Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 991-1002
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24010.2652

On an equation characterizing multi-quartic mappings and its stability

Zohreh Abbasbeygi[a], Abasalt Bodaghi[b,*], Ayoub Gharibkhajeh[a]

[a]*Department of Mathematics, Tehran North Branch, Islamic Azad University, Tehran, Iran*
[b]*Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*

(Communicated by Ali Jabbari)

Abstract

In this paper, we define and investigate the several variables mappings which are quartic in each component. We show that such mappings can be unified as an equation, say, the multi-quartic functional equation. We also establish the Hyers-Ulam stability of multi-quartic functional equation by a fixed point theorem in non-Archimedean normed spaces. Moreover, we generalize some known stability results.

Keywords: Hyers−Ulam stability, multi-quartic mapping, non-Archimedean normed space
2010 MSC: 39B52, 39B72, 39B82, 46B03

1. Introduction and preliminaries

The Ulam stability problem [30] (that was answered by Hyers [17]) has been extended and studied for the several variables mappings in two last decades. Recall that a functional equation $\mathfrak{F}$ is said to be

- *stable* if any mapping ϕ fulfilling $\mathfrak{F}$ approximately, then it is near to an exact solution of $\mathfrak{F}$;
- *hyperstable* if any function ϕ fulfilling $\mathfrak{F}$ approximately (under some conditions), then it is an exact solution of $\mathfrak{F}$.

Here, we indicate some historical notes about some several variables mappings. Suppose that V is a commutative group, W is a linear space, and $n \geq 2$ is an integer. In what follows, consider a several variables mapping $f : V^n \longrightarrow W$. This mapping is called *multi-additive* if it satisfies Cauchy's

*Corresponding author
Email addresses: zohrehabbasbeygi@yahoo.com (Zohreh Abbasbeygi), abasalt.bodaghi@gmail.com (Abasalt Bodaghi), a.gharibkhajeh@gmail.com (Ayoub Gharibkhajeh)

Received: May 2021 *Accepted:* September 2021

functional equation $A(x+y)=A(x)+A(y)$ in each variable. For the convenience of the reader we refer the relevant facts from [21] and many other sources therein. Ciepliński in [12] showed that f is multi-additive if and only if it satisfying the equation

$$f(x_1+x_2)=\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}),$$

where $x_j=(x_{1j},x_{2j},\ldots,x_{nj})\in V^n$ with $j\in\{1,2\}$. Moreover, f is called *multi-quadratic* if it fulfills the quadratic functional equation $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ in each variable [24]. In [33], Zhao et al. proved that a mapping $f:V^n\longrightarrow W$ is multi-quadratic if and only if the following relation holds.

$$\sum_{t\in\{-1,1\}^n} f(x_1+tx_2)=2^n\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}).$$

For the generalized form and Jensen type of multi-quadratic mappings refer to [2], [6] and [29]. Some stabilities for multi-additive and multi-quadratic mappings in miscellaneous Banach spaces have been studied for instance in [10], [11], [12], [13], [24], [32] and [33].

The multi-cubic mappings were introduced for the first time in [15]. In other words, a mapping $f:V^n\longrightarrow W$ is called *multi-cubic* (in the special case of [15]) if it is cubic in each variable, i.e., f satisfies the equation

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x)$$

in each component [19]. In [8], the second author and Shojaee investigated the structure of multi-cubic mappings and proved every multi-cubic functional equation can be stable and hyperstable. For other forms of cubic functional equations and their stabilities we refer to [18] and [28]. Various versions of multi-cubic mappings, functional equations and their stabilities can be found in [14] and [25].

The quartic functional equation

$$\mathcal{Q}(x+2y)+\mathcal{Q}(x-2y)=4\mathcal{Q}(x+y)+4\mathcal{Q}(x-y)-6\mathcal{Q}(x)+24\mathcal{Q}(y) \tag{1.1}$$

was introduced for the first time by Rassias [27]. Motivated by equation (1.1), Bodaghi et al. defined the multi-quartic mappings for the first time in [5] and provided a characterization of such mappings. In fact, they showed that every multi-quartic mapping can be shown as a single functional equation and vice versa. An alternative multi-quartic mapping and its characterization which has been recently studied is available in [26].

The upcoming quartic functional equation which is an equivalent equation to (1.1), defined by Lee et al., in [23].

$$\mathfrak{Q}(2x+y)+\mathfrak{Q}(2x-y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)+24\mathfrak{Q}(x)-6\mathfrak{Q}(y). \tag{1.2}$$

They also established the Hyers−Ulam stability of (1.2); for the generalized forms of quartic functional equations (1.1) and (1.2) we refer to [3], [4] and [22].

Recall that a non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that has the properties as follows: $|a|=0$ if and only if $a=0$, $|ab|=|a||b|$, and $|a+b|\leq\max\{|a|,|b|\}$ for all $a,b\in\mathbb{K}$. Clearly, $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Suppose that $\mathcal{X}$ ia a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. Then, the function $\|\cdot\|:\mathcal{X}\longrightarrow\mathbb{R}$ is called a non-Archimedean norm (valuation) if it satisfying the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|, \quad (x \in \mathcal{X}, a \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad (x, y \in \mathcal{X}).$$

The pair $(\mathcal{X}, \|\cdot\|)$ is said to be a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\}, \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. A complete non-Archimedean normed space is a non-Archimedean normed space $\mathcal{X}$ provided that every Cauchy sequence is convergent.

Some examples of non-Archimedean normed spaces are the p-adic numbers, which are related to quantum physics, p-adic strings and superstrings [20]. For more details of p-adic numbers as a number theoretical analogue of power series in complex analysis, we refer to [16].

It is worth mentioning that an alternative fixed point theorem which was presented in [9] have been considered as a tool for the stability of several variables mappings such as multi-Cauchy-Jensen and multi-additive-quadratic mappings; see [1]. In addition, for the stability of multi-Jensen, multi-additive, multi-quadratic and multi-cubic mappings in non-Archimedean spaces refer to [6], [7], [14], [31] and [32].

In this article, we introduce the new multi-quartic mappings (taken from (1.2)) which are different from what defined in [5] and [26]. We also include a characterization of such mappings. In fact, we prove that every multi-quartic mapping can be shown a single functional equation and vice versa (under some extra conditions). Moreover, we investigate the Hyers−Ulam stability for multi-quartic mappings by applying a fixed point method in non-Archimedean normed spaces [9]. As a result, we show that under some mild conditions this new multi-quartic functional equation can be hyperstable.

2. Characterization of multi-quartic mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $l \in \mathbb{N}_0, m \in \mathbb{N}$, $t = (t_1, \ldots, t_m) \in \{-1, 1\}^m$ and $x = (x_1, \ldots, x_m) \in V^m$, we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. For $x_1, x_2 \in V^n$ and $p_i \in \mathbb{N}_0$ with $0 \leq p_i \leq n$, put $\mathcal{N} = \{\mathfrak{N}_n = (N_1, \ldots, N_n) |\ N_j \in \{x_{1j} \pm x_{2j}, x_{1j}, x_{2j}\}\}$, where $j \in \{1, \ldots, n\}$ and $i \in \{1, 2\}$. Consider the subset $\mathcal{N}^n_{(p_1,p_2)}$ of $\mathcal{N}$ as follows:

$$\mathcal{N}^n_{(p_1,p_2)} := \{\mathfrak{N}_n \in \mathcal{N} |\ \mathrm{Card}\{N_j :\ N_j = x_{ij}\} = p_i \ \ (i \in \{1, 2\})\}.$$

Definition 2.1. *Let V and W be vector spaces over the rational numbers $\mathbb{Q}$. A mapping $f : V^n \longrightarrow W$ is said to be n-multi-quartic or multi-quartic if f satisfies (1.2) in each variable.*

From now on, we use the following notations for the multi-quartic mappings in sense of the above definition.

$$f\left(\mathcal{N}^n_{(p_1,p_2)}\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}^n_{(p_1,p_2)}} f(\mathfrak{N}_n), \tag{2.1}$$

$$f\left(\mathcal{N}^n_{(p_1,p_2)}, z\right) := \sum_{\mathfrak{N}_n\in\mathcal{N}^n_{(p_1,p_2)}} f(\mathfrak{N}_n, z) \qquad (z\in V).$$

Proposition 2.2. *Let $f : V^n \longrightarrow W$ be a multi-quartic mapping. Then, it fulfills the functional equation*

$$\sum_{t\in\{-1,1\}^n} f(2x_1+tx_2) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}\right), \tag{2.2}$$

for all $x_1, x_2 \in V^n$.

Proof . We proceed the proof by induction on n. For $n = 1$, it is obvious that f satisfies equation (1.2). Let (2.2) be valid for some fixed and positive integer $n > 1$. Then

$$\sum_{t\in\{-1,1\}^n} f(2x_1^n+tx_2^n, z) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}, z\right), \tag{2.3}$$

for all $x_1, x_2 \in V^n$ and $z \in V$. Using (2.3) and the fact that (2.2) holds for the case $n = 1$, we arrive

$$\begin{aligned}
&\sum_{t\in\{-1,1\}^{n+1}} f\left(2x_1^{n+1}+tx_2^{n+1}\right)\\
&= 4\sum_{t\in\{-1,1\}^n}\sum_{s\in\{-1,1\}} f\left(2x_1^n+tx_2^n, x_{1,n+1}+sx_{2,n+1}\right)\\
&+ 24\sum_{t\in\{-1,1\}^n} f\left(2x_1^n+tx_2^n, x_{1,n+1}\right) - 6\sum_{t\in\{-1,1\}^n} f\left(2x_1^n+tx_2^n, x_{2,n+1}\right)\\
&= 4\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{s\in\{-1,1\}} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}, x_{1,n+1}+sx_{2,n+1}\right)\\
&+ 24\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}, x_{1,n+1}\right)\\
&- 6\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}, x_{2,n+1}\right)\\
&= \sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1} 4^{n+1-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^{n+1}_{(p_1,p_2)}\right).
\end{aligned}$$

This means that (2.2) holds for $n + 1$. □

It is easily seen that the mapping $f(z_1,\dots,z_n) = c\prod_{j=1}^n z_j^4$ is multi-quartic and so Proposition 2.2 implies that f satisfies (2.2). Therefore, this equation is said to be *multi-quartic functional equation*.

Remark 2.3. It can be verified that if equation (1.2) is true for a mapping $f : V^n \longrightarrow W$, then

$$f(cx) = c^4 f(x) \qquad (c\in\mathbb{Q}). \tag{2.4}$$

Note that the converse is not valid in general. For example, assume that $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra. Fix the vector x_0 in $\mathcal{A}$. Consider the mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ defined through $\varphi(x) = \|x\|^4 x_0$ for all $x \in \mathcal{A}$. Clearly, for each $x \in \mathcal{A}$, (2.4) is true while relation (1.2) does not hold for φ for non-zero elements $x, y \in \mathcal{A}$. Therefore, condition (2.4) does not imply that f is a quartic mapping.

Definition 2.4. *We say a mapping $f : V^n \longrightarrow W$*

(i) *satisfies (has) the 4-power condition or quartic condition in the jth variable if*

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^4 f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $z_1, \cdots, z_n \in V^n$.

(ii) *is even in the jth variable if*

$$f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $z_1, \ldots, z_n \in V^n$.

(iii) *has zero condition if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Note that it follows from Remark 2.3 that the quartic condition does not imply f is quartic in the jth variable.

Put $\mathbf{n} := \{1, \ldots, n\}$, $n \in \mathbb{N}$. For a subset $\mathbf{m} = \{j_1, \ldots, j_i\}$ of $\mathbf{n}$ with $1 \leq j_1 < \cdots < j_i \leq n$ and $x = (x_1, \ldots, x_n) \in V^n$,

$$_{\mathbf{m}}x := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_i}, 0, \ldots, 0) \in V^n$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of $\mathbf{m}$ and whose other components are set equal zero. Note that $_0x = 0$, $_{\mathbf{n}}x = x$.

Here, we recall the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for all $n, k \in \mathbb{N}_0$ with $n \geq k$. We shall show that if a mapping $f : V^n \longrightarrow W$ satisfies equation (2.2), then it is multi-quartic. In order to do this, we need the next fundamental lemma.

Lemma 2.5. *Let a mapping $f : V^n \longrightarrow W$ fulfills equation* (2.2). *Under one of the following assumptions, f has zero condition.*

(i) *f has the quartic condition in all variables;*

(ii) *f is even in each variable.*

Proof . (i) We firstly note that

$$\binom{n-k}{n-k-p_1-p_2}\binom{p_1+p_2}{p_1} = \binom{n-k}{p_1}\binom{n-k-p_1}{p_2} \tag{2.5}$$

for $0 \leq k \leq n-1$. We argue by induction on k that for each $_kx$, $f(_kx) = 0$ where $0 \leq k \leq n-1$. Let $k = 0$. Putting $x_1 = x_2 =_0 x$ in (2.2) and using (2.5) , we have

$$\begin{aligned}
&2^n f(_0x)\\
&= \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2}2^{n-p_1-p_2} f(_0x)\\
&= \left[\sum_{p_1=0}^{n}\binom{n}{p_1} 2^{n-p_1}24^{p_1}\sum_{p_2=0}^{n-p_1}\binom{n-p_1}{p_2} 4^{n-p_1-p_2}(-3)^{p_2}\right] f(_0x)\\
&= \left[\sum_{p_1=0}^{n}\binom{n}{p_1} 2^{n-p_1}24^{p_1}\right] f(_0x)\\
&= 26^n f(_0x).
\end{aligned} \tag{2.6}$$

It follows from (2.6) that $f({}_0x) = 0$. Assume that for each ${}_{k-1}x$, $f({}_{k-1}x) = 0$. We prove that $f({}_kx) = 0$. Without loss of generality, it is assumed that ${}_kx_1 = (x_{11}, \dots, x_{1k}, 0, \dots, 0)$. By our assumption, replacing (x_1, x_2) by $({}_kx_1, {}_0x)$ in equation (2.2), we get

$$\begin{aligned}
&2^{4k+n} f({}_kx) \\
&= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \begin{pmatrix} n-k \\ n-k-p_1-p_2 \end{pmatrix} \begin{pmatrix} p_1+p_2 \\ p_1 \end{pmatrix} 4^{n-p_1-p_2} 24^{p_1} (-6)^{p_2} 2^{n-p_1-p_2} f({}_kx) \\
&= 2^{3k} \left[\sum_{p_1=0}^{n-k} \begin{pmatrix} n-k \\ p_1 \end{pmatrix} 2^{n-k-p_1} 24^{p_1} \sum_{p_2=0}^{n-k-p_1} \begin{pmatrix} n-k-p_1 \\ p_2 \end{pmatrix} 4^{n-k-p_1-p_2} (-3)^{p_2} \right] f({}_kx) \\
&= 2^{3k} \left[\sum_{p_1=0}^{n-k} \begin{pmatrix} n-k \\ p_1 \end{pmatrix} 2^{n-k-p_1} 24^{p_1} \right] f({}_kx) \\
&= 2^{3k} 26^{n-k} f({}_kx).
\end{aligned}$$

Comparing the first and last terms of the above relation, we find $f({}_kx) = 0$. This shows that f has zero condition.

(ii) Similar to part (i), we have $f({}_0x) = 0$. Furthermore, by assumption one can show that $2^{4k+n} f({}_kx) = 2^{3k} 26^{n-k} f({}_kx)$ for all $0 \leq k \leq n-1$. □

Theorem 2.6. *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation* (2.2). *Under one of the following conditions, f is multi-quartic.*

(i) *f has the quartic condition in each variable;*

(ii) *f is even in all variables.*

Proof . (i) Fix $j \in \{1, \dots, n\}$. Set

$$\begin{aligned}
f^*(x_{1j}, x_{2j}) := &\, f(x_{11}, \dots, x_{1,j-1}, x_{1j} + x_{2j}, x_{1,j+1}, \dots, x_{1n}) \\
&+ f(x_{11}, \dots, x_{1,j-1}, x_{1j} - x_{2j}, x_{1,j+1}, \dots, x_{1n}),
\end{aligned}$$

and

$$f^*(x_{1j}) := f(x_1) = f(x_{11}, \dots, x_{1n}), \qquad f^*(x_{2j}) := f(x_{11}, \dots, x_{1,j-1}, x_{2j}, x_{1,j+1}, \dots, x_{1n}).$$

Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$ in (2.2) and using property (2.4) in each variable, we get

$$\begin{aligned}
&2^{n-1} \times 2^{4(n-1)} [f(x_{11}, \dots, x_{1,j-1}, 2x_{1j} + x_{2j}, x_{1,j+1}, \dots, x_{1n}) \\
&+ f(x_{11}, \dots, x_{1,j-1}, 2x_{1j} - x_{2j}, x_{1,j+1}, \dots, x_{1n})] \\
&= 2^{n-1} [f(2x_{11}, \dots, 2x_{1,j-1}, 2x_{1j} + x_{2j}, 2x_{1,j+1}, \dots, 2x_{1n}) \\
&+ f(2x_{11}, \dots, 2x_{1,j-1}, 2x_{1j} - x_{2j}, 2x_{1,j+1}, \dots, 2x_{1n})] \\
&= \sum_{p_1=0}^{n-1} \left[\begin{pmatrix} n-1 \\ p_1 \end{pmatrix} 4^{n-p_1} 24^{p_1} 2^{n-1-p_1} \right] f^*(x_{1j}, x_{2j})
\end{aligned}$$

$$\begin{aligned}
&+\sum_{p_1=1}^{n}\left[\begin{pmatrix} n-1 \\ p_1-1 \end{pmatrix} 4^{n-p_1}24^{p_1}2^{n-p_1}\right] f^*(x_{1j}) \\
&-6\sum_{p_1=1}^{n}\left[\begin{pmatrix} n-1 \\ p_1-1 \end{pmatrix} 4^{n-p_1}24^{p_1-1}2^{n-p_1}\right] f^*(x_{2j}) \\
&=2^{2n}\sum_{p_1=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_1 \end{pmatrix} 6^{p_1}2^{n-1-p_1}\right] f^*(x_{1j},x_{2j}) \\
&+24\sum_{p_1=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_1 \end{pmatrix} 8^{n-1-p_1}24^{p_1}\right] f^*(x_{1j}) \\
&-6\sum_{p_1=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_1 \end{pmatrix} 8^{n-1-p_1}24^{p_1}\right] f^*(x_{2j}) \\
&=2^{2n}8^{n-1}f^*(x_{1j},x_{2j})+24(32)^{n-1}f^*(x_{1j})-6(32)^{n-1}f^*(x_{2j}) \\
&=2^{n+1}2^{4(n-1)}f^*(x_{1j},x_{2j})+24\times 2^{n-1}2^{4(n-1)}f^*(x_{1j})-6\times 2^{n-1}2^{4(n-1)}f^*(x_{2j}).
\end{aligned} \tag{2.7}$$

Now, (2.7) implies that

$$\begin{aligned}
&f\left(x_{11},\ldots,x_{1,j-1},2x_{1j}+x_{2j},x_{1,j+1},\ldots,x_{1n}\right)+f\left(x_{11},\ldots,x_{1,j-1},2x_{1j}-x_{2j},x_{1,j+1},\ldots,x_{1n}\right) \\
&=4f^*(x_{1j},x_{2j})+24f^*(x_{1j})-6f^*(x_{2j}).
\end{aligned}$$

The above equality means that f is quartic in the jth variable.

(ii) Fix $j\in\{1,\ldots,n\}$. Replacing (x_{1k},x_{2k}) by $(0,x_{1k})$ for all $k\in\{1,\ldots,n\}\backslash\{j\}$ in (2.2) and using assumption, we get

$$\begin{aligned}
&2^{n-1}[f\left(x_{11},\ldots,x_{1,j-1},2x_{1j}+x_{2j},x_{1,j+1},\ldots,x_{1n}\right) \\
&+f\left(x_{11},\ldots,x_{1,j-1},2x_{1j}-x_{2j},x_{1,j+1},\ldots,x_{1n}\right)] \\
&=\sum_{p_2=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_2 \end{pmatrix} 4^{n-p_2}(-6)^{p_2}2^{n-1-p_2}\right] f^*(x_{1j},x_{2j}) \\
&+24\sum_{p_2=1}^{n}\left[\begin{pmatrix} n-1 \\ p_2-1 \end{pmatrix} 4^{n-p_2}(-6)^{p_2-1}2^{n-p_2}\right] f^*(x_{1j}) \\
&+\sum_{p_2=1}^{n}\left[\begin{pmatrix} n-1 \\ p_2-1 \end{pmatrix} 4^{n-p_2}(-6)^{p_2}2^{n-p_2}\right] f^*(x_{2j}) \\
&=4\sum_{p_2=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_2 \end{pmatrix} (-6)^{p_2}8^{n-1-p_2}\right] f^*(x_{1j},x_{2j}) \\
&+24\sum_{p_2=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_2 \end{pmatrix} 8^{n-1-p_2}(-6)^{p_2}\right] f^*(x_{1j}) \\
&-6\sum_{p_2=0}^{n-1}\left[\begin{pmatrix} n-1 \\ p_2 \end{pmatrix} 8^{n-1-p_2}(-6)^{p_2}\right] f^*(x_{2j}) \\
&=4\times 2^{n-1}f^*(x_{1j},x_{2j})+24\times 2^{n-1}f^*(x_{1j})-6\times 2^{n-1}f^*(x_{2j}).
\end{aligned} \tag{2.8}$$

It follows from (2.8) that f is quartic in the jth variable. This completes the proof. □

3. Stability Results for (2.2)

In this section, we prove the Hyers−Ulam stability of multi-quartic functional equation (2.2) in non-Archimedean normed by applying a fixed point theorem. We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1+\ldots+1}^{n-\text{times}}=0$. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. The next theorem which is a key tool in obtaining our aim in this paper, is taken from [9, Theorem 1].

Theorem 3.1. *Let the following hypotheses hold.*

(H1) *E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$;*

(H2) *$\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq max_{i\in\{1,\ldots,j\}} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|,$$

for all $\lambda, \mu \in Y^E, x \in E$;

(H3) *$\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through*

$$\Lambda\delta(x) := max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Moreover, a function $\theta : E \longrightarrow \mathbb{R}_+$ and a mapping $\varphi : E \longrightarrow Y$ fulfill the next two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \qquad \lim_{l\to\infty} \Lambda^l\theta(x) = 0, \quad (x \in E).$$

Then, for every $x \in E$, the limit $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ exists and the mapping $\psi \in Y^E$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l\theta(x) \qquad (x \in E).$$

For the rest of this section, given the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\Gamma f : V^n \times V^n \longrightarrow W$ defined via

$$\Gamma f(x_1, x_2) := \sum_{t\in\{-1,1\}^n} f(2x_1 + tx_2) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}\right),$$

for $x_1, x_2 \in V^n$, where $f\left(\mathcal{N}^n_{(p_1,p_2)}\right)$ is defined in (2.1).

In the sequel, it is assumed that all mappings as $f : V^n \longrightarrow W$ satisfying zero condition. In the next theorem, we establish the stability of functional equation (2.2) from linear spaces to complete non-Archimedean normed spaces.

Theorem 3.2. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) = 0, \tag{3.1}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\Gamma f(x_1, x_2)\| \leq \varphi(x_1, x_2), \tag{3.2}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^n|2|^{4n\frac{\beta+1}{2}}} \left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(2^{l\beta+\frac{\beta-1}{2}}, 0\right), \tag{3.3}$$

for all $x \in V^n$.

Proof . Putting $x = x_1$ and $x_2 = 0$ in (3.2) and using our assumptions, we have

$$\left\|2^n f(2x) - \sum_{p_1=0}^{n} \begin{pmatrix} n \\ p_1 \end{pmatrix} 4^{n-p_1} 24^{p_1} 2^{n-p_1} f(x)\right\| \leq \varphi(x, 0),$$

for all $x \in V^n$ (and for the rest of this proof, all the equations and inequalities are valid for all $x \in V^n$), and so

$$\left\|2^n f(2x) - 2^{5n} f(x)\right\| \leq \varphi(x, 0).$$

Thus

$$\left\|f(2x) - 2^{4n} f(x)\right\| \leq \frac{1}{|2|^n}\varphi(x, 0). \tag{3.4}$$

Relation (3.4) can be rewritten as

$$\|f(x) - \mathcal{T} f(x)\| \leq \theta(x), \tag{3.5}$$

where

$$\theta(x) := \frac{1}{|2|^n|2|^{4n\frac{\beta+1}{2}}}\varphi\left(2^{\frac{\beta-1}{2}}x, 0\right), \qquad \mathcal{T}\xi(x) := \frac{1}{2^{4n\beta}}\xi\left(2^\beta x\right),$$

for all $\xi \in W^{V^n}$. Define $\Lambda\eta(x) := \frac{1}{|2|^{4n\beta}}\eta\left(2^\beta x\right)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easily seen that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^\beta x$ for $L_1(x) = \frac{1}{|2|^{4n\beta}}$. On the other hand, we have

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{2^{4n\beta}}\lambda(2^\beta x) - \frac{1}{2^{4n\beta}}\mu(2^\beta x)\right\| \leq L_1(x)\,\|\lambda(g_1(x)) - \mu(g_1(x))\|,$$

for all $\lambda, \mu \in W^{V^n}$. It follows from the above relation that the hypothesis (H2) is true. Moreover, one can check by induction on l that for any $l \in \mathbb{N}$, we find

$$\Lambda^l\theta(x) := \left(\frac{1}{|2|^{4n\beta}}\right)^l \theta\left(2^{l\beta}x\right) = \frac{1}{|2|^n|2|^{4n\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(2^{l\beta+\frac{\beta-1}{2}}, 0\right). \tag{3.6}$$

It concludes from (3.5) and (3.6) that all assumptions of Theorem 3.1 are satisfied and so there exists a unique mapping $\mathcal{Q} : V^n \longrightarrow W$ such that $\mathcal{Q}(x) = \lim_{l\to\infty}\left(\mathcal{T}^l f\right)(x)$, and (3.3) as well. We also can checked by induction on l that

$$\left\|\Gamma\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \leq \left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right), \tag{3.7}$$

for all $x_1, x_2 \in V^n$. Taking $l \to \infty$ in (3.7) and using (3.1), we obtain $\Gamma \mathcal{Q}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$ and hence equation (2.2) is valid for $\mathcal{Q}$. This finishes the proof. □

Here and subsequently, V is a non-Archimedean normed space and W is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. In addition, we assume that $|2| < 1$. The following corollaries are taken Theorem 3.2 regarding the stability of (2.2).

Corollary 3.3. *Given $\delta > 0$. Let $f : V^n \longrightarrow W$ be a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \delta,$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{1}{|2|^n}\delta,$$

for all $x \in V^n$. In addition, under one of the following hypothese, $\mathcal{Q}$ is a multi-quartic mapping.

(i) *$\mathcal{Q}$ satisfies the quartic condition in each variable;*
(ii) *$\mathcal{Q}$ is even in all variables.*

Proof . Note that $|2| < 1$. Choosing $\varphi(x_1, x_2) = \delta$ for the case $\beta = -1$ of Theorem 3.2, we get $\lim_{l\to\infty} |2|^{4nl}\delta = 0$, and hence (3.1) is valid in Theorem 3.2. The last result follows from Theorem 2.6. □

Corollary 3.4. *Let $p \in \mathbb{R}$ fulfills $p \neq 4n$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^p,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \begin{cases} \frac{1}{|2|^n|2|^{4n}} \sum_{j=1}^n \|x_{1j}\|^p & p > 4n, \\ \\ \frac{1}{|2|^n|2|^{p}} \sum_{j=1}^n \|x_{1j}\|^p & p < 4n, \end{cases}$$

for all $x = x_1 \in V^n$. Moreover, if either $\mathcal{Q}$ has the quartic condition in each variable or is even in all variables, then, it is a multi-quartic mapping.

Proof . Set $\varphi(x_1, x_2) = \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|^p$. Then, $\varphi\left(2^l x_1, 2^l x_2\right) = |2|^{lp}\varphi(x_1, x_2)$. Now, Theorem 3.2 and Theorem 2.6 can be applied to arrive the result. □

Under some conditions the functional equation (2.2) can be hyperstable as follows.

Corollary 3.5. *Let $p_{kj} > 0$ for $k \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{k=1}^2 \sum_{j=1}^n p_{kj} \neq 4n$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \prod_{k=1}^{2}\prod_{j=1}^{n} \|x_{kj}\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, then f satisfies (2.2). In particular, under one of the following conditions, f is multi-quartic.

(i) *f has the quartic condition in each variable;*
(ii) *f is even in all variables.*

Proof . Defining $\varphi(x_1, x_2) = \prod_{k=1}^2 \prod_{j=1}^n \|x_{kj}\|^{p_{kj}}$ in Theorem 3.2, and applying Theorem 2.6, we reach the desired result. □

4. Acknowledgments

The author sincerely thank the anonymous reviewers for their careful reading, constructive comments and suggesting some related references to improve the quality of the previous version of paper.

References

[1] A. Bahyrycz, K. Ciepliński and J. Olko, *On Hyers-Ulam stability of two functional equations in non-Archimedean spaces*, J. Fixed Point Theory Appl. 18 (2016) 433–444.
[2] A. Bodaghi, *Functional inequalities for generalized multi-quadratic mappings*, J. Inequal. Appl. 2021 (2021) Paper No. 145.
[3] A. Bodaghi, *Intuitionistic fuzzy stability of the generalized forms of cubic and quartic functional equations*, J. Intel. Fuzzy Syst. 30 (2016) 2309–2317.
[4] A. Bodaghi, *Stability of a quartic functional equation*, Sci. World J. 2014 Art. ID 752146, 9 pages.
[5] A. Bodaghi, C. Park and O. T. Mewomo, *Multiquartic functional equations*, Adv. Diff. Equa. 2019 (2019) Paper No. 312.
[6] A. Bodaghi, C. Park and S. Yun, *Almost multi-quadratic mappings in non-Archimedean spaces*, AIMS Math. 5(5) (2020) 5230–5239.
[7] A. Bodaghi, S. Salimi and G. Abbasi, *Characterization and stability of multi-quadratic functional equations in non-Archimedean spaces*, Ann. Uni. Craiova-Math. Comp. Sci. Ser. 48(1) (2021) 88–97.
[8] A. Bodaghi and B. Shojaee, *On an equation characterizing multi-cubic mappings and its stability and hyperstability*, Fixed Point Theory 22(1) (2021) 83–92.
[9] J. Brzdęk and K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal. 74 (2011) 6861–6867.
[10] K. Ciepliński, *Ulam stability of functional equations in* 2*-Banach spaces via the fixed point method*, J. Fixed Point Theory Appl. 23 (2021) Paper No. 33.
[11] K. Ciepliński, *On the generalized Hyers-Ulam stability of multi-quadratic mappings*, Comput. Math. Appl. 62 (2011) 3418–3426.
[12] K. Ciepliński, *Generalized stability of multi-additive mappings*, Appl. Math. Lett. 23 (2010) 1291–1294.
[13] M. Dashti and H. Khodaei, *Stability of generalized multi-quadratic mappings in Lipschitz spaces*, Results Math. 74 (2019) Paper No. 163.
[14] N. Ebrahimi Hoseinzadeh, A. Bodaghi and M.R. Mardanbeigi, *Almost multi-cubic mappings and a fixed point application*, Sahand Commun. Math. Anal. 17(3) (2020) 131–143.
[15] M.B. Ghaemi, M. Majani and M.E. Gordji, *General system of cubic functional equations in non-Archimedean spaces*, Tamsui Oxford J. Inf. Math. Sci. 28(4) (2012) 407–423.
[16] K. Hensel, *Uber eine neue Begrndung der Theorie der algebraischen Zahlen, Jahresber,* Deutsche Mathematiker-Vereinigung. 6 (1897) 83–88.
[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. 27 (1941) 222–224.
[18] K.W. Jun and H.M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequ. Appl. 6(2) (2003) 289–302.
[19] K.W. Jun and H.M. Kim, *The generalized Hyers-Ulam-Russias stability of a cubic functional equation*, J. Math. Anal. Appl. 274(2) (2002) 267–278.
[20] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Mathematics and its Applications, vol. 427, Kluwer Academic Publishers, Dordrecht, 1997.
[21] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Birkhauser Verlag, Basel, 2009.
[22] Y. Lee and S. Chung, *Stability of quartic functional equations in the spaces of generalized functions*, Adv. Diff. Equ. 2009 (2009) Art. ID. 838347.
[23] S. Lee, S. Im and I. Hwang, *Quartic functional equations*, J. Math. Anal. Appl. 307 (2005) 387–394.
[24] C.-G. Park, *Multi-quadratic mappings in Banach spaces*, Proc. Amer. Math. Soc. 131 (2002) 2501–2504.
[25] C. Park and A. Bodaghi, *Two multi-cubic functional equations and some results on the stability in modular spaces*, J. Inequ. Appl. 2020 (2020), Paper No. 6.
[26] C. Park, A. Bodaghi and T.-Z. Xu, *On an equation characterizing multi-Jensen-quartic mappings and its stability*, J. Math. Inequa. 15(1) (2021) 333–347.
[27] J.M. Rassias, *Solution of the Ulam stability problem for quartic mappings*, Glasnik Matematicki Series III. 34(2) (1999) 243–252.

[28] J.M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Matematicki. Serija III. 36(1) (2001) 63–72.

[29] S. Salimi and A. Bodaghi, *A fixed point application for the stability and hyperstability of multi-Jensen-quadratic mappings*, J. Fixed Point Theory Appl. 22 (2020) Paper No. 9.

[30] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York, 1964.

[31] T.-Z. Xu, *Stability of multi-Jensen mappings in non-Archimedean normed spaces*, J. Math. Phys. 53 (2012) Art. ID. 023507.

[32] T.-Z. Xu, Ch. Wang and Th. M. Rassias, *On the stability of multi-additive mappings in non-Archimedean normed spaces*. J. Comput. Anal. Appl. 18 (2015), 1102–1110.

[33] X. Zhao, X. Yang and C.-T. Pang, *Solution and stability of the multiquadratic functional equation*, Abstr. Appl. Anal. 2013 (2013), Art. ID 415053, 8 pp.