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Numerical investigation for solving non-linear partial differential equation using Sumudu-Elzaki transform decomposition method

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Abstract

In this study, we used a powerful method, named as Sumudu-Elzaki transform method (SETM) together with Adomian polynomials (APs), which can be used to solve non-linear partial differential equations. We will give the essential clarification of this method by expanding some numerical examples to exhibit the viability and the effortlessness of this technique which can be used to solve other non-linear problems.

Keywords: Sumudu-Elzaki transform, Adomin decomposition, Non-linear partial differential equation.
2010 MSC: Please write mathematics subject classification of your paper here.

1. Introduction

Years ago, many differential equations have been solved using integral transforms, such as Laplace Transform (LT), Fourier Integral Transform (FIT), Sumudu Transform (ST), Elzaki Transform (ET) which they are the most commonly used in the literature [3, 25, 8, 27, 21, 6, 7]. Adomian decomposition method (ADM) is developed by George Adomian (USA) for solving ordinary and nonlinear partial differential equations [14, 10, 11, 24, 22, 4]. The non-linear partial differential equations (NLPDEs) show up in numerous utilizations of math, physical science, science and designing, thus the specialist presents various strategies for settling it, for example, Homotopy Perturbation Method

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(HPM) [26], Variation Iteration Method (VIM) [19]. The above strategies with integral transform are utilized in a number of techniques like Laplace Variation Iteration Method (LVIM) [12], Sumudu Homotopy Perturbation Method (SHPM) [5], Elzaki Variation Iteration Method (EVIM) [13]. Many authors combined these transformation with the (ADM) for solving (NLPDEs) such as Laplace Decomposition Method (LDM) [11, 15, 16], Sumudu Decomposition Method (SDM) [24], Elzaki Decomposition Method (EDM) [28, 23]. While other authors combined two of these transforms together for solving some kinds of differential equation. Shams A. et al. [1] used (Laplace-Sumudu) for solving integral differential equation; whereas Alla M. et al. [9] solved Hirota Schrodinger and Complex MKDV equations by using (Laplace-Elzaki).
In this study, another strategy for solving (NLPDEs) which is called Sumudu-Elzaki Transform Decomposition Method (SETDM) is presented. The entire build-up of the present study involves the following: definitions of the (SETM) covered by Section 2, while section 3 highlights the basic derivative properties of (SETM). Section 4, however, is a proof of the convergence theorem of (SETM), while section 5 presents (SETDM). Then in section 6, approximate solutions of the non-linear equations are shown to be close to the exact solutions. Accordingly then, few examples are given as such to elucidate this process and to prove its effectiveness. The study is rounded-up with several conclusions.

2. Basic Definitions and theorems of (SETM)

Definition 2.1. *Consider $h(x,t)$, a function of two variables $x,t \in R^+$, which can be expressed as an infinite convergent series, The (SETM) of the function $h(x,t)$ is denoted by:*

$$SE[h(x,t)] = H(\alpha,\beta) = \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}dxdt \tag{2.1}$$

It is clear that (SETM) is a linear integral transformation as:

$$\begin{aligned} SE[\gamma h(x,t)+\delta g(x,t)] =& \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}[\gamma h(x,t)+\delta g(x,t)]dxdt \\ &= \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}\gamma h(x,t)dxdt \\ &\quad + \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}\delta g(x,t)dxdt \\ &= \frac{\beta\gamma}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}h(x,t)dxdt \\ &\quad + \frac{\beta\delta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}g(x,t)dxdt \\ &= \gamma SE[h(x,t)] + \delta SE[g(x,t)] \end{aligned} \tag{2.2}$$

Where γ and δ are constants, and consider that β and α be enough large constants. The inverse of (SETM) $SE^{-1}[H(\alpha,\beta)] = h(x,t)$is defined by:

$$\begin{aligned} SE^{-1}[H(\alpha,\beta)] =& h(x,t) \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{\alpha}e^{-\frac{x}{\alpha}}d\alpha.\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\beta e^{-\frac{t}{\beta}}H(\alpha,\beta)d\beta \end{aligned} \tag{2.3}$$

$H(\alpha,\beta)$ should be analytic function defined by the inequalities Re $\alpha \le a$ and Re $\beta \le b$, for all α and β in the region and a and b are real constants that must be chosen carefully.

Definition 2.2. *A function $h(x,t)$ is of exponential order for $a > 0, b > 0$ on $0 \le x < \infty$, $0 \le t < \infty$, if $\exists K > 0, s.t \mid h(x,t) \mid \le Ke^{ax+bt}, \forall x > X, t > T$, where K is constant and we write $h(x,t) = O(e^{ax+bt})$ as $x \longrightarrow \infty$, $t \longrightarrow \infty$, or, equivalently*
$\lim_{x\longrightarrow\infty,t\longrightarrow\infty} e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)} \mid h(x,t) \mid = \lim_{x\longrightarrow\infty,t\longrightarrow\infty} e^{-\left(\frac{1}{\alpha}-a\right)x-\left(\frac{1}{\beta}-b\right)t} = 0, \frac{1}{\alpha} > a, \frac{1}{\beta} > b.$
The function $h(x,t)$ is called an exponential order as $x \longrightarrow \infty$, $t \longrightarrow \infty$, and obviously, it doesn't grow faster than Ke^{ax+bt} as $x \longrightarrow \infty$, $t \longrightarrow \infty$.

Theorem 2.3. *If a function $h(x,t)$, continuous in finite interval $(0,X)$ and $(0,T)$, is of exponential order e^{ax+bt}, then the (SETM) $h(x,t)$ of exist for all $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ provided Re$[\frac{1}{\alpha}] > a$ and Re$[\frac{1}{\beta}] > b$.*

Proof . From the (Def. 2.2), we have

$$\begin{aligned}
\mid H(\alpha,\beta) \mid &= \mid \frac{\beta}{\alpha}\int_0^\infty \int_0^\infty e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)} h(x,t)dxdt \mid \\
&\le K\frac{1}{\alpha}\int_0^\infty e^{-x\left(\frac{1}{\alpha}-a\right)}dx\beta\int_0^\infty e^{-t\left(\frac{1}{\beta}-b\right)}dt \\
&\frac{K\beta^2}{(1-\alpha a)(1-\beta b)}, Re[\frac{1}{\alpha}] > a, Re[\frac{1}{\beta}] > b
\end{aligned} \tag{2.4}$$

Then, from Eq. 2.4 we have $\lim_{x\longrightarrow\infty,t\longrightarrow\infty} \mid H(\alpha,\beta) \mid = 0$, or $\lim_{x\longrightarrow\infty,t\longrightarrow\infty} H(\alpha,\beta) = 0$. □

3. Basic Derivative Properties of the (SETM)

If $H(\alpha,\beta) = SE[h(x,t)]$, then

1 $SE\left[\frac{\partial h(x,t)}{\partial x}\right] = \frac{1}{\alpha}H(\alpha,\beta) - \frac{1}{\alpha}E(h(0,t))$

Proof . $SE\left[\frac{\partial h(x,t)}{\partial x}\right] = \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}\frac{\partial h(x,t)}{\partial x}dxdt = \beta\int_0^\infty e^{-\frac{t}{\beta}}dt\frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}\frac{\partial h(x,t)}{\partial x}dx$
Using integration by parts, let $u = e^{-\frac{x}{\alpha}}, dv = \frac{\partial h(x,t)}{\partial x}dx$, then
$SE\left[\frac{\partial h(x,t)}{\partial x}\right] = \beta\int_0^\infty e^{-\frac{t}{\beta}}dt\left\{\frac{1}{\alpha}e^{-\frac{x}{\alpha}}h(x,t) \mid_0^\infty + \frac{1}{\alpha}\frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\beta}}h(x,t)dx\right\} = \frac{1}{\alpha}H(\alpha,\beta) - \frac{1}{\alpha}E(h(0,t))$. □

2 $SE\left[\frac{\partial \Phi(x,t)}{\partial t}\right] = \frac{1}{\beta}H(\alpha,\beta) - \beta S(h(x,0))$

Proof . $SE\left[\frac{\partial h(x,t)}{\partial t}\right] = \frac{\beta}{\alpha}\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}}\left[\frac{\partial h(x,t)}{\partial t}\right]dxdt = \frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}dx\beta\int_0^\infty e^{-\frac{t}{\beta}}\left[\frac{\partial h(x,t)}{\partial t}\right]dt$
Using integration by parts, let $u = e^{-\frac{t}{\beta}}, dv = \frac{\partial h(x,t)}{\partial t}dx$, then
$SE\left[\frac{\partial h(x,t)}{\partial t}\right] = \frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}dx\left\{\beta e^{-\frac{t}{\beta}}h(x,t) \mid_0^\infty + \frac{1}{\beta}\int_0^\infty e^{-\frac{t}{\beta}}\frac{\partial h(x,t)}{\partial t}dt\right\} = \frac{1}{\beta}H(\alpha,\beta) - \beta S(\Phi(x,0))$ □

Similarly

3 $SE\left[\frac{\partial^2 h(x,t)}{\partial x^2}\right] = \frac{1}{\alpha^2}H(\alpha,\beta) - \frac{1}{\alpha^2}E(h(0,t)) - \frac{1}{\alpha}E\left(\frac{\partial h(0,t)}{\partial x}\right)$

4 $SE\left[\frac{\partial^2 h(x,t)}{\partial t^2}\right] = \frac{1}{\beta^2}H(\alpha,\beta) - S(h(x,0)) - \beta S\left(\frac{\partial h(x,0)}{\partial t}\right)$

5 $SE\left[\frac{\partial^2 h(x,t)}{\partial x \partial t}\right] = \frac{1}{\alpha\beta}H(\alpha,\beta) - \frac{1}{\alpha\beta}E(h(x,0)) - \beta S\left(\frac{\partial h(x,0)}{\partial x}\right)$

4. The Convergence Theorems of (SETM)

Theorem 4.1. *Let the function $h(x,t)$ is continuous in the $xt-$plane, if the integral converges at $\alpha=\alpha_0, \beta=\beta_0$ then the integral, $\frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}dxdt$ is convergence for $\alpha<\alpha_0, \beta<\beta_0$. For the proof, we will utilize the accompanying theorems.*

Theorem 4.2. *Suppose that $\beta\int_0^\infty e^{-\frac{t}{\beta}}h(x,t)dt$, converges at $\beta=\beta_0$, then the integral converges for $\beta<\beta_0$.*

Proof . For the proof see [20]. □

Theorem 4.3. *Suppose that $\frac{1}{\alpha}\int_0^\infty e^{-\frac{t}{\alpha}}h(x,t)dt$, converges at $\alpha=\alpha_0$, then the integral converges for $\alpha<\alpha_0$.*

Proof . For the proof see [2]. □

Now the proof of the Th. 4.1 is as follows

$$\begin{aligned}\frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}dxdt &= \frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}\left(\beta\int_0^\infty e^{-\frac{t}{\beta}}h(x,t)dt\right)dx\\ &= \frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}\varphi(x,t)dx \end{aligned}\tag{4.1}$$

Where $\varphi(x,\beta)=\beta\int_0^\infty e^{-\frac{t}{\beta}}h(x,t)dt$, by using Th.4.2 the integral $\beta\int_0^\infty e^{-\frac{t}{\beta}}h(x,t)dt$ converges for $\beta<\beta_0$, and by using Th. 4.3 the integral $\frac{1}{\alpha}\int_0^\infty e^{-\frac{x}{\alpha}}\varphi(x,\beta)dx$ converge for $\alpha<\alpha_0$, we see the integral in RHS of Eq. 4.1 is converges for $\alpha<\alpha_0$, $\beta<\beta_0$, hence the integral $\frac{\beta}{\alpha}\int_0^\infty\int_0^\infty h(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}dxdt$ Converge for $\alpha<\alpha_0$, $\beta<\beta_0$, and this complete the proof of Th. 4.1.

5. Descriptions of the Method

This method is described as in the following manner. Let us consider the (NLPDEs) with the initial condition (I.C) of the following form:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),\ u(x,0)=h(x), u_t(x,t)=f(x)\tag{5.1}$$

Where, L is a second order partial differential operator with respect to t $L=\frac{\partial^2}{\partial t^2}$, R is a remaining differential linear operator, N represents a general nonlinear differential operator, and $g(x,t)$ is a source term.

Using the linearity and the differentiation properties of the (SETM) for Eq. 5.1 and (ST) for the (I.C) yields:

$$SE(Lu(x,t))+SE(Ru(x,t))+SE(+Nu(x,t))=SE(g(x,t))\tag{5.2}$$

$$S(u(x,0))=S(h(x))=H(\alpha,0), S(u_t(x,0))=S(f(x))=\frac{\partial}{\partial t}H(\alpha,0)\tag{5.3}$$

to substitute Eq.5.2 in Eq.5.3, in the wake of utilizing derivative property (2), we get:

$$SE(u(x,t))=\tau^2SE(g(x,t))+\tau^4S(h(x))+\tau^3S(f(x))-\tau^2SE(Ru(x,t))-\tau^2SE(Nu(x,t))\tag{5.4}$$

now, by using the inverse (SETM) to Eq. 5.1 we get:

$$u(x,t) = G(x,t) - SE^{-1}\left[\tau^2[SE(Ru(x,t)) + SE(Nu(x,t))]\right] \tag{5.5}$$

where $G(x,t)$ illustrates the terms arising from the source term and the prescribed initial conditions.
After this step, we use the following decomposition series for the linear term:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \tag{5.6}$$

and also, the infinite series defined by

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u(x,t)) \tag{5.7}$$

is used for the nonlinear terms.
Here A_n represents the (APs), described by the formula given below:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty}\lambda^i u_i)]_{\lambda=0}, n = 0,1,2,3,... \tag{5.8}$$

now, substitute Eqs. (5.6-5.7) in Eq. 5.5, we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - SE^{-1}\left[\tau^2[SE(R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n)]\right] \tag{5.9}$$

then from Eq.5.9 we have:

$$\begin{cases} u_0(x,t) = G(x,t), \\ u_1(x,t) = -SE^{-1}(\tau^2[SE(Ru_0(x,t) + A_0)]), \\ u_2(x,t) = -SE^{-1}(\tau^2[SE(Ru_1(x,t) + A_1)]), \end{cases} \tag{5.10}$$

then from Eq.5.10 we can get the general recursive formula as:

$$u_n(x,t) = -SE^{-1}\left[\tau^2[SE(Ru_{n-1}(x,t) + A_{n-1})]\right], n \geq 1 \tag{5.11}$$

so, the approximate solution $u(x,t)$ is given by this series: $u(x,t) = \lim_{n\to\infty}\sum_{n=0}^{\infty} u_n(x,t)$.

6. Illustrative examples

Example 6.1. *[17] Consider the following (NLPDE)*

$$u_t + uu_x - u_{xx} = 0 \tag{6.1}$$

Subject to the (I.C):
$u(x,0) = x$.
By applying (SETM) to Eq. 6.1 we have:

$$\frac{1}{\beta}H(\alpha,\beta) - \beta S(h(\alpha,0)) = SE(u_{xx} - uu_x) \tag{6.2}$$

by using (ST) to (I.C) we have:

$$S(u(x,0)) = H(\alpha,0) = S(x) = \alpha \tag{6.3}$$

from Eq. 6.3 and Eq. 6.2, we obtain:

$$H(\alpha,\beta) = \alpha\beta^2 + \beta SE(u_{xx} - uu_x) \tag{6.4}$$

by using the inverse (SETM) to Eq. 6.4, we get:

$$u(x,t) = x + SE^{-1}(\beta SE(u_{xx} - uu_x)) \tag{6.5}$$

After using the (ADM), we can write Eq. 6.5 as,

$$\sum_{n=0}^{\infty} u_n(x,t) = x + SE^{-1}[\tau^2 SE(\sum_{n=0}^{\infty}(u_n)_{xx} - \sum_{n=0}^{\infty} A_n(u))]. \tag{6.6}$$

Where, $A_n(u)$ are (APs) that illustrate the nonlinear terms.
The first few Ingredients of $A_n(u)$ are shown as:

$$\begin{cases} A_0(u) = u_0(u_0)_x, \\ A_1(u) = (u_0)_x u_1 + u_0(u_1)_x, \\ A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0, \\ A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0, \\ . \\ . \\ . \end{cases} \tag{6.7}$$

by contrasting the two sides of Eq. 6.6, we have:

$$u_0(x,t) = xu_{n+1}(x,t) = SE^{-1}[\tau^2 SE((u_n)_{xx} - A_n(u))], n \geq 0. \tag{6.8}$$

then:
$u_1(x,t) = SE^{-1}[\tau^2 SE((u_0)_{xx} - A_0(u))] = SE^{-1}[\tau^2 SE(-x)] = -SE^{-1}[\alpha\tau^2] = -xt$
$u_2(x,t) = SE^{-1}[\tau^2 SE((u_1)_{xx} - A_1(u))] = SE^{-1}[\tau^2 SE(2xt)] = -SE^{-1}[2\alpha\tau^4] = xt^2$
by the similar way we get:
$u_3(x,t) = -xt^3$
and so on. Then the first four terms of the decomposition series for Eq. 6.1, is given as:
$u(x,t) = x - xt + xt^2 - xt^3 + ...,$
the solution in a closed form is given as:
$u(x,t) = \frac{x}{1+t}, \mid t \mid < 1.$
Along with any numerical verification of the proposed method which is definitely conducive to a higher accuracy, we may resort to evaluating the numerical solutions by using the 10-term approximation for Eq. 6.1. In this respect, we can see that while Table 1 show the difference of the absolute errors of between the approximate solution and exact solution. However, there are 10-terms used to evaluate the approximate solutions. On the same footing, with the actual solution of the equations, we used a subtle approximation. This is accomplished by manipulating only the first ten terms of the above-mentioned decomposition. No doubt that the corpus of the errors can be minimized through the addition of new terms of the decomposition series.

Table 1: The Numerical Result of the Absolute Error of Example 6.1 by Comparison between the exact solution with approximate solution for 10-term approximation

t \ x	0.2	0.4	0.6	0.8	1.0
0.01	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
0.02	2.7756e-17	5.5511e-17	3.33307-16	1.1102e-16	0.0000e+00
0.03	2.7756e-17	5.5511e-17	1.1102e-16	1.1102e-16	0.0000e+00
0.04	8.3267e-17	1.6653e-16	2.2204e-16	3.33307-16	4.4409e-16
0.05	9.4369e-16	1.8874e-15	2.6645e-15	3.7748e-15	4.6629e-15

Example 6.2. *[18] Consider the following KdV equations*

$$u_t - 6uu_x + u_{xxx} = 0, \tag{6.9}$$

Subject to (I.C):
$u(x,0) = \frac{1}{6}(x-1)$
Applying (SETM) to Eq. 6.9, we have:

$$\frac{1}{\beta}H(\alpha,\beta) - \beta S(h(\alpha,0)) = SE(6uu_x - u_{xxx}), \tag{6.10}$$

by using (ST) to (I.C) we have:

$$S(u(x,0)) = H(\alpha,0) = S(x) = \frac{1}{6}(\alpha - 1), \tag{6.11}$$

from Eq.6.11 and Eq.6.10, we obtain:

$$H(\alpha,\beta) = \frac{1}{6}(\alpha\beta^2 - \beta^2) + \beta SE(6uu_x - u_{xxx}), \tag{6.12}$$

by using the inverse (SETM) to Eq. 6.12, we get:

$$u(x,t) = \frac{1}{6}(x-1) + SE^{-1}(\beta SE(6uu_x - u_{xxx})). \tag{6.13}$$

After using the (ADM) we can write Eq. 6.13 as,

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{6}(x-1) + SE^{-1}[\beta SE(6\sum_{n=0}^{\infty} A_n(u) - \sum_{n=0}^{\infty}(u_n)_{xxx})]. \tag{6.14}$$

by contrasting the two sides of Eq. 6.14, we have:

$$u_0(x,t) = \frac{1}{6}(x-1)u_{n+1}(x,t) = SE^{-1}[\beta SE(6A_n(u) - (u_n)_{xxx})], n \geq 0. \tag{6.15}$$

then:
$u_1(x,t) = SE^{-1}[\beta SE(6A_0(u) - (u_0)_{xxx})] = SE^{-1}[\beta SE\left(6\frac{1}{36}(x-1)\right)] = SE^{-1}\left[\frac{1}{6}(\alpha\beta^3 - \beta^3)\right] = \frac{1}{6}(x-1)t.$
$u_2(x,t) = SE^{-1}[\beta SE(6A_1(u) - (u_1)_{xxx})] = SE^{-1}[\beta SE\left(6\frac{1}{36}(2xt - 2t)\right)] = SE^{-1}\left[\frac{1}{6}(4\alpha\beta^4 - 4\beta^4)\right] = \frac{1}{6}(x-1)t^2.$
by the same way we get:
$u_3(x,t) = \frac{1}{6}(x-1)t^3$
and so on.
Where the first few ingredients of $A_n(u)$ are shown as:

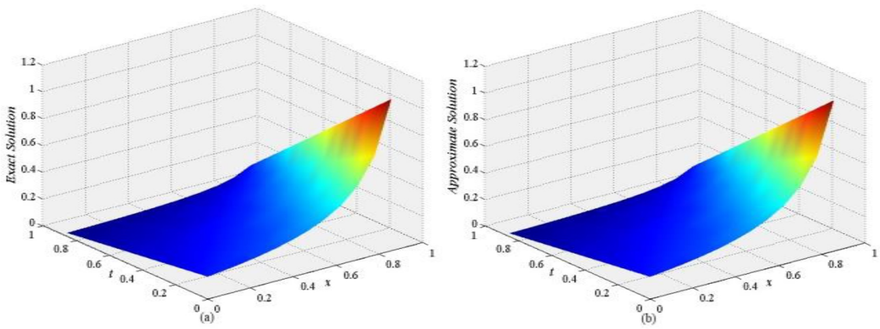

Figure 1: The solution of Eq. (25) by the proposed method (a-Exact solution), (b-Approximate solution).

$$\begin{cases} A_0(u) = u_0(u_0)_x, \\ A_1(u) = (u_0)_x u_1 + u_0(u_1)_x, \\ A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0, \\ A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0, \\ . \\ . \\ . \end{cases} \tag{6.16}$$

Then the first four terms of the decomposition series for Eq. 6.9, is given as:
$u(x,t) = \frac{1}{6}(x-1)(1+t+t^2+t^3+...)$,
the solution in a closed form is given as:
$u(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right), \mid t \mid < 1.$

Along with any numerical verification of the proposed method which is definitely conducive to a higher accuracy, we may resort to evaluating the numerical solutions by using the 10-term approximation for Eq. 6.9. In this respect, we can see that while Figure 1a shows the exact solution, Figure 1b reflects the approximate solution. However, there are ten terms used to evaluate the approximate solutions. On the same footing, with the actual solution of the equations, we used a subtle approximation. This is accomplished by manipulating only the first ten terms of the above-mentioned decomposition. No doubt that the corpus of the errors can be minimized through the addition of new terms of the decomposition series. The numerical approximations appear to have a high degree of accuracy, especially in the majority of u_n cases. Meanwhile, for very low values of n, the n-term approximation is accurate.

Example 6.3. *[17] Consider the following (NLPDE):*

$$u_{tt} - \frac{2x^2}{t} u u_x = 0, \tag{6.17}$$

subject to (I.C):
$u(x,0) = 0, u_t(x,t) = x.$
By applying (SETM) to Eq. 6.17, we have:

$$\frac{1}{\beta^2} H(\alpha,\beta) - \beta^2 S(h(\alpha,0)) - \beta S\left(\frac{\partial h(\alpha,0)}{\partial t}\right) = SE\left(\frac{2x^2}{t} u u_x\right), \tag{6.18}$$

by using (ST) to (I.C) we have:

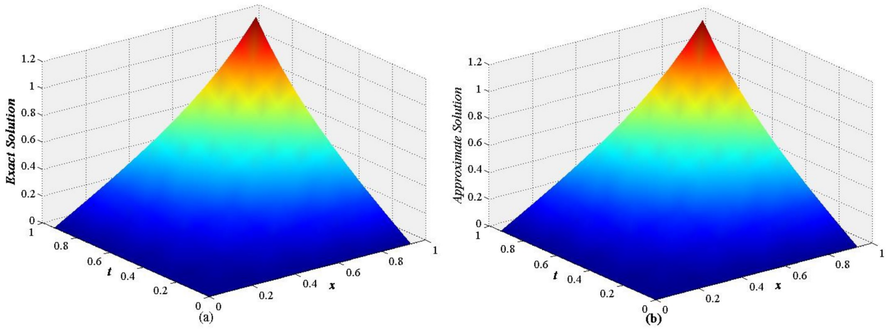

Figure 2: The solution of Eq. 6.17 by the proposed method (a-Exact solution), (b-Approximate solution).

$$S(u(x,0)) = 0 \text{ and } S(u_t(x,0)) = \frac{\partial h(\alpha,0)}{\partial t} = S(x) = \alpha, \tag{6.19}$$

From Eq. 6.19 and Eq. 6.18, we obtain:

$$H(\alpha,\beta) = \beta^3\alpha + \beta^2 SE\left(\frac{2x^2}{t}uu_x\right), \tag{6.20}$$

by using the inverse (SETM) to Eq. 6.20, we get:

$$u(x,t) = xt + SE^{-1}\left(\beta^2 SE\left(\frac{2x^2}{t}uu_x\right)\right). \tag{6.21}$$

After using the (ADM), we can write Eq. 6.21 as,

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}\sum_{n=0}^{\infty} A_n(u)\right)\right]. \tag{6.22}$$

Where, $A_n(u)$ are (APs) that illustrate the nonlinear terms.
By comparing both sides of Eq. 6.22, we have:

$$u_0(x,t) = xtu_{n+1}(x,t) = SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}A_n\right)\right], n \geq 0. \tag{6.23}$$

then:
$u_1(x,t) = SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}A_0\right)\right] = SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}xt^2\right)\right] = SE^{-1}[\beta^2 SE(2x^3t)] = SE^{-1}[12\alpha^3\beta^5] = \frac{1}{3}x^3t^3,$
$u_2(x,t) = SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}A_1\right)\right] = SE^{-1}\left[\beta^2 SE\left(\frac{2x^2}{t}(\frac{1}{3}x^3t^4 + x^3t^4)\right)\right] = SE^{-1}\left[\beta^2 SE\left((\frac{2}{3}x^5t^3 + 2x^5t^3)\right)\right]$
$= SE^{-1}[480\alpha^5\beta^7 + 1440\alpha^5\beta^7] = SE^{-1}[1920\alpha^5\beta^7] = \frac{2}{15}x^5t^5,$

In a similar way we get:
$u_3(x,t) = \frac{17}{315}x^7t^7$
and so on. Then the first four terms of the decomposition series for Eq. 6.17, is given as:
$u(x,t) = xt + \frac{1}{3}x^3t^3 + \frac{2}{15}x^5t^5 + \frac{17}{315}x^7t^7 + ...,$
the solution in a closed form is given as:
$u(x,t) = \tan(xt).$
Moreover, Figures 2a,b shows the exact and approximate solutions respectively.

7. Conclusions

In this paper, the collection between (ADM) and (SETM) is proposed. We use the advantage of this method to obtain the numerical approximate solutions as compared with the exact solution of some (NLPDEs) such as third Order Korteweg-De Vries Equations (KdV) equations. It is shown that this method is simple and direct very efficient. At last, we can say that this method is actually dependable and applicable to all (NLPDEs).

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