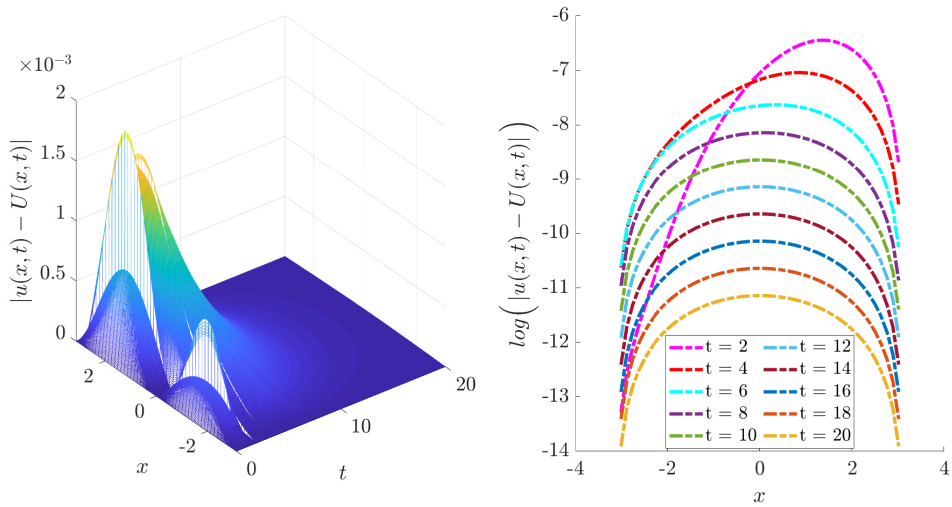

Figure 2: The absolute errors in all mesh grids (left) and the logarithm of absolute errors in integer time levels (right) using $N = 100$, $M = 50$ and θ for Example 4.1.

Table 1: The $E_{N,M}$ error norms for different values of N, M and θ for Example 4.1.

		$E_{N,M}$		
θ	N	$M = 10$	$M = 20$	$M = 50$
0.3		$1.989816e-02$	$4.477085e-03$	$8.544827e-04$
0.5	100	$2.148904e-02$	$6.104647e-03$	$1.935009e-03$
0.7		$2.308263e-02$	$7.744292e-03$	$3.588892e-03$
0.3		$2.026016e-02$	$4.839383e-03$	$5.352103e-04$
0.5	200	$2.105541e-02$	$5.649270e-03$	$1.363847e-03$
0.7		$2.185134e-02$	$6.460539e-03$	$2.192068e-03$
0.3		$2.047630e-02$	$5.058768e-03$	$6.869624e-04$
0.5	500	$2.079435e-02$	$5.382041e-03$	$1.018635e-03$
0.7		$2.111252e-02$	$5.705563e-03$	$1.350240e-03$

Table 2: The $L_\infty(t)$ error norms for different values of N and t using $M = 50$ and $\theta = 0.5$ for Example 4.1.

	$L_\infty(t)$		
t	$N = 100$	$N = 200$	$N = 500$
0.5	$5.013596e-04$	$5.037531e-04$	$5.044232e-04$
1.5	$1.683969e-03$	$1.257683e-03$	$1.000757e-03$
2.5	$1.433337e-03$	$1.044193e-03$	$8.110082e-04$
3.5	$1.030662e-03$	$7.330615e-04$	$5.562178e-04$
4.5	$7.423716e-04$	$5.165676e-04$	$3.819745e-04$
5.5	$5.543977e-04$	$3.799266e-04$	$2.755322e-04$
6.5	$4.250325e-04$	$2.888744e-04$	$2.074885e-04$
7.5	$3.293611e-04$	$2.232741e-04$	$1.596718e-04$
8.5	$2.560948e-04$	$1.735161e-04$	$1.239634e-04$
9.5	$1.994653e-04$	$1.350674e-04$	$9.642891e-05$

Example 4.2. *This example is inspired by the generalized Fisher equation [16]. In the Eq (1.1), assume that*

$$\begin{aligned} C\left(x,t,u\left(x,t\right),u\left(x,[t]\right)\right) &= su(x,t)(1-u(x,t)) + qu(x,[t])(1-u(x,[t])) \\ &+ \frac{(10+2a^2)\exp(x-5t)}{(\exp(x-5t)+1)^3} - a^2\frac{6\exp(2x-10t)}{(\exp(x-5t)+1)^4} \\ &+ \frac{s}{(1+\exp(x-5t))^2}\left(\frac{1}{(1+\exp(x-5t))^2}-1\right) \\ &+ \frac{q}{(1+\exp(x-5[t]))^2}\left(\frac{1}{(1+\exp(x-5[t]))^2}-1\right). \end{aligned}$$

where s and q are positive constants.

The exact solution of this problem is $u(x,t) = \frac{1}{(1+\exp(x-5t))^2}$, where $0 \le x \le 1$. The initial and boundary conditions can be obtain using exact solution. For numerical computations we assume that $s = q = 6$.

In Figure 3, the exact and numerical solutions on the time interval [0,10] are plotted. The error figures is also drawn in Figure 4. Also for a more detailed review, the absolute error estimates are reported by different parameters in the Tables 3 and 4. As you can see, the results from the graphs and tables show that the proposed method works very well.

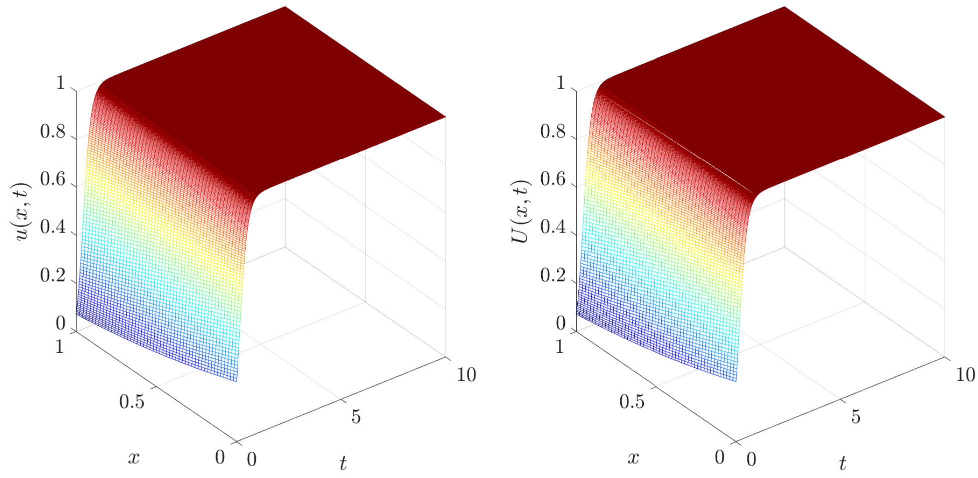

Figure 3: The exact solutions (left) and the numerical solutions (right) on all mesh grids using $N = 100$ and $M = 50$ for Example 4.2.

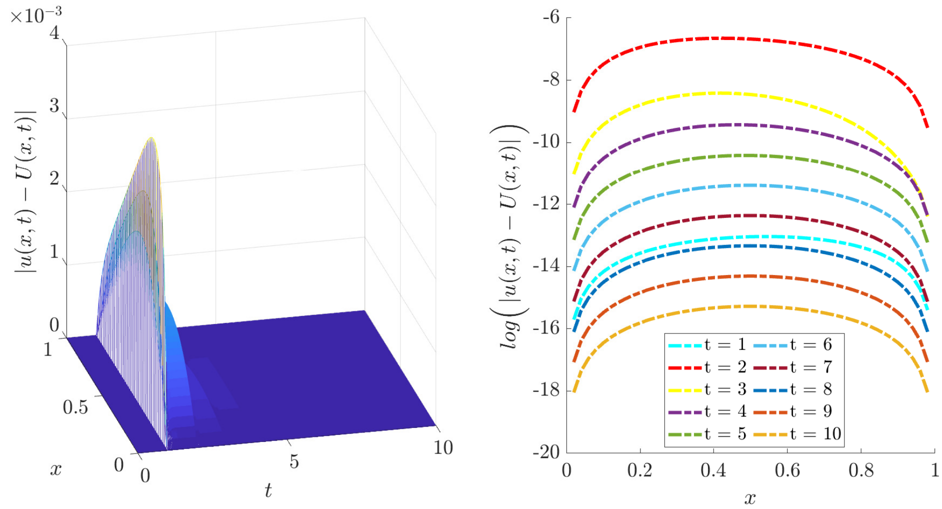

Figure 4: The absolute errors in all mesh grids (left) and the logarithm of absolute errors in integer time levels (right) using $N = 100$ and $M = 50$ for Example 4.2.

Table 3: The $E_{N,M}$ error norms for different values of N, M and θ for Example 4.2.

		$E_{N,M}$		
θ	N	$M = 10$	$M = 20$	$M = 50$
0.3		$1.432407e-04$	$1.453905e-04$	$1.485814e-04$
0.5	2000	$2.333241e-04$	$2.380192e-04$	$2.420419e-04$
0.7		$3.214901e-04$	$3.292817e-04$	$3.341418e-04$
0.3		$9.587827e-05$	$9.707846e-05$	$9.974485e-05$
0.5	3000	$1.568234e-04$	$1.593426e-04$	$1.633430e-04$
0.7		$2.168762e-04$	$2.210983e-04$	$2.251891e-04$
0.3		$5.761418e-05$	$5.912604e-05$	$6.018532e-05$
0.5	5000	$9.462335e-05$	$9.642043e-05$	$9.896447e-05$
0.7		$1.312988e-04$	$1.332348e-04$	$1.373292e-04$

Table 4: The $L_\infty(t)$ error norms for different values of N and t using $M = 50$ and $\theta = 0.5$ for Example 4.2.

	$L_\infty(t)$		
t	$N = 2000$	$N = 3000$	$N = 5000$
0.5	$4.495196e-07$	$4.572568e-07$	$4.612182e-07$
1.5	$6.869937e-05$	$4.584562e-05$	$2.752666e-05$
2.5	$1.196958e-05$	$7.989056e-06$	$4.796314e-06$
3.5	$4.281064e-06$	$2.856610e-06$	$1.714577e-06$
4.5	$1.594541e-06$	$1.063907e-06$	$6.385296e-07$
5.5	$6.015322e-07$	$4.013463e-07$	$2.408741e-07$
6.5	$2.274286e-07$	$1.517417e-07$	$9.106998e-08$
7.5	$8.600112e-08$	$5.738044e-08$	$3.443770e-08$
8.5	$3.252186e-08$	$2.169877e-08$	$1.302283e-08$
9.5	$1.229840e-08$	$8.205562e-09$	$4.924686e-09$

Example 4.3. *Consider the nonlinear parabolic equation with piecewise constant arguments*

$$\begin{cases} u_t\,(x,t) = u_{xx}\,(x,t) + 2u(x,t) + u^3(x,[t]) \\ \qquad\qquad -2\exp(-2t-3)(\sin(\pi x)+1) + a^2p^2\exp(-2t-3)\sin(\pi x) \\ \qquad\qquad -2(1+\sin(\pi x))\exp(-2t-3) \\ \qquad\qquad -(1+\sin(\pi x))^3\exp(-6t-9), \quad -1 < x < 1, \quad 0 < t \le 10, \\ u\,(x,0) = (1+\sin(\pi x))\exp(-3), \quad -1 \le x \le 1, \\ u(-1,t) = u(1,t) = \exp(-2t-3), \quad 0 \le t \le 10. \end{cases} \tag{4.2}$$

The exact solution of (4.2) *is* $u(x,t) = (1+\sin(\pi x))\exp(-2t-3)$*. The finite difference schemes* (2.4) *is used to solve* (4.2) *and the exact and numerical solutions on all mesh grids are shown in Figure 5 . You can see the absolute errors in all mesh grids and the logarithm of absolute errors in integer time levels in Figure 6. Tables 5 and 6 report* $E_{N,M}$ *and* $L_\infty(t)$*, respectively. In this example, the figures and tables also support and confirm our theoretical results.*

Table 5: The $E_{N,M}$ error norms for different values of N, M and θ for Example 4.3.

		$E_{N,M}$		
θ	N	$M = 10$	$M = 20$	$M = 50$
0.3		$1.256031e-03$	$3.309113e-04$	$5.988944e-05$
0.5	500	$1.245338e-03$	$3.233433e-04$	$5.146612e-05$
0.7		$1.264275e-03$	$3.425828e-04$	$7.210778e-05$
0.3		$1.252955e-03$	$3.287159e-04$	$5.720910e-05$
0.5	700	$1.245324e-03$	$3.233368e-04$	$5.145979e-05$
0.7		$1.258848e-03$	$3.370581e-04$	$6.604691e-05$
0.3		$1.250658e-03$	$3.270844e-04$	$5.536010e-05$
0.5	1000	$1.245319e-03$	$3.233335e-04$	$5.145660e-05$
0.7		$1.254782e-03$	$3.329272e-04$	$6.158071e-05$

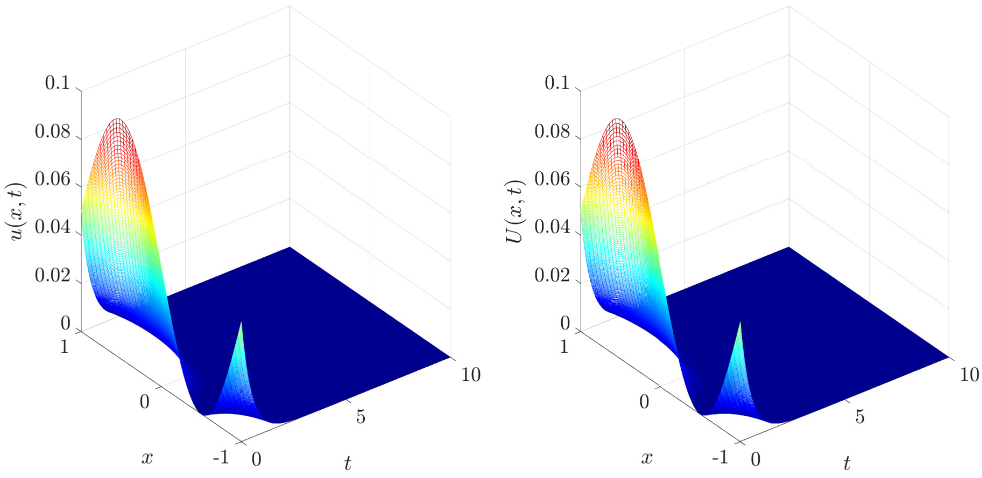

Figure 5: The exact solutions (left) and the numerical solutions (right) on all mesh grids using $N = 100$ and $M = 50$ for Example 4.3.

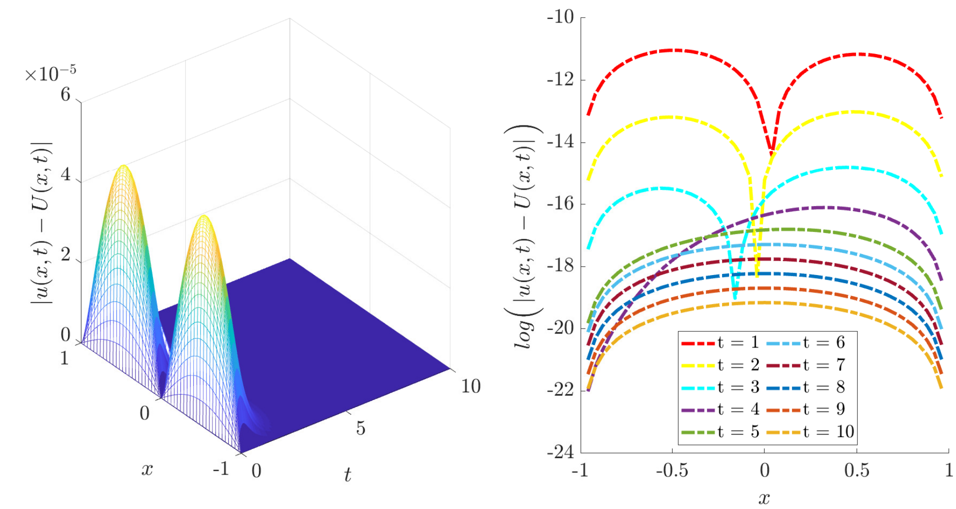

Figure 6: The absolute errors in all mesh grids (left) and the logarithm of absolute errors in integer time levels (right) using $N = 100$ and $M = 50$ for Example 4.3.

Table 6: The $L_\infty(t)$ error norms for different values of N and t using $M = 50$ and $\theta = 0.5$ for Example 4.3.

	$L_\infty(t)$		
t	$N = 500$	$N = 700$	$N = 1000$
0.5	$3.837580e-05$	$3.836263e-05$	$3.835563e-05$
1.5	$5.690931e-06$	$5.639367e-06$	$5.596700e-06$
2.5	$8.668563e-07$	$8.366163e-07$	$8.112880e-07$
3.5	$1.823262e-07$	$1.613985e-07$	$1.446629e-07$
4.5	$7.183382e-08$	$5.664441e-08$	$4.422819e-08$
5.5	$4.223182e-08$	$3.212265e-08$	$2.365444e-08$
6.5	$2.639148e-08$	$2.001526e-08$	$1.466487e-08$
7.5	$1.655096e-08$	$1.255225e-08$	$9.195397e-09$
8.5	$1.037971e-08$	$7.871980e-09$	$5.766774e-09$
9.5	$6.509502e-09$	$4.936813e-09$	$3.616560e-09$

Example 4.4. *In this example we consider an extended of generalized Burgers'–Fisher equation [2] with assuming*

$$
\begin{aligned}
&C\left(x,t,u\left(x,t\right),u\left(x,[t]\right)\right)\\
&=\frac{r}{\gamma}u(x,t)(1-u^{\gamma}(x,t))+\frac{q}{\lambda}u(x,[t])(1-u^{\lambda}(x,[t]))\\
&\quad-2\exp(-2t-3)(\sin(\pi x)+1)+a^2\pi^2\exp(-2t-3)\sin(\pi x)\\
&\quad-\frac{r}{\gamma}\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{2}{\gamma+\lambda}x+\frac{1}{\gamma+\lambda}t\right)\right)\left(1-\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{2}{\gamma+\lambda}x+\frac{1}{\gamma+\lambda}t\right)\right)^{\gamma}\right)\\
&\quad-\frac{q}{\lambda}\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{2}{\gamma+\lambda}x+\frac{1}{\gamma+\lambda}[t]\right)\right)\left(1-\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{2}{\gamma+\lambda}x+\frac{1}{\gamma+\lambda}[t]\right)\right)^{\lambda}\right).
\end{aligned}
$$

where r, γ, q and λ are real known constants. The exact solution of this example is given as

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{2}{\gamma+\lambda}x+\frac{1}{\gamma+\lambda}t\right),$$

where $-1\le x\le 1$.

The results of exact and numerical solutions on all mesh grids are plotted in Figure 7. The $E_{N,M}$ and $L_\infty(t)$ are illustrated for different parameters in Table 7 and 8, respectively . In this example, we have tried to investigate the behavior of this type of equation over a longer period of time. Fortunately, the results were satisfactory and this shows the positive performance of the proposed method.

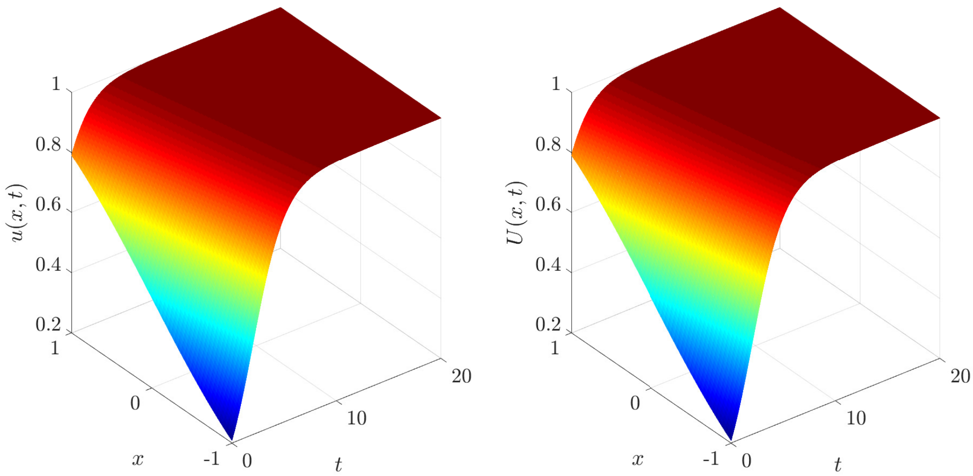

Figure 7: The exact solutions (left) and the numerical solutions (right) on all mesh grids using $N = 100$ and $M = 50$ for Example 4.4.

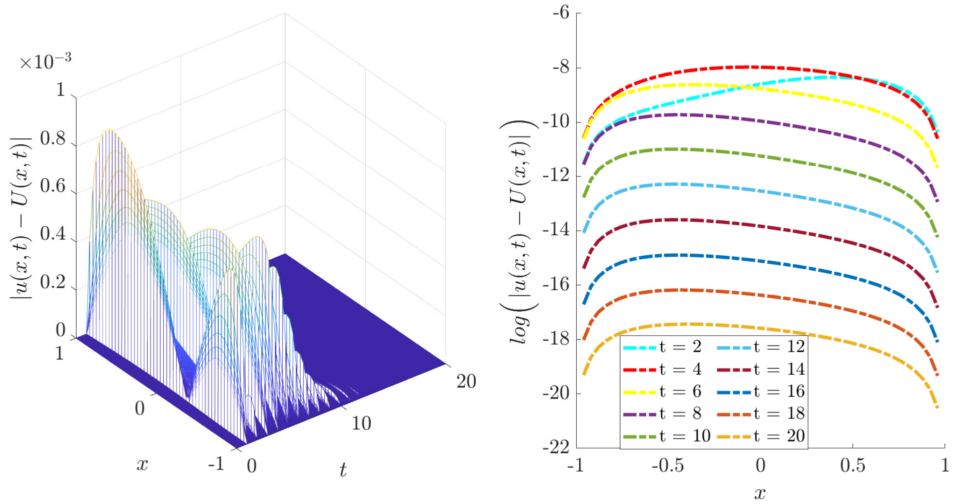

Figure 8: The absolute errors in all mesh grids (left) and the logarithm of absolute errors in integer time levels (right) using $N = 100$ and $M = 50$ for Example 4.4.

Table 7: The $E_{N,M}$ error norms for different values of N, M and θ for Example 4.4.

		$E_{N,M}$		
θ	N	$M = 10$	$M = 20$	$M = 50$
0.3		$3.654497e-04$	$1.636857e-04$	$1.211278e-04$
0.5	500	$4.098334e-04$	$2.311309e-04$	$1.936102e-04$
0.7		$4.721719e-04$	$3.009616e-04$	$2.663118e-04$
0.3		$3.279388e-04$	$1.122344e-04$	$6.411813e-05$
0.5	1000	$3.493955e-04$	$1.435832e-04$	$9.999333e-05$
0.7		$3.708342e-04$	$1.767315e-04$	$1.362908e-04$
0.3		$3.091622e-04$	$9.098579e-05$	$3.606308e-05$
0.5	2000	$3.199010e-04$	$1.026646e-04$	$5.353636e-05$
0.7		$3.306353e-04$	$1.180200e-04$	$7.141037e-05$

Table 8: The $L_\infty(t)$ error norms for different values of N and t using $M = 50$ and $\theta = 0.5$ for Example 4.4.

	$L_\infty(t)$		
t	$N = 500$	$N = 1000$	$N = 2000$
0.5	$7.139638e-06$	$7.141298e-06$	$7.141713e-06$
1.5	$2.985599e-05$	$1.233810e-05$	$7.709319e-06$
2.5	$3.677496e-05$	$1.704852e-05$	$7.388486e-06$
3.5	$4.290270e-05$	$2.080779e-05$	$9.847334e-06$
4.5	$3.938572e-05$	$1.991772e-05$	$1.018735e-05$
5.5	$2.989411e-05$	$1.546726e-05$	$8.226404e-06$
6.5	$1.835514e-05$	$9.609810e-06$	$5.223582e-06$
7.5	$1.068857e-05$	$5.618041e-06$	$3.070219e-06$
8.5	$5.653797e-06$	$2.985283e-06$	$1.645639e-06$
9.5	$3.073078e-06$	$1.621247e-06$	$8.916753e-07$

5. Conclusion

In this work, a θ finite difference scheme was introduced to the numerical solution of nonlinear parabolic PDEs with piecewise constant arguments. The scheme leads to a nonlinear system of algebraic equations which was solved using the trust-region-dogleg method. The convergence and stability of the presented method were investigated. Also, we considered four examples to test the presented method. The numerical results show that the solutions are very accurate and the presented scheme is efficient.

References

[1] D. Agirseven, *Approximate Solutions of delay parabolic equations with the Dirichlet condition*, Abstr. Appl. Anal., Article Number 682752, 2012 (2012), `https://doi.org/10.1155/2012/682752`.

[2] J. Alavi, H. Aminikhah, *Numerical Study of the Inverse Problem of Generalized Burgers– Fisher and Generalized Burgers–Huxley Equations*, Adv. Math. Phys.,2021(2021), `https://doi.org/10.1155/2021/6652108`.

[3] A. Ashyralyev, D. Agirseven, *Stability of parabolic equations with unbounded operators acting on delay terms*, Electron. J. Differential Equations, 2014(160) (2014) 1–13.

[4] A. Ashyralyev, D. Agirseven , *On source identification problem for a delay parabolic equation*, Nonlinear Anal. Model. Control, 19 (3) (2014) 335–349.

[5] A. Ashyralyev, D. Agirseven, *Stability of delay parabolic difference equations*, Filomat 28(5)(2014) 995–1006.

[6] A. Ashyralyev, D. Agirseven, *Well-posedness of delay parabolic equations with unbounded operators acting on delay terms*, Bound. Value Probl., 2014(126) (2014), `https://doi.org/10.1186/1687-2770-2014-126`.

[7] A. Ashyralyev, D. Agirseven, *Well-posedness of delay parabolic difference equations*, Adv. Difference Equ., 2014(18) (2014), `https://doi.org/10.1186/1687-1847-2014-18`.

[8] A. Ashyralyev, D. Agirseven, *On convergence of difference schemes for delay parabolic equations*, Comput. Math. Appl., 66 (7) (2013), 1232–1244, `https://doi.org/10.1016/j.camwa.2013.07.018`.

[9] A. Ashyralyev, P. E. Sobolevskii, *On the stability of the linear delay differential and difference equations* , Abstr. Appl. Anal., 6(5) (2001), 267–297.

[10] A. Ashyralyev, D. Agirseven, *Bounded solutions of nonlinear hyperbolic equations with time delay*, Electron. J. Differential Equations, 2018(21) (2018) 1–15.

[11] H. Bereketoglu, M. Lafci *Behavior of the solutions of a partial differential equation with a piecewise constant argument*, Filomat, 31(19) (2017) 5931–5943.

[12] M. L. Büyükkahraman, H. Bereketoglu, *On a partial differential equation with piecewise constant mixed arguments*, Iran J. Sci. Technol. Trans. Sci., 44 (2020), 1791–1801.

[13] R. H. Byrds, R. B. Schnabel, G. A. Shultz, *A trust region algorithm for nonlinearly constrained optimization*, SIAM J. Numer. Anal., 24(5) (1987) 1152–1170.

[14] M. Esmaeilzadeh, H. S. Najafi, H. Aminikhah, *A numerical scheme for diffusion-convection equation with piecewise constant arguments* , Comput. Methods Differ. Equ., 8(3) (2020) 573–584.

[15] M. Esmaeilzadeh, H. S. Najafi, H. Aminikhah, *A numerical method for solving hyperbolic partial differential equations with piecewise constant arguments and variable coefficients*, J. Difference Equ. Appl., 27(2) (2021) 172–194.

[16] N. A. Kudryashov, A.S. Zakharchenko, *A note on solutions of the generalized Fisher equation*, Appl. Math. Lett., 32 (2014) 53–56.

[17] H. Poorkarimi, J. Wiener, *Bounded solutions of nonlinear parabolic equations with time delay*, 15th Annu. Conf. Appl. Math., Univ. of Central Oklahoma, Electron. J. Differential Equations , Conference 02 (1999) 87–91.

[18] H. Poorkarimi, J. Wiener, *Bounded solutions of non-linear hyperbolic equations with delay*, Proceedings of the VII International Conference on Non-Linear Analysis, V. Lakshmikan-tham, Ed. 1 (1986) 471–478.

[19] S. M. Shah, H. Poorkarimi, J. Wiener, *Bounded solutions of retarded nonlinear hyperbolic equations*, Bull. Allahabad Math. Soc., 1 (1986) 1–14.

[20] H. Poorkarimi, J. Wiener, *Almost periodic solutions of nonlinear hyperbolic equations with time delay*, Conf. 07 (2001), 99–102.

[21] S. M. Shah, J. Wiener, *Advanced differential equations with piecewise constant argument deviations*, Internat. J. Math. Sci. 6 (1983), 671–703.

[22] M. H. Song, Z. W. Yang, M. Z. Liu, *Stability of θ-methods for advanced differential equations with piecewise continuous arguments*, Comput. Math. Appl. 49 (2005) 1295–1301.

[23] T. Veloz, M. Pinto, *Existence, Computability and stability for solutions of the diffusion equation with general piecewise constant argument*, J. Math. Anal. Appl. 426 (2015) 330–339.

[24] Q. Wang, *Stability of numerical solution for partial differential equations with piecewise constant arguments*, Adv. Diff. Eq. (2018), doi.org/10.1186/s13662-018-1514-1.

[25] Q. Wang, J. Wen, *Analytical and numerical stability of partial differential equations with piecewise constant arguments*, Num. Meth. Partial Diff. Eq., 30 (2014) 1–16.

[26] Q. Wang, *Stability analysis of parabolic partial differential equations with piecewise continuous arguments*, Inc. Numer Meth. Partial Differential Eq., 33 (2017) 531–545.

[27] J. Wiener, *Generalized Solutions of Functional Differential Equations*, World Scientific, Singapore, 1993.