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The class of D(T)–operators on Hilbert spaces

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Abstract

The purpose of this paper is to introduce a new class operator on separable Hilbert space. The operator $T \in B(H)$ is called D(T)-operator if there exist $U \in B(H), U \neq 0, I$ such that $T^{*}TU = UT^{*}T$, where T^{*} adjoint operator of T. Then some main properties of the class of D(T)-operator are studied in this research.

Keywords: D(T)-operator, quasi-normal operator, The spectrum of normal operator, Hilbert space.
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1. Introduction

Hilbert space is an inner product space which is complete with respect to the norm induced by its inner. Many authors have studied the operator theory on Hilbert space (for more details [2, 4, 7]). In this paper, we assume that H is separable Hilbert space. In 2018, Alpay and Colombo [1] are studied the spectrum of T is in the form:

$$\sigma(T) = \{\lambda \in \mathbb{C} \backslash T - \lambda I \text{ is not invertible}\}.$$

Rynne and Martin [3] in 2008 are studied orthogonal set for which if $u, v \in H$, then $< u, v >= 0$. The projection set are studied in [2] and [4] as form if $\mathcal{W} \in H$, then $\{x \in H :< x, y >= 0 \text{ for all } y \in \mathcal{W}\}$.

This paper contains three sections. In section two we study most of the basic properties of D(T)-operator and we investigated many new results of this subject.

In the third section, the most important results that we reached through this research were reviewed.

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2. Main Results

In this section, we investigated some of new results in this subject through some of the basic theorems and propositions.

Definition 2.1. *Let $T \in B(H)$. Then T is called the class of D(T)-operator if there exist $U \in B(H), U \neq 0, I$ such that $T^*TU = UT^*T$. In this case we put $D(T) = \{U \in B(H) : T^*TU = UT^*T$.*

Remark 2.2. *$D(T) \neq \emptyset$ because $0, I \in D(T)$.*

Proposition 2.3. *$D(T)$ is a closed subspace of $B(H)$.*

Proof . Let $U_1, U_2 \in D(T)$. Then

$$T^*T(\alpha U_1 + \beta U_2) = \alpha U_1 T^*T + \beta U_2 T^*T = (\alpha U_1 + \beta U_2)T^*T.$$

Therefore $(\alpha U_1 + \beta U_2) \in D(T)$. Let $U_n \in D(T)$ such that $U_n \longrightarrow U$. Therefore,

$$T^*TU_n \longrightarrow T^*TU \quad \text{and} \quad U_nT^*T \longrightarrow UT^*T.$$

Since $T^*TU_n = U_nT^*T$, for all n, $T^*TU = UT^*T$. Then $U \in D(T)$. □

Proposition 2.4. 1. *If $U_1, U_2 \in D(T)$, then $U_1U_2 \in D(T)$.*
2. *If $U \in D(T)$, $U^n \in D(T)$, for all n.*
3. *$U \in D(T)$ iff $U^* \in D(T)$.*
4. *If $U \in D(T)$ is an invertible operator, then $U^{-1} \in D(T)$.*

Theorem 2.5. *If $T_1^*T_2^* + T_2^*T_1^* = 0$, then $D(T_1) \cap D(T_2) \subseteq D(T_1 + T_2)$.*

Proof . Let $U \in D(T_1) \cap D(T_2)$. Then

$$\begin{aligned}(T_1 + T_2)^*(T_1 + T_2)U =&(T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2)U \\ =&T_1^*T_1U + T_2^*T_2U \\ =&UT_1^*T_1 + UT_2^*T_2 \\ =&U(T_1 + T_2)^*(T_1 + T_2).\end{aligned}$$

□

Proposition 2.6. *If $U \in D(T)$, then $TUT^* \in D(T^*)$ i.e. $TD(T)T^* \subseteq D(T^*)$].*

Proof . We have,

$$TT^*(TUT^*) = (TUT^*)TT^*.$$

Then $TUT^* \in D(T^*)$. □

Remark 2.7. 1. *If $U \in D(T)$, then $U^* \in D(T)$ and $U^{*n} \in D(T)$, for all n.*
2. *$(T^*T)^n \in D(T)$, for all n.*

Proposition 2.8. 1. *T is a quasi-normal operator [5] if and only if $T \in D(T)$.*
2. *If $T \in D(T)$, then $T^n \in D(T)$.*

3. $(D(T), +, 0)$ *is a ring with identity.*

Theorem 2.9. *$T \in D(T)$ if and only if $T + T^*$ and $T - T^* \in D(T)$.*

Proof . Assume that $T \in D(T) \Rightarrow T(T^*T) = (T^*T)T$. Therefore

$$T^*TT^* = T^{*2}T.$$

So,

$$(T + T^*)T^*T = T(T^*T + T^{*2}T = T^*TT + T^*TT^* = (T^*T)[T + T^*].$$

Then $(T + T^*) \in D(T)$. Moreover,

$$(T - T^*)T^*T = TT^*T - T^{*2}T = T^*TT - T^*TT^* = (T^*T)[T - T^*].$$

Thus $(T - T^*) \in D(T)$.

On the other hand,Assume that

$$(T + T^*)T^*T = T^*T(T + T^*), (T - T^*)T^*T = T^*T(T - T^*).$$

Then

$$TT^*T + T^{*2}T = T^*T^2 + T^*TT^* ... \tag{2.1}$$

and

$$TT^*T - T^{*2}T = T^*T^2 - T^*TT^* ... \tag{2.2}$$

From (2.1) and (2.2), we get $2T(T^*T) = 2(T^*T)T$. Then $T \in D(T)$. □

The following theorem is general from above theorem.

Theorem 2.10. *$T \in D(T)$ if and only if $\alpha T + \beta T^*$ and $\alpha T - \beta T^* \in D(T)$, where $\alpha, \beta \in \mathbb{C}$ are not both equal zero.*

Proof . Let $T \in D(T)$. Then $T(T^*T) = T^*TT$. So $T^{*2}T = T^*TT^*$. Then

$$\begin{aligned}(\alpha T + \beta T^*)T^*T =& \alpha TT^*T + \beta T^{*2}T \\ =& \alpha T^*TT + \beta T^*TT^* \\ =& (T^*T)[\alpha T + \beta T^*].\end{aligned}$$

Therefore $\alpha T + \beta T^* \in D(T)$. By the same way $\alpha T - \beta T^* \in D(T)$.

($\Leftarrow$) Let $\alpha T + \beta T^*, alphaT - \beta T^* \in D(T)$. If $\alpha = 0$, then $\beta T^* \in D(T)$. Since $\beta = 0$, $T^* \in D(T)$. Therefore $T \in D(T)$.

Assume that $\alpha \neq 0$. Then

$$(\alpha T \mp \beta T^*)T^*T = T^*T(\alpha T \mp \beta T^*).$$

Thus,

$$\alpha TT^*T + \beta T^{*2}T = \alpha T^*T^2 + \beta T^*TT^* ... \tag{2.3}$$

and

$$\alpha TT^*T - \beta T^{*2}T = \alpha T^*T^2 - \beta T^*TT^* ... \tag{2.4}$$

These imply that $2\alpha TT^*T = 2\alpha T^*T^2$. Since $\alpha \neq 0$, we have $T(T^*T) = (T^*T)T$. This shows that $T \in D(T)$. □

Proposition 2.11. *$\sigma(T) \cap \sigma(T^*) = \emptyset$ and $T + T^* \in D(T)$ if and only if $T \in D(T)$.*

Theorem 2.12. *If* ker $T^*T = H$, *then* $D(T) = B(H)$.

Proof . Let $S \in B(H)$. Since $ST^*T(x) = 0$ and $T^*TS(x) = 0$, for all $x \in H$, $ST^*T = T^*TS$. Thus, $S \in D(T)$, for all $S \in B(H)$ □

Corollary 2.13. 1. *If T is a zero operator, then $D(T) = B(H)$*

2. *$T^*T = 0$ if and only if $T = 0$.*

Theorem 2.14. *If $\{0\} \neq \ker(T^*T) \subsetneq H$, then $D(T) \neq B(H)$.*

Proof . Assume that $D(T) = B(H)$. Let $N =$ ker T^*T, then $\{0\} \neq N \subsetneq H$. Therefore, $\{0\} \neq N^{\perp} \neq H$. Let $0 \neq y \in N, 0 \neq \acute{y} \in N^{\perp}$. Let $x = y + \acute{y}$, $M = span\{x\}$ and P_M be the projection onto M. Assume that $P_M T^*T = T^*TP_M$. Then

$$P_M T^*T(y) = T^*TP_M(y).$$

Thus, $T^*TP_M(y) = 0$. Then $P_M(y) \in N$. But $P_M(y) = \alpha x$, where $\alpha \in \mathbb{C}$.

If $\alpha \neq 0$, then $\alpha x \in N$. So $x \in N$ and therefore, $\acute{y} \in N \cap N^{\perp}$. Hence, $\acute{y} = 0$. This is a contradiction with $\acute{y}$ is a non-zero vector.

If $\alpha = 0$, then $P_M(y) = 0 \Rightarrow y \in M^{\perp}$. So $< y, x >= 0$. But,

$$< y, y >= 0 \Rightarrow y = 0.$$

A contradiction, therefore $P_M \notin D(T)$. So $D(T) \neq B(H)$. □

Remark 2.15. *If T is a self adjoint operator [6], then $D(T) = (\acute{T}^2)$, where $(\acute{T}^2)$ is the commute of T^2.*

Theorem 2.16. *Let T be an operator with* ker $T^*T = \{0\}$. *Then $D(T) = B(H)$ if and only if $T^*T = \alpha I$, where α is a non-zero constant.*

Proof . Let $\{e_i\}$ be a basis for H. Suppose that $UT^*T = T^*TU$, for all $U \in B(H)$. Let $M = span\{T^*T(e_1)\}$, then $H = M \bigoplus M^{\perp}$ and

$$P_{M^{\perp}} T^*T(e_1) = T^*TP_{M^{\perp}}(e_1).$$

This implies that $T^*TP_{M^{\perp}}(e_1) = 0$, but ker $T^*T = 0$. So $P_{M^{\perp}}(e_1) = 0$. Thus, $e_1 \in M$. Therefore $e_1 = \alpha_1 T^*T(e_1), \alpha_1 \neq 0$.

By the same way $e_i = \alpha T^*T(e_i), i = 1, 2, ...$ Also

$$e_i + e_j = \alpha T^*T(e_i + e_j) = \alpha T^*Te_i + \alpha T^*Te_j.$$

So,

$$\begin{aligned} \alpha_i T^*Te_i + \alpha_j T^*Te_j =& \alpha T^*Te_i + \alpha T^*Te_j \\ =& T^*T(\alpha_i e_i + \alpha_j e_j) \\ =& T^*T(\alpha e_i + \alpha e_j). \end{aligned}$$

Therefore $(\alpha_i e_i + \alpha_j e_j) = (\alpha e_i + \alpha e_j)$. Hence, $(\alpha_i - \alpha)e_i + (\alpha_j - \alpha)e_j = 0$. Therefore, $(\alpha_i - \alpha) = 0$ and $(\alpha_j - \alpha) = 0$. Hence, $\alpha_i = \alpha = \alpha_j$. Thus, $T^*T(e_i) = \frac{1}{\alpha}e_i$, for all i. This means that $T^*T = \frac{1}{\alpha}I$.

The converse, clearly holds. □

Proposition 2.17. *$D(T)$ is invariant subspace of T if and only if $T \in D(T)$.*

Proof . Since $I \in D(T)$, $TI \in D(T) \Rightarrow T \in D(T)$. On the other hand, let $U \in D(T)$. Since $T \in D(T)$, $TU \in D(T)$. Hence, $D(T)$ is invariant subspace. □

Theorem 2.18. *Let $\{0\} \neq ker\ T^*T \subsetneq H$. Then $\{0\} \neq \overline{D(T)(h)} \subsetneq H$, for all $h \in \ker\ T^*T$ and $h \neq 0$.*

Proof . Let $0 \neq h \in \ker\ T^*T$. Since $I \in D(T)$, $h = I(h) \in D(T)(h) \subseteq \overline{D(T)(h)}$. Therefore,

$$\{0\} \neq \overline{D(T)(h)}.$$

If $U \in D(T)$, then $T^*TU = UT^*T$. Therefore,

$$T^*TU(h) = UT^*T(h).$$

This implies that $T^*TU(h) = 0$. Therefore,

$$U(h) \in \ker\ T^*T.$$

Thus, $D(T)(h) \subseteq \ker\ T^*T$ and so $\overline{D(T)(h)} \subseteq \overline{ker\ T^*T}$. Therefore, $\overline{D(T)(h)} \subsetneq H$. □

Proposition 2.19. *If T is quasi normal operator [5], then $T\overline{D(T)(h)} \subseteq overline D(T)(h)$, for all $h \in H$.*

Proof . Since $T \in D(T)$, $T \in D(T) \subseteq D(T)$. Let $z \in \overline{D(T)(h)}$, so there exist $U_n(h) \in D(T)(h)$ such that $U_n(h) \to Z$ and

$$TU_n(h) \to T(Z).$$

Therefore $\underline{U_n \in D(T)} \Rightarrow \underline{TU_n \in} D(T)$. But $TU_n(h) \in D(T)(h)$. Therefore $T(Z) \in \overline{D(T)(h)}$ and consequently, $T\overline{D(T)(h)} \subseteq \overline{D(T)(h)}$. □

Theorem 2.20. *$D(T)(h)$ is a subspace of H, $\forall h \in H$.*

Proof . Let $U_1(h), U_2(h) \in D(T)(h) \Rightarrow U_1, U_2 \in D(T)$. Hence, $U_1 + U_2 \in D(T)$ and

$$U_1(h) + U_2(h) = (U_1 + U_2)(h) \in D(T)(h).$$

Let $\alpha \in \mathbb{C}, U(h) \in D(T)(h) \Rightarrow \alpha(U(h)) = (\alpha U)(h) \in D(T)(h)$. □

Corollary 2.21. *$\overline{D(T)(h)}$ is a subspace of $H, \forall h \in H$.*

Proof . Let $Z_1, Z_2 \in \overline{D(T)(h)}$. Then there exist $U_n^1(h), U_n^2(h) \in D(T)(h)$ such that

$$U_n^1(h) \to Z_1, U_n^2(h) \to Z_2.$$

Therefore

$$U_n^1(h) + U_n^2(h) \to Z_1 + Z_2$$

and

$$(U_n^1 + U_n^2)(h) \to Z_1 + Z_2.$$

Since $(U_n^1 + U_n^2) \in D(T) \to Z_1 + Z_2 \in \overline{D(T)(h)}$. Let $Z \in \overline{D(T)(h)}$ and $\overline{\alpha \text{ is}}$ constant. Therefore $U_n(h) \to Z$ and $\alpha U_n(h) \to \alpha Z$. This shows that $\alpha U_n \in D(T) \Rightarrow \alpha Z \in \overline{D(T)(h)}$. □

3. Discussion and Conclusion

The present paper discusses some essentially properties of a new class operator which is called D(T)-operator. Many important results have been reached in this research, which are:

1. If $ker\ T^*T = H$, then $D(T) = B(H)$.
2. $T \in D(T)\ iff\ \alpha T + \beta T^*$ and $\alpha T - \beta T^* \in D(T)$.
3. If $ker\ T^*T = \{0\}$, then $D(T) = B(H)\ iff\ T^*T = \alpha I$.
4. If T is quasi normal operator, then $T\overline{D(T)(h)} \subseteq \overline{D(T)(h)}\ \ \forall h \in H$.
5. T is quasi-normal operator $iff\ T \in D(T)$.
6. If $T \in D(T)$, then $T^n \in D(T)$.

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