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Solving second order ordinary differential equations by using Newton's interpolation and Aitken's methods

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Abstract

In this paper, the methods called Newton's interpolation and Aitken's methods were developed and examined. We use Newton's interpolation and Aitken's methods to find the exact and analytic results for three different types of nonlinear ordinary differential equations (NLODEs) of first and second order through illustrative examples. By using the new method, we successfully handle some class of nonlinear ordinary differential equations of first and second order in a simple and elegant way compared to Newton's and Lagrange methods in previous studies. One can conclude that Newton's interpolation and Aitken's methods are easy to yield and implement actual precise results.

Keywords: Ordinary differential equations, Newton's interpolation method, Aitken's method
2010 MSC: 65B05, 65J15, 65L06

1. Introduction

In theoretical physics and other applied mathematics associated fields, much research awareness has been focused toward finding and deliberating the exact and estimated solutions of nonlinear ordinary differential equations of second order for different potential, throughout the previous years. There are many numerical problems in real-life phenomena that can be expressed by class of ordinary

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differential equations (ODEs), specifically the equations of second order, hence the need to solve the differential equations is very important. The study of precise and estimated numerical solutions of nonlinear ordinary differential equations of second order get an actual vital part in numerous branches of physical-mathematical sciences, namely applied mathematics, mechanical engineering, electrical engineering, fluid mechanics, condensed matter physics, theoretical physics and consequently [8]. Lately, several methods have been in place to analyze and get the exact solutions of nonlinear ordinary differential equations with the support of software, like Mathematica, Mathematical Laboratory (MATLAB), and Maple. Consequently, there is constantly demand to develop consistent and effective approaches to get an approximate result of ordinary differential equations of second order. Previous studies have problem focusing on first order differential equation. Faith [5] used the combination of Newton's interpolation and Lagrange methods to obtain the subsequent dual terms and formerly use the three values for y to form a quadratic equation using Lagrange method. IDE [4] used the combination of Aitken's and Newton's interpolation methods to obtain the solution of ordinary differential equation of first order to obtain the second two terms then use the three values for y to form a linear or quadratic equation using Aitken interpolation method. Consequently, there are several approaches that show an estimated result for dissimilar kinds of differential equations. Temimi and Ansari [14] proposed a semi-analytical iterative technique to solve nonlinear problems. It has been applied to obtain several differential equations that arise in physics such as second-order nonlinear ordinary differential equations [1]. Fatoorehchia and Abolghasemia [6] studied the exact analytical solution of the nonlinear differential equations by using the Laplace transform method. Mahmoud and Shehu [2] studied the exact solution of the nonlinear ordinary differential equations by using the natural decomposition method (NDM) to obtain exact solutions for three different types of nonlinear ordinary differential equations (NLODEs). Mbagwu, Madububa and Nwamba [10] studied the series solution of nonlinear ordinary differential equations using single Laplace transform method in mathematical physics. A well-posed differential equations problem comprises of at minimum single differential equation and at slightest one extra equation such that the system organized has one and only one result (existence and uniqueness) named the exact or analytic solution to differentiate it from the estimated numerical solutions. Additionally, this analytic solution must be contingent endlessly on the statistics in the sense that if the equations are altered slightly then also the solution does not change too much. The study in this regard needs to find the analytic or exact solution of ordinary differential equation(s) of second order, we studied this problem(s) by using combination of Aitken's and Newton's interpolation methods [9, 7, 13, 12, 3]. For some recent qualitative results of solutions of nonlinear differential equations of second order, see, also, [11, 16, 15, 17]. Lastly, we complete on a numeral of problems and mathematical solutions gotten express the competence of the technique specified by current study. We study the following initial value problems:

$$y'' = f\left(x, y'\right), \quad y\left(x_o\right) = y_o, \quad y'\left(x_o\right) = y_o, \tag{1.1}$$

where $f\left(x, y'\right) \in C\left(R, R\right)$,

$$xy'' + 2y' + x = 1, \; y(1) = 2, \; y'(1) = 1 \tag{1.2}$$

and

$$y'' + \pi^2 e^y = 0, \; y\left(1\right) = 0, \; y'\left(0\right) = a. \tag{1.3}$$

The objective of the present paper is to extend the application of the Newton's interpolation and Aitken's method to provide estimated results for the above initial value problems with respect to the nonlinear ordinary differential equations of second order. The paper is prepared as follows: The Lagrange method, Newton's interpolation method and Aitken's method in Section 2. In Section 3, we solved some numerical problems to validate our methods. A conclusion is given in Section 4.

2. The Lagrange Method, Newton's Interpolation Method and Aitken's Method

In this section, we studied Lagrange method, Newton's interpolation method and Aitken's method. We applied Newton's interpolation method to get the next two relations, then apply the three values for y to form a quadratic equation by means of Lagrange interpolation method as shown below.

2.1. Lagrange Method

This method is defined by the following formula (see [5]):

$$y_n=\frac{(x-x_1)-(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)-(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1+\frac{(x-x_0)-(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2. \tag{2.1}$$

2.2. Newton's Interpolation Method

This method is defined by the following formula (see [5]):

$$\begin{aligned} f_n(x)=&a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\cdots+ \\ &a_n(x-x_0)(x-x_1)\cdots a_2(x-x_{n-1}), \end{aligned} \tag{2.2}$$

where

$$a_0=y_0,\ a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)},\ a_2=\frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)}. \tag{2.3}$$

2.3. Explanation of the Technique

Here, we combined Aitken's and Newton's interpolation methods ([5]) together. We applied Newton's interpolation method to obtain the subsequent two terms before using the three values for y to form quadratic or a linear equation by applying Aitken interpolation method as follows:

$$f_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\cdots \tag{2.4}$$

$$+a_n(x-x_0)(x-x_1)\cdots a_2(x-x_{n-1}), \tag{2.5}$$

where

$$a_0=y_0,\ a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)},\ a_2=\frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)}, \tag{2.6}$$

$$y_1=a_0+a_1(x-x_0), \tag{2.7}$$

$$y_2=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1). \tag{2.8}$$

2.4. Aitken Interpolation Method

This method is defined by the following formula (see [5]):

$$P_{0,1}(x)=\frac{1}{x_1-x_0}\left[(x_1-x)f(x_0)-(x_0-x)f(x_1)\right] \tag{2.9}$$

$$=\frac{1}{x_1-x_0}\begin{vmatrix} I_0(x) & x_0-x \\ I_1(x) & x_1-x \end{vmatrix}, \tag{2.10}$$

$$P_{0,1,2}(x)=\frac{1}{x_2-x_1}\begin{vmatrix} P_{0,1(x)} & x_1-x \\ P_{0,2(x)} & x_2-x \end{vmatrix}, \tag{2.11}$$

where $I_0(x)=f(x_0)$ and $I_1(x)=f(x_1)$.

We may regard $I_0(x)$ and $I_1(x)$ as two independents zero-degree interpolating polynomials to $f(x)$. It is easily verified that $I_{0,1}(x_0)=f(x_0)$ and $I_{0,1}(x_1)=f(x_1)$, and

$$y_n=P_{0,1,2,\cdots,n}(x)=\frac{1}{x_n-x_{n-1}}\begin{vmatrix} P_{0,1,\cdots,(n-1),n}(x) & x_{n-1}-x \\ P_{0,1,\cdots,(n-2),n}(x) & x_n-x \end{vmatrix}. \tag{2.12}$$

3. Mathematical Examples

Here, several problems were applied to clarify the application of Newton's interpolation method and Aitken's method. These examples are used in a case that they have precise logical results for simpler authentication. In the following problems, the nonlinearities elaborate are encountered in the mathematical models from different disciplines such as physics, mathematics, chemistry and engineering.

Problem 1. Let solve the following initial value problem:

$$y'(x) - 1 = y^2(x), \; y(0) = 0 \tag{3.1}$$

We suppose that $h = 0.10$.

Here, we used Newton's interpolation method first, i.e.

$$a_0 = 0 = y_0,$$
$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,0} = 1,$$
$$y_1 = 0 + 1(0.10 - 0) = 0.10$$

,

$$a_2 = \frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2 - x_0)} = \frac{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0.10,0.10} - \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,0}}{0.20 - 0} = 0.05,$$
$$y_2 = 1 - 1(0.20 - 0) + 0.05(0.20 - 0)(0.20 - 0.10) = 0.801.$$

Let form the linear and quadratic equations by means of Aitken's technique, i.e.

$$P_{0,1}(x) = 1 - x,$$
$$P_{0,2}(x) = 1 - 0.994x,$$
$$P_{0,1,2}(x) = 0.05x^2 - 1.0051x + 1.$$

Problem 2. Let us consider the following initial value problem:

$$xy'' + 2y' + y = 1, \; y(1) = 2, y'(1) = 1. \tag{3.2}$$

We suppose the step $h = 0.50$ and take the first initial condition, $y(1) = 2$. Using Newton's interpolation method, we have:

$$a_0 = 2 = y_0,$$
$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{1,2} = 2,$$
$$y_1 = 2 + 2(0.50 - 0) = 2 + 1 = 3.$$

Next, the formula

$$a_2 = \frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2 - x_0)}$$

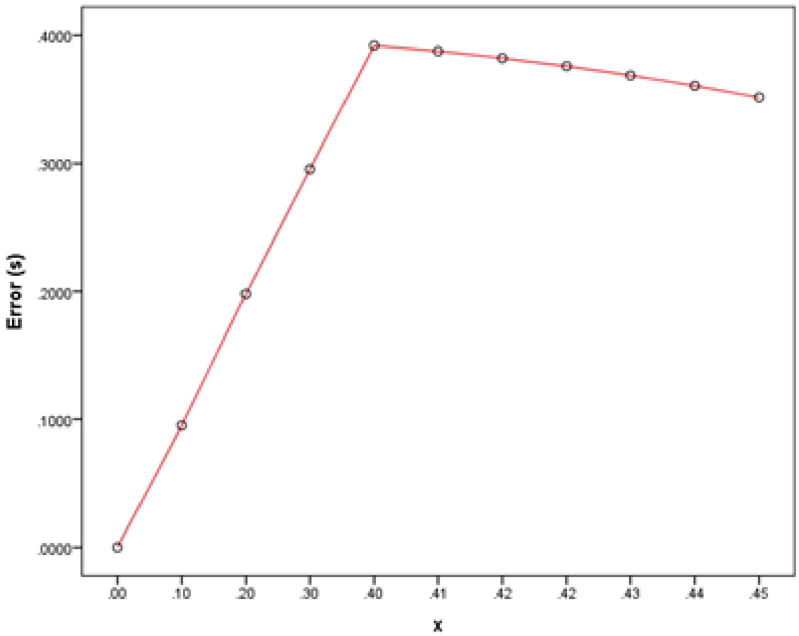

Figure 1: This figure shows the orbit of the solution of $\frac{\mathrm{dy}}{\mathrm{dx}}-1=y^2(x)$ with initial condition $y(0)=0$. Here, the values are obtained by proposed methods (Newton and Aitken methods).

implies that

$$a_2=\frac{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0.50,3}-\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{1,2}}{0.60-0}=-0.83.$$

$$y_2=2+2\,(0.60-0)-0.83\,(0.60-0)\,(0.60-0.50)=3.1502.$$

Let form the linear and quadratic equations by means of Aitken method. From equation (3.2), we obtain:

$$xy''+2y'=1-y, \tag{3.3}$$

$$P_{0,1}(x)=2-2x,$$

$$P_{0,2}(x)=2-0.994x,$$

$$P_{0,1,2}(x)=-0.83x^2-3.3431x+2,$$

From equation (3.2), we now take the second initial condition, which is $y'(1)=1$.

Choose that $h=0.50$. Let use Newton's interpolation method first. Then, we have

$$a_0=1=y_0,$$

$$a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)}=\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{1,1}=1,$$

$$y_1=1+1\,(0.50-0)=1+0.5=1.5,$$

$$a_2=\frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)}$$

$$=\frac{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0.50,1.5}-\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{1,1}}{0.60-0}=-0.41,$$

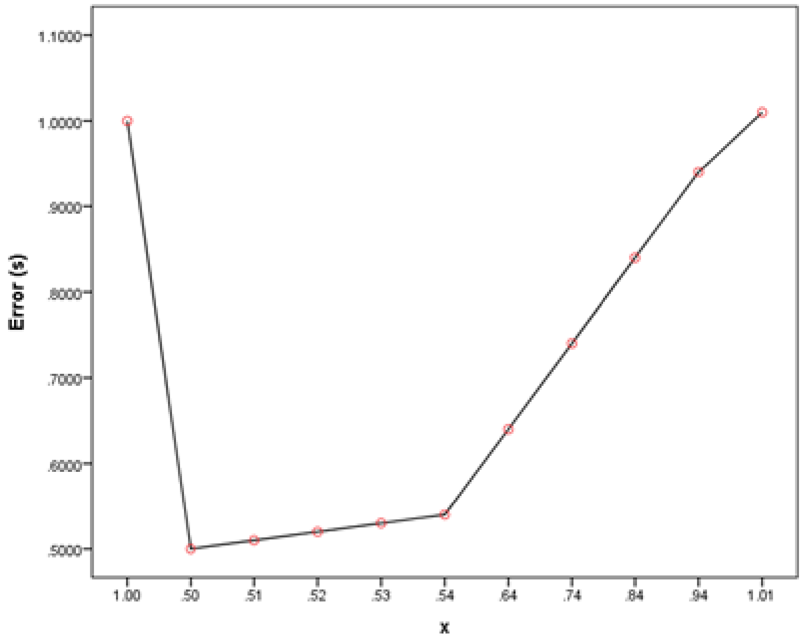

Figure 2: This figure shows the orbit of the solution of x$y'' + 2y' + y = 1$ with initial condition $(1) = 2$. Here, values are obtained by proposed methods (Newton and Aitken methods).

$$y_2 = 1 + 1\,(0.60 - 0) - 0.41\,(0.60 - 0)\,(0.60 - 0.50) = 1.5754.$$

Now, we establish the linear and quadratic equations by means of Aitken method. Hence, from equation (3.3), we obtain the following relations, respectively:

$$xy'' + 2y' = 1 - y,$$

$$P_{0,1}\,(x) = 1 - x,$$
$$P_{0,2}\,(x) = 1 - 1.4965x,$$
$$P_{0,1,2}\,(x) = -0.41x^2 + 1.005x + 1.$$

Problem 3. We now solve the following initial value problem:

$$y'' + \frac{a}{x}y' = e^y, \quad y\,(0) = 0, \quad y'\,(0) = 1. \tag{3.4}$$

We choose the stage $h = 0.40$ and take the first initial condition, $y\,(0) = 0$.

Let use Newton's interpolation method first. Then, we have:

$$a_0 = 2 = y_0,$$
$$a_1 = \frac{f\,(x_1) - f\,(x_0)}{(x_1 - x_0)} = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,0} = 1,$$
$$y_1 = 0 + 1\,(0.40 - 0) = 0.40,$$
$$a_2 = \frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2 - x_0)} = \frac{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0.40,0.40} - \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,0}}{0.50 - 0} = 0.32,$$

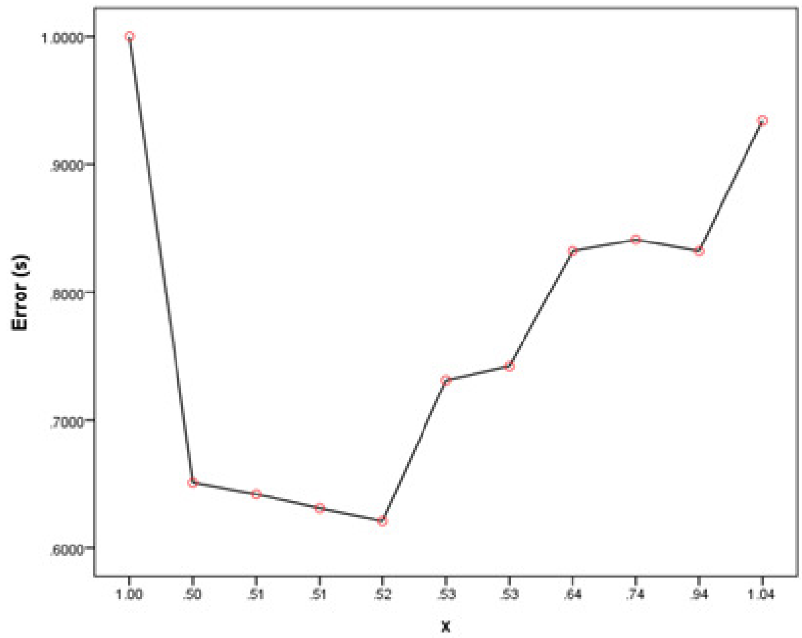

Figure 3: This figure shows the orbit of the solution of $xy'' + 2y' + y = 1$ with initial condition $y'(1) = 1$. Here, the values obtained by proposed methods (Newton and Aitken methods).

$$y_2 = 0 + 1(0.50 - 0) + 0.32(0.50 - 0)(0.50 - 0.40) = 0.5160.$$

Now, we form the linear and quadratic equations by means of Aitken's method:
From equation (3.2), we obtain:

$$P_{0,1}(x) = x,$$

$$P_{0,2}(x) = 0.996x,$$

$$P_{0,1,2}(x) = 0.32x^2 + 1.00606x + 1.$$

From equation (3.4), we take the second initial condition, $y'(0) = 1$.
Choose the stage $h = 0.40$. Let use Newton's interpolation method. Then, we have:

$$a_0 = 1 = y_0,$$

$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,0} = -1,$$

$$y_1 = 1 - 1(0.40 - 0) = 0.60,$$

$$a_2 = \frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2 - x_0)} = \frac{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0.40,0.60} - \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{0,1}}{0.50 - 0} = -0.152,$$

$$y_2 = 1 - 1(0.50 - 0) - 0.152(0.50 - 0)(0.50 - 0.40) = 0.4924.$$

Now, we form the linear and quadratic equations by means of Aitken's method. Then, we obtain

$$P_{0,1}(x) = 1 - x,$$

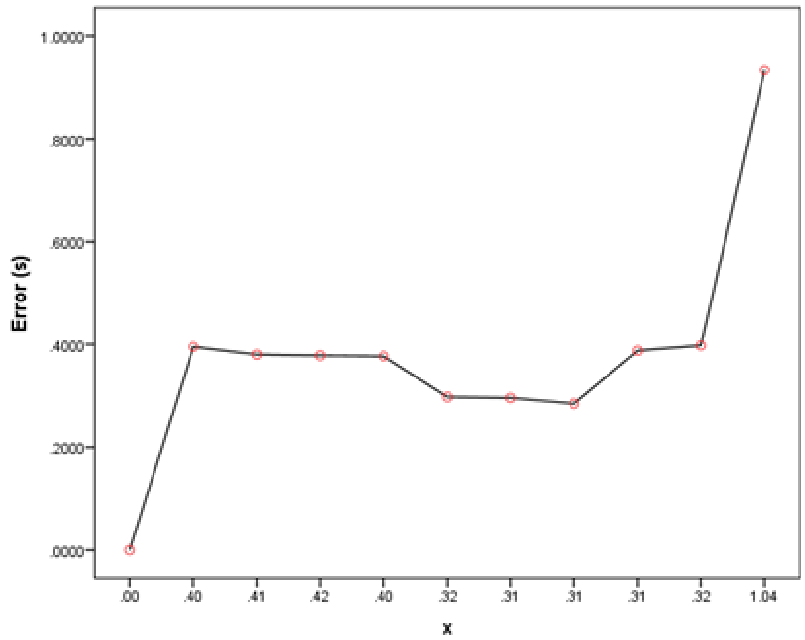

Figure 4: This figure shows the orbit of the solution of $y'' + \frac{a}{x} y' = e^y$ with initial condition $y(0) = 0$. Here, the values are obtained by proposed methods (Newton and Aitken methods).

$$P_{0,2}(x) = 1 - 0.99496x,$$
$$P_{0,1,2}(x) = -0.152x^2 - 1.00707x + 1.$$

The values of both (numerical and analytic solution) for different nonlinear ordinary differential equations (ODEs) are shown in the above figures. Figure 1 represents the orbit of the solution of $\frac{\mathrm{dy}}{\mathrm{dx}} - 1 = y^2(x)$ (Problem 1), Figure 2 represents the orbit of the solution of $\mathrm{xy}'' + 2y' + y = 1$ (Problem 2), with initial condition $y(1) = 2$, Figure 3 represents the orbit of the solution of $\mathrm{xy}'' + 2y' + y = 1$ (problem 2), with initial condition $y'(1) = 1$, Figure 4 represents the orbit of the solution of $y'' + \frac{a}{x} y' = e^y$ (Problem 3), with initial condition $y(0) = 0$, Figure 5 represents the orbit of the solution of $y'' + \frac{a}{x} y' = e^y$ (Problem 3), with initial condition $y'(0) = 1$. A good performance of the proposed methods, Newton's interpolation method and Aitken's interpolation methods are observed from the figures. All errors are decreased by increasing the number of known functions (x) and errors (s). A vital property of Newton's interpolation and Aitken's methods, which is observed from all figures, is that if we take more known functions (x) and error (s), then the accuracy gets better.

4. Conclusions

In this article, Newton's and Aitken's interpolation methods were proposed for obtaining the three nonlinear ordinary differential equations. We successfully found exact and analytic solutions to all three applications through illustrative examples. The Newton's interpolation and Aitken's interpolation methods present a substantial development in the fields over present methods. In the future, our goal is to use Aitken's and Newton's interpolation methods to obtain other differential equations (PDEs, ODEs) of nonlinear type that arise in additional areas of science, engineering, and another related field. The results obtained show that this approach can solve the problem effectively and is more reliable compared with Newton's interpolation and Lagrange methods.

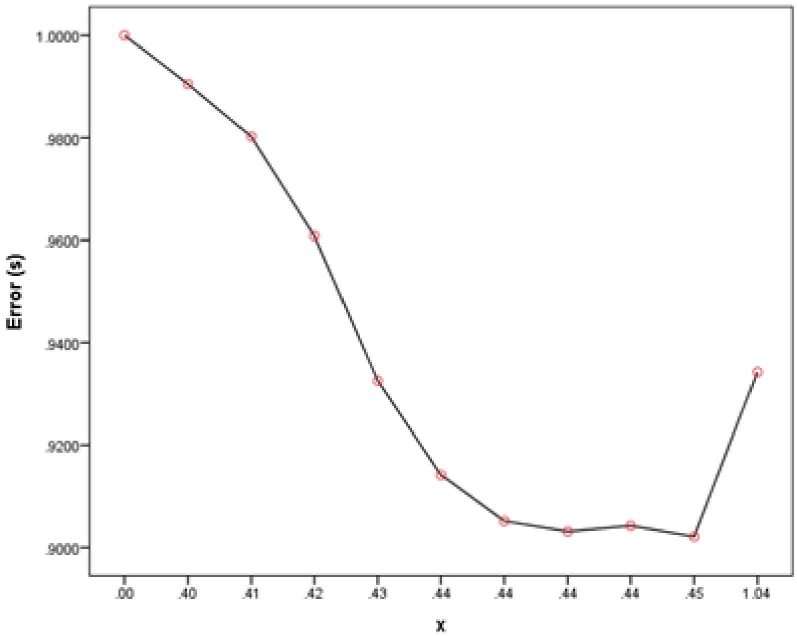

Figure 5: This figure shows the orbit of the solution of $y'' + \frac{a}{x}y' = e^y$ with initial condition $y'(0) = 1$. Here, we obtain the values by the proposed methods (Newton and Aitken methods).

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