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Fixed point on generalized dislocated metric spaces

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Abstract

In the present paper, we introduce new types of convergence of a sequence in left dislocated and right dislocated metric spaces. Also, we generalize the Banach contraction principle in these newly defined generalized metric spaces.

Keywords: Fixed point, left dislocated metric, right dislocated metric, contraction.
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1. Introduction

Soon after Maurice Fréchet [2] seminal paper on metric spaces researchers have started to generalize extend his idea. Menger [5] was the first to propose probabilistic metric spaces, a generalization of metric spaces. Afterward a generalization pseudometric spaces/dislocated metric spaces of metric spaces was proposed by Hitzler and Seda [4], Hitzler [3], Hitzler and Seda [4] and Beg et al. [1] studied generalization of Banach contraction principle in dislocated metric spaces. Their results were applied in the area of programming language semantics.

Following Waszkiewicz [6, 7], let (X, d) be a distance space where d is a function from X into $[0, \infty)$. Define the distance topology on (X, d) as follows:

(1) Let $x \in X$ and $\epsilon > 0$. Then the set $B_d(x, \epsilon) := \{y \in X : d(x, y) < d(x, x) + \epsilon\}$ is called ball with centre x and radius ϵ.

(2) $N_x := \{A \subseteq X : \exists \text{ some } \epsilon > 0 \text{ such that } B_d(x, \epsilon) \subseteq A\}$.

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(3) The distance topology on (X, d) is denoted and defined by $\tau_d := \{A \subseteq X : \forall x \in A, A \in N_x\}$.

We denote and define the inverse distance topology on (X, d) as follows:
$\tau_{d_1} := \tau_d^{-1}$, where $d_1(x, y) = d(y, x)$.
Furthermore, Waszkiewicz [7] established the following proposition.

Proposition 1.1. *Let (X, d) be a distance space, (x_n) be a sequence of elements of X and $x \in X$. Then $d(x, x_n) \to d(x, x) \Rightarrow x_n \to_{\tau_d} x$.*

In a similar way, we state and prove the following proposition.

Proposition 1.2. *Let (X, d) be a distance space, (x_n) be a sequence of elements of X and $x \in X$. Then $d(x_n, x) \to d(x, x) \Rightarrow x_n \to_{\tau_d^{-1}} x$.*

Proof . Let U be anyτ_d^{-1}-open set around x. Then $\exists\ \epsilon > 0$ such that $x \in B_d^{-1}(x, \epsilon) \subseteq U$. Suppose that $d(x_n, x) \to d(x, x)$. Then $\exists\ n_\epsilon \in N(N :=$ the set of all positive integers) such that $\forall n \geq n_\epsilon$, $|d(x_n, x) - d(x, x)| < \epsilon$. If $|d(x_n, x) - d(x, x)| \geq 0$, then $d(x_n, x) < d(x, x) + \epsilon$ and so $x_n \in U$. If $d(x_n, x) - d(x, x) \leq 0$, then $d(x_n, x) \leq d(x, x)$ and so $d(x_n, x) < d(x, x) + \epsilon$, i.e., $x_n \in U$. In the present paper, we introduce new types of convergence of a sequence in distance space. Mainly we aim to generalize Banach contraction principle in special types of these spaces, namely, q-left-Hausdorff q-left-complete ld-metric spaces and q-right-Hausdorff q-right-complete rd-metric spaces. Also, we give two counterexamples to illustrate that the converse of Proposition 1.1 (Proposition 2.5 [7]) and Proposition 1.2 may not be true in these spaces. □
Let (X, d) be a distance space. Consider the following conditions, for all $x, y, z \in X$,

(Mi) $d(x, x) = 0$,

(Mii) $d(x, y) = d(y, x) = 0$, then $x = y$,

$(Miii)$ $d(x, y) = d(y, x)$,

(Miv) $d(x, y) \leq d(x, z) + d(z, y)$,

(Mv) $d(x, y) \leq d(z, x) + d(z, y)$,

(Mvi) $d(x, y) \leq d(x, z) + d(y, z)$.

If d satisfies conditions $(Mi) - (Miv)$, then it is called a metric. If it satisfies conditions $(Mii), (Miii)$ and (Miv), it is called a dislocated metric [4] (or simply d-metric). If it satisfies conditions (Mii) and (Mv), it is called a left dislocated metric [9] (or simply ld-metric). If it satisfies conditions (Mii) and (Mvi), it is called a right dislocated metric [9] (or simply rd-metric).
The following theorem is established by Hitzler and Seda [4].

Theorem 1.3. *Let (X, d) be a complete d-metric space and let $f : X \to X$ be a Banach contraction function. Then f has a unique fixed point.*

We use the following lemma due to Ahmed, Zeyada and Hassan [9].

Lemma 1.4. *Let $(X.d)$ be a ld-metric space. If $f : (X, d) \to (X, d)$ is a Banach contraction function, then $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$.*

Lemma 1.5. *Let $(X.d)$ be a rd-metric space. If $f : (X, d) \to (X, d)$ is a Banach contraction function, then $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$.*

Theorem 1.3 was generalized in [9] by the following theorems.

Theorem 1.6. *Let $(X.d)$ be a complete ld-metric space and let $f : X \to X$ be a Banach contraction function. Then f has a unique fixed point.*

Theorem 1.7. *Let $(X.d)$ be a complete rd-metric space and let $f : X \to X$ be a Banach contraction function. Then f has a unique fixed point.*

2. Definitions in distance spaces

In this section, we introduce definitions needed for our results in a distance space. As it turns out, these notions can be carried over directly from conventional metrics.

Definition 2.1. *A sequence (x_n) in a distance space (X, d) is called a Cauchy sequence if $\forall \epsilon > 0$, $\exists\ n_0 \in N$ such that $d(x_m, x_n) < \epsilon\ \forall m, n \geq n_0$.*

Definition 2.2. *A sequence (x_n) q-left-converges to x iff $\lim\limits_{n\to\infty} d(x_n, x) = d(x, x)$. In this case x is called a q-left-limit of (x_n).*

Definition 2.3. *A sequence (x_n) q-right-converges to x iff $\lim\limits_{n\to\infty} x, d(x_n) = d(x, x)$. In this case x is called a q-right-limit of (x_n).*

Definition 2.4. *A distance space (X, d) is called q-left (resp. q-right) complete if every Cauchy sequence is q-left (resp. q-right) convergent.*

Definition 2.5. *Let (X, d_1) and (Y, d_2) be distance spaces and let $f : (X, d_1) \to (Y, d_2)$. Then f is q-left-continuous iff $\forall x_0 \in X, \forall \epsilon > 0\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\varepsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| < \varepsilon$$

.

Definition 2.6. *Let (X, d_1) and (Y, d_2) be distance spaces and let $f : (X, d_1) \to (Y, d_2)$. Then f is q-left-continuous iff $\forall x_0 \in X, \forall \epsilon > 0\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|d_1(x_0, x) - d_1(x_0, x_0)| < \delta(\varepsilon) \Rightarrow |d_2(f(x_0), f(x)) - d_2(f(x_0), f(x_0))| < \varepsilon$$

.

Definition 2.7. *[8] A function $f : X \to X$ is called a Banach contraction function if there exists $0 \leq \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.*

Lemma 2.8. *Every subsequence of q-left (resp. q-right) convergent sequence to x_0 is a q-left (resp. q-right) convergent to x_0.*

Lemma 2.9. *Let (X, d_1) and (Y, d_2) be distance spaces. A mapping $f : (X, d_1) \to (Y, d_2)$ is q-left-continuous iff $\forall (x_n)$ in X q-left- d_1-converges to $x_0 \in X, (f(x_n))$ in Y q-left-d_2-converges to $f(x_0) \in Y$.*

Proof . Let f be q-left-continuous and (x_n) be a sequence in X. Suppose that (x_n) q-left-d_1-converges to $x_0 \in X$. Let $\epsilon > 0$. Then $\exists\delta(\epsilon) > 0$ such that

$$|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\varepsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| < \epsilon$$

.

Then $\exists\delta(\epsilon) > 0$ and $\exists\ n_0 \in N$ such that $\forall\ n \geq n_0$, $|d_1(x_n, x_0) - d_1(x_0, x_0)| < \delta(\varepsilon)$. Thus

$$|d_2(f(x_n), f(x_0)) - d_2(f(x_0), f(x_0))| < \epsilon$$

.

Hence, $(f(x_n))$ in Y q-left-d_2-converges to $f(x_0) \in Y$.

Conversely, suppose that f is not q-left-continuous. Then $\exists\ x_0 \in X$, $\exists\ \epsilon > 0$ such that $\forall\delta > 0$,

$$|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\varepsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| \geq \epsilon$$

.

Then the sequence $(xn_)$ $(x_n = x \forall n \in N)$ q-left-d1-converges to x_0 but $(f(x_n))$ does not q-left-d_2-converges to $f(x_0)$. $\square$

We state the following lemma without proof:

Lemma 2.10. *Let (X, d_1) and (Y, d_2) be distance spaces. A mapping $f : (X, d_1) \to (Y, d_2)$ is q-right continuous iff $\forall(xn)$ in X q-right- d1-converges to $x_0 \in X$, $(f(x_n))$ in Y q-right-d_2-converges to $f(x_0) \in Y$.*

3. A generalization of Banach contraction mapping in left-d-metric space

In this section, we give a generalization of the Banach contraction mapping in left d-metric space.

Definition 3.1. *A left-d-metric space (X, d) is called a q-left-Hausdorff space iff every left-q-convergent sequence (x_n) in X left-q-converges to a unique point in X.*

Theorem 3.2. *Let (X, d) be a q-left-Hausdorff q-left-complete ld-metric space and let $f : X \to X$ be a q-left-continuous Banach contraction mapping. Then f has a unique fixed point.*

Proof . Existence: from Lemma 1.4, $(f_n(x_0))$ is a Cauchy sequence for each $x_0 \in X$. Since (X, d) is q-left complete, then $(f^n(x_0))$ q-left-converges to a point $x \in X$, say. From the q-left-continuity of the mapping f and Lemma 2.9, $(f^{n+1}(x_0))$ q-left-converges to $f(x)$. From Lemma 2.8, $(f^{n+1}(x_0))$ q-left-converges to x. Since (X, d) is a q-left-Hausdorff, then $f(x) = x$. $\square$

Uniqueness: suppose that there are two fixed points x and y. Then

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y) =)(1\text{âË†}'\lambda)d(x, y) \leq 0,$$
$$d(y, x) = d(f(y), f(x)) \leq \lambda d(y, x) =)(1\text{âË†}'\lambda)d(y, x) \leq 0.$$

Since $(1 - \lambda) > 0$, then we have $d(x, y) = d(y, x) = 0$. Hence, we obtain from (Mii) that $x = y$.

The following counterexample illustrates that there exists a q-left-Hausdorff q-left-complete ld-metric space in which the converse of Proposition 1.1 [8] is not true.

Counterexample: Let $X = \{x, y, z\}$. Define $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = d(z, x) = d(z, y) = \frac{1}{8}, d(y, x) = d(x, z) = d(y, z) = \frac{1}{6}, d(x, x) = \frac{1}{7}, d(y, y) = 0, d(z, z) = \frac{1}{4}$$

(1) One can easily verifies that (X, d) is an ld-metric space.

(2) Any sequence (x_n) in X is one of the following forms:

(a) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = x$;

(b) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = y$;

(c) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = z$;

(d) $\forall n \in N$ such that $x_n = x$ $\exists$ $n \in N$ such that $m > n$ and $x_m = z$ and $\forall k \in N$ such that $x_k = z$ $\exists l \in N$ such that $l > k$ and $x_l = x$;

(e) $\forall n \in N$ such that $x_n = y$ $\exists$ $n \in N$ such that $m > n$ and $x_m = z$ and $\forall k \in N$ such that $x_k = z$ $\exists l \in N$ such that $l > k$ and $x_l = y$;

(f) $\forall n \in N$ such that $x_n = x$ $\exists$ $n \in N$ such that $m > n$ and $x_m = x$ and $\forall k \in N$ such that $x_k = x$ $\exists k \in N$ such that $l > k$ and $x_l = y$.

Since only any sequence of form (a) is a Cauchy sequence and q-left-converges to x, then (X, d) is q-leftcomplete.

(3) One can deduce that any sequence of from (a) which are the only q-left-convergent sequences in X, q-left-converges to the unique point x. Hence (X, d) is q-left-Hausdorff.

(4) One can verifies that $\tau_d = \{X, \varnothing, \{y\}, \{x, y\}\}$ and note that any sequence of the form (b) τ_d-converges to x but does not q-left-converges to x.

Remark 3.3. *Note that although (X, d) in Counterexample 3.1 is q-left-Hausdorff but (X, τ_d) is not Hausdorff.*

4. A generalization of Banach contraction mapping in right-d-metric space

We give a generalization of the Banach contraction mapping in rd-metric space.

Definition 4.1. *A right-d-metric space (X, d) is called a q-right-Hausdorff space iff every right-q-convergent sequence (x_n) in X right-q-converges to a unique point in X.*

Theorem 4.2. *Let (X, d) be a q-left-Hausdorff q-right-complete rd-metric space and let $f : X \to X$ be a q-right-continuous Banach contraction mapping. Then f has a unique fixed point.*

Proof . Existence: from Lemma 1.2, $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$. Since (X, d) is q-right complete, then $(f^n(x_0))$ q-right-converges to a point $x \in X$, say. From the q-right-continuity of the mapping f and Lemma 2.2, $(f^{n+1}(x_0))$ q-right-converges to $f(x)$. From Lemma 2.1, $(f^{n+1}(x_0))$ q-right-converges to x. Since (X, d) is a q-left-Hausdorff, then $f(x) = x$. Uniqueness: suppose that there are two fixed points x and y. Then

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y) =)(1 - \lambda)d(x, y) \leq 0,$$
$$d(y, x) = d(f(y), f(x)) \leq \lambda d(y, x) =)(1 - \lambda)d(y, x) \leq 0.$$

Since $(1-\lambda) > 0$, then we have $d(x, y) = d(y, x) = 0$. Hence we obtain from (Mii) that $x = y$. The following counterexample illustrate that there exists a q-left Hausdorff q-right-complete rd-metric space in which the converse of Proposition 1.1 [8] is not true. □

Counterexample: Let $X = \{x, y, z\}$. Define $d_1 : X \times X \to [0, \infty)$ by $d_1(a, b) = d(b, a)$ $\forall a, b \in X$, where d is defined as in Counterexample 3.1. One can verifies that (X, d) is a q-right-Hausdorff q-right-complete rd-metric space. One can verifies that $\tau_d^{-1} = \{X, \varnothing, \{y\}, \{x, y\}\}$. Note that any sequence of the form (c) τ_d^{-1}-converges to x but does not q-right-converge to x.

Remark 4.3. *Note that although (X, d_1) in Counterexample 4.1 is q-right-Hausdorff but (X, τ_{d_1}) is not Hausdorff.*

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