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The general implicit-block method with two-points and extra derivatives for solving fifth-order ordinary differential equations

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Abstract

In this work, a general implicit block method (GIBM) with two points for solving general fifth-order initial value problems (IVPs) has been derived. GIBM is proposed by adopting the basis functions of Hermite interpolating polynomials. GIBM is presented to be suitable with the numerical solutions of fifth-order IVPs. Hence, the derivation of GIBM has been introduced. Numerical implementations compared with the existing numerical GRKM method are used to prove the accuracy and efficiency of the proposed GIBM method. The impressive numerical results of the test problems using the proposed GIBM method agree well with the approximated solutions of them using the existing GRKM method.

Keywords: Implicit numerical method, ODEs, IVPs, Block method, Order, RKM, GRKM, Fifth-order, Ordinary differential equations.

1. Introduction

Differential equations (DEs) have powerful rule in the different fields of applied-mathematics such as physics, engineering, biology, economic, medicine and chemistry. The mathematical models of truth problems in engineering and applied science are modeled by using the tools of DEs. However, computing the solutions of different types of DEs, analytically or numerically, have been challenged the minds and intelligence of mathematicians. At present, the powerful modern or classical, analytical

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or numerical methods be available to use by mathematicians, engineers and scientists. The literature review(LR) of different classical or modern methods for finding the analytical or approximated solutions of mathematical models that contain fifth-order IVPs or boundary value-problems(BVPs) are listed as follows: [1] developed second-, fourth-, sixth- and eighth-orders finite-difference methods for solving IVPs while [2, 3, 4] derived numerical method of second-order for solving IVPs, [5] solved BVPs using the technique of spline which is non-polynomial and [6] developed new integrator for solving ODEs of fifth-order. Accordingly, the numerical methods are sometimes not able to approximate the solutions of some types of DEs directly or indirectly. However, the propose of this study is to introduce the derivation of direct GIBM method. Many researchers like: [7, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] have derived one-step numerical integrators for solving IVPs of orders less than order ten while other authors derived multistep numerical methods for solving these problem [24, 25, 26, 27, 28, 29, 30]. In this work, a new implicit block method with two points second derivative has been proposed. GIBM is derived by adopting Hermite polynomials to enhance the accuracy of approximated solutions of IVPs by incorporating the first derivative of $\psi(\xi, \eta(\xi), \eta'(\xi), \eta''(\xi), \eta'''(\xi), \eta^{(4)}(\xi))$. The purpose of including more derivatives in the formula of IVP is that, more generalized and accurate numerical results can be obtained. Some test examples have been solved to show the effectiveness of GIBM method. As well as the numerical results compared with similar numerical results which obtained by existing GRKM method. Numerical results are carried out to verify the efficiency and accuracy of the constructed method compared with general existing GRKM method. Applications of IVPs are also, introduced to yield impressive numerical results for the new two-points block-method. The numerical solutions of test problems using proposed GIBM method are agree well with the numerical solutions using GRKM method.

The new method is derived using Hermite polynomials $P_2(\xi)$, which can be defined by the following:

$$P_2(\xi) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} \psi_i^{(k)} L_{i,k}(\xi), \tag{1.1}$$

where $\psi_i = \psi(\xi_i), \xi_j = a + jh, i = 0, 1, \ldots; j = 0, 1, \ldots, m$ and $h = \frac{b-a}{n}$, n is a positive integer. $L_{i,k}(\xi)$ can be defined by

$$L_{i,m_i}(\xi) = \ell_{i,m_i}(\xi), \; i = 0, 1, \ldots, n,$$

$$\ell_{i,k}(\xi) = \frac{(\xi - \xi_i)^k}{k!} \prod_{j=0, j\neq i}^{n} (\frac{\xi - \xi_j}{\xi_i - \xi_j})^{m_j}, i = 0, 1, \ldots, n, k = 0, 1, \ldots, m_i.$$

And recursively for $k = m_i - 2, m_i - 3, \ldots, 0$.

$$L_{i,k}(\xi) = \ell_{i,k}(\xi) - \sum_{v=k+1}^{m_i-1} \ell_{i,k}^{(v)}(\xi_i) L_{i,v}(\xi).$$

In this paper, block method with some derivatives is derived for directly solving the IVPs for following class of general quasi-linear fifth order ODEs

$$\eta^{(5)}(\xi) = \psi(\xi, \eta(\xi), \eta'(\xi), \eta''(\xi), \eta'''(\xi), \eta^{(4)}(\xi)), \qquad \xi_0 \leq \xi \leq \xi_1 \tag{1.2}$$

with the initial conditions(ICs),

$$\eta^{(j)}(\xi_0) = \alpha^j = [\eta_1(\xi), \eta_2(\xi), \ldots, \eta_N(\xi)], j = 0, 1, 2, 3, 4; \tag{1.3}$$

2. Preliminary

In this section, we have introduced some concepts and background which related with the problem of study.

2.1. Quasi-Linear Fifth-Order ODEs

In this subsection, we will study the quasi-linear fifth-order ODEs

2.1.1. Class One of General Quasi-Linear Fifth-Order ODEs

The class one of general quasi-linear fifth-order ODEs can be written in the Equation (1.2) with the ICs in Equation (1.3).

2.1.2. Class Two of General Quasi-Linear Fifth-Order ODEs

The class two of general quasi-linear fifth-order ODE with no appearance of $\eta^{(3)}(\xi)$ and $\eta^{(4)}(\xi)$ which has following form:

$$\eta^{(5)}(\xi) = \psi(\xi, \eta(\xi), \eta'(\xi), \eta''(\xi)), \qquad \xi_0 \leq \xi \leq \xi_1, \tag{2.1}$$

with the ICs in Equation (1.3)

2.1.3. Class Three of Quasi-Linear Fifth-Order ODEs

The third class of general quasi-linear fifth-order ODE with no appearance of $\eta''(\xi)$, $\eta^{(3)}(\xi)$ and $\eta^{(4)}(\xi)$ which has the following form:

$$\eta^{(5)}(\xi) = \psi(\xi, \eta(\xi), \eta'(\xi)), \qquad \xi_0 \leq \xi \leq \xi_1, \tag{2.2}$$

with the ICs in Equation (1.3).

2.1.4. Special Class Quasi-Linear Fifth-Order ODEs

The special quasi-linear fifth-order ODE has written as follows form:

$$\eta^{(5)}(\zeta) = \psi(\xi, \eta(\xi)), \qquad \xi_0 \leq \xi \leq \xi_1, \tag{2.3}$$

with the ICs in Equation (1.3), for $j = 0, 1, \ldots, 5$.

These ODEs in equations (1.2) and (2.1)-(2.3) are found in several of engineering and physical models. Some of researchers used to solve the ODEs in equations (1.2) and (2.1)-(2.3) using multistep methods or by converting these ODEs to system of 1^{st}-order ODEs. Hence, it would be more significate if ODEs of fifth-order in equations (1.2) and (2.1)-(2.3) can be solved directly using the GIBM method which be more importunate since it has less functions of computational time and evaluations in the running of implementation. In this study, we are consider the multi-step block method for finding the solutions of fifth-order ODEs. However, we developed the derivation of GIBM method and the proposed-method has been derived using the intepolation of Hermite polynomials.

2.2. RKM and GRKM Methods

In this subsection, RKM and GRKM methods with s-stages which introduced by [6] and [31] and proposed for solving special and general classes of quasi-linear fifth-order ODEs in equations (2.1)-(2.3) which have the following forms:

2.2.1. RKM Method for Solving Class Three

The RKM method has the following form:

$$z_{n+1} = z_n + hz_n^{'} + \frac{h^2}{2!}z_n^{''} + \frac{h^3}{3!}z_n^{'''} + \frac{h^4}{4!}z_n^{(4)} + h^5\sum_{i=1}^{s} b_i k_i \quad (2.4)$$

$$z_{n+1}^{'} = z_n^{'} + hz_n^{''} + \frac{h^2}{2!}z_n^{'''} + \frac{h^3}{3!}z_n^{(4)} + h^4\sum_{i=1}^{s} b_i^{'} k_i \quad (2.5)$$

$$z_{n+1}^{''} = z_n^{''} + hz_n^{'''} + \frac{h^2}{2!}z_n^{(4)} + h^3\sum_{i=1}^{s} b_i^{''} k_i \quad (2.6)$$

$$z_{n+1}^{'''} = z_n^{'''} + hz_n^{(4)} + h^2\sum_{i=1}^{s} b_i^{'''} k_i \quad (2.7)$$

$$z_{n+1}^{(4)} = z_n^{(4)} + h\sum_{i=1}^{s} b_i^{''''} k_i \quad (2.8)$$

where, the s-stages RKM method for approximating the solutions of special quasi-linear ODEs of fifth-order in Equation (2.3) with ICs (1.3),

and,

$$k_1 = \psi(\zeta_n, z_n) \quad (2.9)$$

and

$$k_i = \psi(\zeta_n + c_i h, z_n + hc_i z_n^{'} + \frac{h^2}{2}c_i^2 z_n^{''} + \frac{h^3}{6}c_i^3 z_n^{'''} + \frac{h^4}{24}c_i^4 z_n^{''''} + h^5\sum_{j=1}^{i-1} a_{ij}k_j) \quad (2.10)$$

for $i = 2, 3, \ldots, s$. where h is the step-size of the interval of definition.

2.2.2. GRKM method for Solving Class Two

The GRKM integrator with s-stages for solving class two quasi-linear fifth-order ODEs in equations (2.1)-(2.2) with ICs in Equation (1.3) are the equations formulas in equations (2.4)-(2.9) and the following forms:

$$\begin{aligned} k_i &= \psi(\zeta_n + c_i h, z_n + c_i h z_n^{'} + c_i^2\frac{h^2}{2!}z_n^{''} + c_i^3\frac{h^3}{3!}z_n^{'''} + c_i^4\frac{h^4}{4!}z_n^{(4)} + c_i^5\frac{h^5}{5!}z_n^{(5)} + h^6\sum_{m=1}^{i-1} a1_{im}k_m, z_n^{\prime} \\ &+ c_n h z_n^{''} + c_n^2\frac{h^2}{2!}z_n^{'''} + c_i^3\frac{h^3}{3!}z_n^{''''} + h^4\sum_{l=1}^{i-1} a2_{im}k_m, z_n^{\prime\prime} + c_n h z_n^{'''} + c_n^2\frac{h^2}{2!}z_n^{''''} + h^3\sum_{l=1}^{i-1} a3_{im}k_m), \end{aligned} \quad (2.11)$$

for $i = 2, 3, 4, \cdots, s$.

While, GRKM integrator with s-stages for solving class three of general quasi-linear fifth-order ODEs in Equation (2.2) with ICs in Equation (1.3) are the equations formulas in equations (2.4)-(2.9) and the following form:

$$\begin{aligned} k_i &= \psi(\zeta_n + c_i h, z_n + c_i h z_n^{'} + c_i^2\frac{h^2}{2!}z_n^{''} + c_i^3\frac{h^3}{3!}z_n^{'''} + c_i^4\frac{h^4}{4!}z_n^{(4)} + c_i^5\frac{h^5}{5!}z_n^{(5)} \\ &+ h^6\sum_{m=1}^{i-1} a1_{im}k_m, z_n^{\prime} + c_n h z_n^{''} + c_n^2\frac{h^2}{2!}z_n^{'''} + c_i^3\frac{h^3}{3!}z_n^{''''} + h^4\sum_{l=1}^{i-1} a2_{im}k_m) \end{aligned} \quad (2.12)$$

for $i = 2, 3, \ldots, s$. where h. is the step-size of the interval of definition.

[10] and [32] have derived the RKD and RKT methods for solving special-quasi linear ODEs of third-order while [33] have derived RKFD method for solving special-quasi linear ODEs of fourth-order. The parameters of RKM method are $c_i, a_{ij}, b_i^{(k)}$ for $i, j = 1, 2, \ldots, s$ and $k = 0, 1, 2, 3$ are obtained by solving the system of OCs. Butcher tableaus of Three-stages RKM and GRKM integrators have shown in the Tables 1 and 2, resp.

Table 1: Butcher Tableau(BT) of RKM Integrator

0	0		
$\frac{3}{5} - \frac{\sqrt{6}}{10}$	$\frac{1}{2}$	0	
$\frac{3}{5} + \frac{\sqrt{6}}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	1	0	$-\frac{119}{120}$
	$-\frac{1}{40} - \frac{\sqrt{6}}{360}$	$\frac{1}{60} + \frac{\sqrt{6}}{360}$	0
	$\frac{1}{18}$	$\frac{1}{18} - \frac{\sqrt{6}}{48}$	$\frac{1}{18} + \frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36} - \frac{\sqrt{6}}{48}$	$\frac{7}{18} - \frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{7}{36} - \frac{\sqrt{6}}{48}$	$\frac{7}{18} - \frac{\sqrt{6}}{18}$

Table 2: Butcher Tableau(BT) of GRKM Integrator

0	0		
$\frac{3}{5} - \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	0	
$\frac{3}{5} + \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	$-\frac{1}{2}$	0
	0		
	$\frac{12}{625} - \frac{3}{2500}\sqrt{6}$	0	
	$\frac{1}{2}$	$\frac{-1}{2}$	0
	0		
	$\frac{27}{500} - \frac{19}{100}\sqrt{6}$	0	
	$\frac{33}{2500} + \frac{51}{5000}\sqrt{6}$	$\frac{51}{1250} + \frac{11}{1250}\sqrt{6}$	0
	$-\frac{37}{360}$	$\frac{1}{18}$	$\frac{1}{18}$
	$\frac{17}{360} - \frac{7}{360}\sqrt{6}$	$\frac{1}{18}$	$-\frac{11}{180} + \frac{7}{360}\sqrt{6}$
	$\frac{1}{18}$	$\frac{1}{18} + \frac{1}{48}\sqrt{6}$	$\frac{1}{18} - \frac{1}{48}\sqrt{6}$
	$\frac{1}{9}$	$\frac{7}{36} + \frac{1}{18}\sqrt{6}$	$\frac{7}{36} - \frac{1}{18}\sqrt{6}$
	$\frac{1}{9}$	$\frac{4}{9} + \frac{1}{36}\sqrt{6}$	$\frac{4}{9} - \frac{1}{36}\sqrt{6}$

3. Proposed GIBM Method

In this section, we have introduced the proposed method.

3.1. Analysis of Proposed GIBM Method

In this section, the proposed-method has been derived and introduced. The derivation of the proposed-method based on interpolating of Hermite polynomial named by $P_2(\xi)$ which interpolates at two-points. This Hermite polynomial has the following form:

$$P_2(\xi) = \sum_{i=0}^{m} \sum_{k=0}^{m_i-1} \psi_i^{(k)} L_{ik}(\xi)$$

where, $\psi_i(\xi) = \psi(\xi, \eta_i(\xi), \eta_i'(\xi), \eta_i''(\xi), \eta_i'''(\xi), \eta_i^{(4)}(\xi))$, $\xi_j = a + jh; i, j = 0, 1, 2, \ldots, m$ and $h = \frac{b-a}{m}$, $L_{ik}(\xi) =$ the generalized Lagrange-polynomial; $i = 0, 1, 2, \ldots, m; k = 0, 1, 2, \ldots, m_i$ and m is positive integer. We use

$$P_2(\xi) = \psi_0 L_{00}(\xi) + \psi_1 L_{10}(\xi) + \psi_2 L_{20}(\xi) + \psi_0' L_{01}(\xi) + \psi_1' L_{11}(\xi) + \psi_2' L_{21}(\xi),$$

where ψ' is the first-derivative of the function ψ with respect to ξ. The approximation computes the approximated solutions, η_{m+1} and η_{m+2} at two-points ξ_{m+1} and ξ_{m+2} respectively where ξ_m =the starting point and ξ_{m+2} =the end point in the block-interval $[\xi_m, \xi_{m+2}]$ with step-size $2h$. The approximated solution of η_{n+2} at the end-point ξ_{m+2} should be given as initial value in the new iteration.

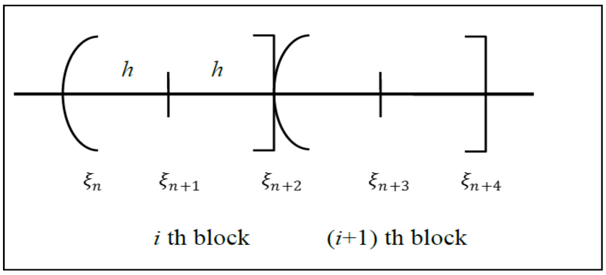

Figure 1: Two Points Block Method

3.1.1. Hermite Polynomials

In this paper, we have used Hermite polynomials which defined as follows:

$$L_{00}(\xi) = \left(\frac{\xi-\xi_{n+1}}{\xi_n-\xi_{n+1}}\right)^2\left(\frac{\xi-\xi_{n+2}}{\xi_n-\xi_{n+2}}\right)^2\left(1+\left(\frac{2}{\xi_n-\xi_{n+2}}+\frac{2}{\xi_n-\xi_{n+1}}\right)(\xi-\xi_n)\right), \tag{3.1}$$

$$L_{10}(\xi) = \left(\frac{\xi-\xi_{n}}{\xi_{n+1}-\xi_{n}}\right)^2\left(\frac{\xi-\xi_{n+2}}{\xi_{n+1}-\xi_{n+2}}\right)^2\left(1+\left(\frac{2}{\xi_{n+1}-\xi_{n+2}}+\frac{2}{\xi_{n+1}-\xi_{n}}\right)(\xi-\xi_{n+1})\right), \tag{3.2}$$

$$L_{20}(\xi) = \left(\frac{\xi-\xi_{n}}{\xi_{n+2}-\xi_{n}}\right)^2\left(\frac{\xi-\xi_{n+1}}{\xi_{n+2}-\xi_{n+1}}\right)^2\left(1+\left(\frac{2}{\xi_{n+2}-\xi_{n+1}}+\frac{2}{\xi_{n+2}-\xi_{n+1}}\right)(\xi-\xi_{n+2})\right), \tag{3.3}$$

$$L_{01}(\xi) = (\xi-\xi_n)\left(\frac{\xi-\xi_{n+1}}{\xi_n-\xi_{n+1}}\right)^2\left(\frac{\xi-\xi_{n+2}}{\xi_n-\xi_{n+2}}\right)^2, \tag{3.4}$$

$$L_{11}(\xi) = (\xi-\xi_{n+1})\left(\frac{\xi-\xi_{n}}{\xi_{n+1}-\xi_{n}}\right)^2\left(\frac{\xi-\xi_{n+2}}{\xi_{n+1}-\xi_{n+2}}\right)^2, \tag{3.5}$$

$$L_{21}(\xi) = (\xi-\xi_{n+2})\left(\frac{\xi-\xi_{n}}{\xi_{n+2}-\xi_{n}}\right)^2\left(\frac{\xi-\xi_{n+1}}{\xi_{n+2}-\xi_{n+1}}\right)^2. \tag{3.6}$$

Using the assumption $\zeta = \frac{\xi-\xi_{n+2}}{h}$,, then, Hermite polynomials can written in the independent variable ζ as follows:

$$L_{00}(\zeta) = \frac{7+3\zeta}{4}(\zeta(\zeta+1))^2 \tag{3.7}$$

$$L_{10}(\zeta) = (\zeta(2+\zeta))^2 \tag{3.8}$$

$$L_{20}(\zeta) = \frac{1-3\zeta}{4}(2+\zeta)^2(\zeta+1)^2 \tag{3.9}$$

$$L_{01}(\zeta) = \frac{h(1+\zeta)}{4}(\zeta(\zeta+2))^2 \tag{3.10}$$

$$L_{11}(\zeta) = h(1+\zeta)(\zeta(\zeta+1))^2 \tag{3.11}$$

$$L_{21}(\zeta) = \frac{h\zeta}{4}((2+\zeta)(\zeta+1))^2 \tag{3.12}$$

3.2. Derivation of Proposed Two Points Implicit Block Method

In this section, we presented the construction of two-points implicit block-method with second derivatives for solving general quasi linear fifth order ODEs. In this proposed method, the domain of definition $[a, b]$ contains only two points for each block. The approximated-solution $z^{(j)}_{n+1}$, for $j = 0, 1, 2, 3, 4$ at the first point ξ_{n+1} of Equation (ch4/1) can be obtained by multiple integrating of Equation (1.2) up to fifth-times with respect to the independent-variable ξ resp. over the interval $[\xi_m, \xi_{m+1}]$. The integral formulas can be written as follows:

$$\int_{\xi_n}^{\xi_{+1}} z^{(5)}(\zeta)d\zeta = \int_{\xi_n}^{\xi_{+1}} \psi(\zeta,\phi(\zeta))d\zeta, \tag{3.13}$$

$$\int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi} z^{(5)}(\zeta)d\zeta d\zeta = \int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi} \psi(\zeta,\phi(\zeta))d\zeta d\zeta, \tag{3.14}$$

$$\int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} z^{(5)}(\zeta)d\zeta d\zeta d\zeta = \int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} \psi(\zeta,\phi(\zeta))d\zeta d\zeta d\zeta, \tag{3.15}$$

$$\int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} z^{(5)}(\zeta)d\zeta d\zeta d\zeta d\zeta d\zeta = \int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} \psi(\zeta,\phi(\zeta))d\zeta d\zeta d\zeta d\zeta, \tag{3.16}$$

$$\int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} z^{(5)}(\zeta)d\zeta d\zeta d\zeta d\zeta d\zeta = \int_{\xi_n}^{\xi_{+1}}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi}\int_{\xi_n}^{\xi} \psi(\zeta,\phi(\zeta))d\zeta d\zeta d\zeta d\zeta d\zeta, \tag{3.17}$$

where $\phi(\zeta) \equiv \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))$. and $\xi_{n+1} = \xi_n + h$ By integrate the equations (3.13)-(3.17) to get the following equations:

$$z_{n+1}^{(4)} = z_n^{(4)} + \int_{\xi_n}^{\xi_{n+1}} \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))d\zeta \tag{3.18}$$

$$z_{n+1}^{(3)} = z_n^{(3)} + hz_n^{(4)} + \int_{\xi_n}^{\xi_{n+1}} \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))d\zeta \tag{3.19}$$

$$z_{n+1}^{''} = z_n^{''} + hz_n^{'''} + \frac{h^2}{2!}z_n^{(4)} + \int_{\xi_n}^{\xi_{n+1}} \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))d\zeta \tag{3.20}$$

$$z_{n+1}^{'} = z_n^{'} + hz_n^{''} + \frac{h^2}{2!}z_n^{'''} + \frac{h^3}{3!}z_n^{(4)} + \int_{\xi_n}^{\xi_{n+1}} \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))d\zeta \tag{3.21}$$

$$z_{n+1} = z_n + hz_n + \frac{h^2}{2!}z_n^{'} + \frac{h^3}{3!}z_n^{''} + \frac{h^4}{4!}z_n^{'''} + \int_{\xi_n}^{\xi_{n+1}} \phi(z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{''''}(\zeta))d\zeta \tag{3.22}$$

Let $\zeta_{n+1} = \zeta_n + h$ and the change of coordinate $s = \frac{\zeta - \zeta_{n+2}}{h}$, $dt = hds$ where, ψ in (1.2) will be replaced by the following Hermite interpolating polynomial $\Theta(s) = \psi_0 L_{00}(s) + \psi_1 L_{10}(s) + \psi_2 L_{20}(s) + g_0 L_{01}(s) + g_1 L_{11}(s) + g_2 L_{21}(s)$. where $\psi = \psi(\zeta, z(\zeta), z^{'}(\zeta), z^{''}(\zeta), z^{'''}(\zeta), z^{(4)}(\zeta))$ Using the the approximation which gives the following formulas:

$$z_{n+1}^{(4)} = z_n^{(4)} + \int_{-2}^{-1} \Theta(s)hds \tag{3.23}$$

$$z_{n+1}^{(3)} = z_n^{(3)} + hz_n^{(4)} - \int_{-2}^{-1} h(s+1)\Theta(s)hds \tag{3.24}$$

$$z_{n+1}^{''} = z_n^{''} + hz_n^{'''} + \frac{h^2}{2!}z_n^{(4)} + \int_{-2}^{-1} \frac{(h(s+1))^2}{2!}\Theta(s)hds \tag{3.25}$$

$$z_{n+1}^{'} = z_n^{'} + hz_n^{''} + \frac{h^2}{2!}z_n^{(3)} + \frac{h^3}{3!}z_n^{(4)} - \int_{-2}^{-1} \frac{(h(s+1))^3}{3!}\Theta(s)hds \tag{3.26}$$

$$z_{n+1} = z_n + hz_n^{'} + \frac{h^2}{2!}z_n^{''} + \frac{h^3}{3!}z_n^{(3)} + \frac{h^4}{4!}z_n^{(4)} + \int_{-2}^{-1} \frac{(h(s+1))^4}{4!}\Theta(s)hds \tag{3.27}$$

By integration the equations (3.23)-(3.27), we obtained the following new formulas:

$$z^{(4)}_{n+1} = \Delta_1 + \frac{h}{240}(101\psi_n + 128\psi_{n+1} + 11\psi_{n+2} + h(13g_n - 40g_{n+1} - 3g_{n+2})), \tag{3.28}$$

$$z^{(3)}_{n+1} = \Delta_2 + \frac{h^2}{1680}(520\psi_n + 7\psi_{n+1} + \psi_{n+2} + h(59g_n - 128g_{n+1} - 11g_{n+2})), \tag{3.29}$$

$$z^{''}_{n+1} = \Delta_3 + \frac{h^3}{6720}(817\psi_n + 256\psi_{n+1} + 47\psi_{n+2} + h(83g_n - 140g_{n+1} - 13g_{n+2})), \tag{3.30}$$

$$z^{'}_{n+1} = \Delta_4 + \frac{h^4}{2016}(67\psi_n + 14\psi_{n+1} + 3\psi_{n+2}) + \frac{h^5}{60480}(185g_n - 256g_{n+1} - 25g_{n+2}), \tag{3.31}$$

$$z_{n+1} = \Delta_5 + \frac{h^5}{241920}(1699\psi_n + 256\psi_{n+1} + 61\psi_{n+2} + h(143g_n - 168g_{n+1} - 17g_{n+2})), \tag{3.32}$$

where

$$\Delta_1 = z^{(4)}_n, \tag{3.33}$$

$$\Delta_2 = z^{(3)}_n + hz^{(4)}_n, \tag{3.34}$$

$$\Delta_3 = z^{''}_n + hz^{(3)}_n + \frac{h^2}{2!}z^{(4)}_n, \tag{3.35}$$

$$\Delta_4 = z^{'}_n + hz^{''}_n + \frac{h^2}{2!}z^{(3)}_n + \frac{h^3}{3!}z^{(4)}_n, \tag{3.36}$$

$$\Delta_5 = z_n + hz^{'}_n + \frac{h^2}{2!}z^{''}_n + \frac{h^3}{3!}z^{(3)}_n + \frac{h^4}{4!}z^{(4)}_n. \tag{3.37}$$

Evaluating the $P_2(\zeta)$ at the point z_{n+2} over $[\zeta_{n+1}, \zeta_{n+2}]$ to have a two points implicit method. Using the same steps as in the previous formula z_{n+1}, we have the second formula at ζ_{n+2} :

$$z^{(4)}_{n+2} = \triangle 6 + \frac{h}{240}(11\psi_n + 128\psi_{n+1} + 101\psi_{n+2} + h(3g_n + 40g_{n+1} - 13g_{n+2})). \tag{3.38}$$

$$z^{'''}_{n+2} = \triangle 7 + \frac{h^2}{1680}(37\psi_n + 616\psi_{n+1} + 187\psi_{n+2} + 2h(5g_n + 76g_{n+1} - 16g_{n+2})). \tag{3.39}$$

$$z^{''}_{n+2} = \triangle 8 + \frac{h^3}{6720}(41\psi_n + 928\psi_{n+1} + 151\psi_{n+2} + h(11g_n + 188g_{n+1} - 29g_{n+2})). \tag{3.40}$$

$$z^{'}_{n+2} = \triangle 9 + \frac{h^4}{30240}(350\psi_n + 148\psi_{n+1} + 15\psi_{n+2} + h(10g_n + 190g_{n+1} - 23g_{n+2})). \tag{3.41}$$

$$z_{n+2} = \triangle 10 + \frac{h^5}{241920}(49\psi_n + 1840\psi_{n+1} + 127\psi_{n+2} + h(13g_n + 272g_{n+1} - 27g_{n+2})). \tag{3.42}$$

where

$$\Delta_6 = z^{(4)}(\zeta_{n+1}) \tag{3.43}$$

$$\Delta_7 = z^{'''}(\zeta_{n+1}) + hz^{(4)}(\zeta_{n+1}) \tag{3.44}$$

$$\Delta_8 = z^{''}(\zeta_{n+1}) + hz^{'''}(\zeta_{n+1}) + \frac{h^2}{2}z^{(4)}(\zeta_{n+1}) \tag{3.45}$$

$$\Delta_9 = z^{'}(\zeta_{n+1}) + hz^{''}(\zeta_{n+1}) + \frac{h^2}{2}z^{'''}(\zeta_{n+1}) + \frac{h^3}{6}z^{(4)}(\zeta_{n+1}) \tag{3.46}$$

$$\Delta_{10} = z(\zeta_{n+1}) + hz^{'}(\zeta_{n+1}) + \frac{h^2}{2}z^{''}(\zeta_{n+1}) + \frac{h^3}{6}z^{'''}(\zeta_{n+1}) + \frac{h^4}{24}z^{(4)}(\zeta_{n+1}) \tag{3.47}$$

3.3. The Order and Error Constant of the Proposed Method

The formulae of two points implicit block method is given in the equations (3.28)-(3.32) have been written in matrix form as follows:

$$\alpha Z_m = h\beta Z_m^{'} + h^2\gamma Z_m^{''} + h^3\psi Z_m^{'''} + h^4\delta Z_m^{(4)} + \varphi h^5 F_m + h^6\lambda G_m \tag{3.48}$$

where,

$$\alpha_{ij} = \begin{cases} 1 & (i,j) \in (5,9),(10,10), \\ -1 & (i,j) \in (5,8),(10,9), \\ 0 & o.w. \end{cases} ; \quad \beta_{ij} = \begin{cases} 1 & (i,j) \in (4,8),(5,8),(9,9),(10,9), \\ -1 & (i,j) \in (4,9),(9,10), \\ 0 & o.w. \end{cases}$$

$$\gamma_{ij} = \begin{cases} 1 & (i,j) \in (3,8),(4,8),(8,9),(9,9), \\ -1 & (i,j) \in (3,9),(8,10), \\ \frac{1}{2} & (i,j) \in (5,8),(10,9), \\ 0 & o.w. \end{cases} ; \quad \psi_{ij} = \begin{cases} 1 & (i,j) \in (2,8),(3,8),(7,9),(8,9), \\ -1 & (i,j) \in (2,9),(7,10), \\ \frac{1}{2} & (i,j) \in (4,8),(9,9), \\ \frac{1}{6} & (i,j) \in (5,8),(10,9), \\ 0 & o.w. \end{cases}$$

$$\delta_{ij} = \begin{cases} 1 & (i,j) \in (1,8),(2,8),(6,9),(7,9), \\ -1 & (i,j) \in (1,8),(6,10), \\ \frac{1}{2} & (i,j) \in (3,8),(8,9), \\ \frac{1}{6} & (i,j) \in (4,8),(9,9), \\ \frac{1}{24} & (i,j) \in (5,8),(10,9), \\ 0 & o.w. \end{cases}$$

$$\varphi_{e_8} = [\frac{101}{240}, \frac{13}{42}, \frac{817}{6720}, \frac{67}{2016}, \frac{1699}{241920}, \frac{11}{240}, \frac{37}{1680}, \frac{41}{6720}, \frac{5}{4032}, \frac{49}{241920}]^T,$$

$$\varphi_{e_9} = [\frac{128}{240}, \frac{7}{42}, \frac{256}{6720}, \frac{14}{2016}, \frac{256}{241920}, \frac{128}{240}, \frac{616}{1680}, \frac{928}{6720}, \frac{128}{4032}, \frac{1840}{241920}]^T,$$

$$\varphi_{e_{10}} = [\frac{11}{240}, \frac{1}{42}, \frac{47}{6720}, \frac{3}{2016}, \frac{61}{241920}, \frac{101}{240}, \frac{187}{1680}, \frac{151}{6720}, \frac{15}{4032}, \frac{127}{241920}]^T,$$

$$\lambda_{e_8} = [\frac{13}{240}, \frac{59}{1680}, \frac{83}{6720}, \frac{185}{60480}, \frac{143}{241920}, \frac{3}{240}, \frac{5}{840}, \frac{11}{6720}, \frac{10}{30240}, \frac{13}{241920}]^T,$$

$$\lambda_{e_9} = [\frac{-40}{240}, \frac{-128}{1680}, \frac{-140}{6720}, \frac{-256}{60480}, \frac{-168}{241920}, \frac{40}{240}, \frac{76}{840}, \frac{188}{6720}, \frac{190}{30240}, \frac{272}{241920}]^T,$$

and,

$$\lambda_{e_{10}} = [\frac{-3}{240}, \frac{-11}{1680}, \frac{-13}{6720}, \frac{-25}{60480}, \frac{-17}{241920}, \frac{-13}{240}, \frac{-16}{840}, \frac{-29}{6720}, \frac{-23}{30240}, \frac{-27}{241920}]^T.$$

where $\varphi = \left(\varphi_{e_j}\right)$ and $\lambda = \left(\lambda_{e_j}\right)$ for j=1,2,...,10 and $\varphi_{e_j} \equiv \lambda_{e_j} \equiv 0$ for j=1,2,...,7.

$$Z_m^{(i)} = (z_{n-7}^{(i)}, z_{n-6}^{(i)}, z_{n-5}^{(i)}, z_{n-4}^{(i)}, z_{n-3}^{(i)}, z_{n-2}^{(i)}, z_{n-1}^{(i)}, z_{n}^{(i)}, z_{n+1}^{(i)}, z_{n+2}^{(i)})^T,$$

$$F_m^{(i)} = (\psi_{n-7}^{(i)}, \psi_{n-6}^{(i)}, \psi_{n-5}^{(i)}, \psi_{n-4}^{(i)}, \psi_{n-3}^{(i)}, \psi_{n-2}^{(i)}, \psi_{n-1}^{(i)}, \psi_{n}^{(i)}, \psi_{n+1}^{(i)}, \psi_{n+2}^{(i)})^T.$$

for $i = 0, 1$. It can be define the linear operator in equation (3.48) as

$$L[Z(\zeta); h] = \alpha Z_m - h\beta Z_m^{'} - h^2\gamma Z_m^{''} - h^3\psi Z_m^{'''} - h^4\delta Z_m^{(4)} - \varphi h^5 F_m - h^6\lambda G_m. \tag{3.49}$$

where $Z(\zeta)$ is an arbitrary function that is continuous and differentiable, expanding equation (3.49) in the Taylor series at point ζ yields

$$L[Z(\zeta;h] = C_0Z(\zeta) + C_1hZ'(\zeta) + C_ph^pZ^p(\zeta) + \cdots + C_{p+1}h^{p+1}Z^{p+1}(\zeta) + \ldots \tag{3.50}$$

The proposed GIBM method has order= P if the linear operator in Equation (3.50) satisfy that $C_j = 0; j = 0, 1, \ldots, P+4$ and $C_{P+5} \neq 0$ where C_{P+5} =the error constant of the method. In GIBM method, we have $C_j = 0; j = 0, 1, \ldots, 10$ and

$$C_{11} = [\frac{1}{9450}, \frac{1}{17280}, \frac{1}{56700}, \frac{1}{259200}, \frac{1}{1496880}, \frac{1}{9450}, \frac{29}{604800}, \frac{23}{1814400}, \frac{1}{403200}, \frac{47}{119750400}]^T.$$

So, it can be concluded that the order of the two points implicit method is 6.

3.4. Zero-Stability of GIBM Method

In this subsection, the zero-stability of the GIBM method is discussed. The formulas of the new method in equations (3.28)-(3.32) and (3.38)-(3.42) are considered as a zero stable in case the roots $r_i = 1, 2, \ldots, N$ of the first characteristic polynomial $\rho(R) = \det\,[RA^{(0)} - A^{(1)}] = 0$ is found to satisfy $\mid R \mid \leq 1$.

Hence, we will use the following technique to find the matrix form of the first characteristic polynomial. Substituting Equation (3.28) into Equation (3.38), we have

$$y^{(4)}_{n+2} = y^{(4)} + \frac{h}{15}[7f_n + 16f_{n+1} + 7f_{n+2}] + \frac{h^2}{15}[g_n - g_{n+2}]. \tag{3.51}$$

By replacing equations (3.28)-(3.29) into (3.39) , we have

$$z'''_{n+2} = z'''_n + 2hz^{(4)}_n + \frac{h^2}{105}[79\psi_n + 112\psi_{n+1} + 19\psi_{n+2}] + \frac{h^3}{105}[10g_n - 16g_{n+1} - 4g_{n+2}]. \tag{3.52}$$

Also, by substituting equations (3.28)-(3.30) into (3.40) , we get

$$z''_{n+2} = z''_n + 2hz'''_n + 2h^2z^{(4)}_n + \frac{h^3}{105}[68\psi_n + 64\psi_{n+1} + 8\psi_{n+2}] + \frac{h^4}{105}(8g_n - 16g_{n+1} - 2g_{n+2}). \tag{3.53}$$

Substituting equations (3.28)-(3.31) into (3.41) , we have

$$z'_{n+2} = z'_n + 2hz''_n + 2h^2z'''_n + \frac{4h^3}{3}z^{(4)}_n + \frac{h^4}{63}(24\psi_n + 16\psi_{n+1} + 2\psi_{n+2}) + \frac{h^5}{945}(40g_n - 80g_{n+1} - 8g_{n+2}). \tag{3.54}$$

By substituting equations (3.28)-(3.32) into (3.42), we get

$$z_{n+2} = z_n + 2hz'_n + 2h^2z''_n + \frac{4h^3}{3}z'''_n + \frac{2h^4}{3}z^{(4)}_n + \frac{h^5}{945}[161\psi_n + 80\psi_{n+1} \tag{3.55}$$

$$+ 11\psi_{n+2} + h(17g_n - 32g_{n+1} - 3g_{n+2})). \tag{3.56}$$

Using the equations (3.28)-(3.32) and the equations (3.51)-(3.55)).

The general form of the matrices $A^{(i)}$ for i=0,1 can be written as

$$A^{(0)}_{ij} = \delta_{ij} \quad \text{and} \quad A^{(1)}_{ij} = \begin{cases} 1 & j = i+5; i < 5, \\ & i = j; i > 5 \\ 0 & o.w. \end{cases}$$

where δ is Kroneker delta and $A^{(i)}; i = 0, 1$ are 10x10 matrices. Then, $\rho(R) = |RA^{(0)} - A^{(1)}| = 0$ implies that $\rho(R) = R^5(R-1)^5,\ R = 0, 1$ 5-times.

Hence, it can conclude that the new method is zero stable.

3.5. Implementation

In this subsection, we are going to explain the implementation of the two points implicit block method. The values of $z^{(4)}_{n+1}, z'''_{n+1}, z''_{n+1}, z'_{n+1}, z_{n+1}, z^{(4)}_{n+2}, z'''_{n+2}, z''_{n+2}, z'_{n+2}$ and z_{n+2} in the equations (3.28)-(3.32) and the equations (3.38)-(3.42) have been approximated using the predictor-corrector equations.

The predictor equations using Taylor method have the following forms:

$$\begin{aligned}
z^{(4)p}_{n+m} &= z^{(4)p}_{n+(m-1)} + h\psi^c_{n+(m-1)}, \\
z'''^{p}_{n+m} &= z'''^{p}_{n+(m-1)} + hz^{(4)p}_{n+(m-1)} + \frac{h^2}{2!}\psi^c_{n+(m-1)}, \\
z''^{p}_{n+m} &= z''^{p}_{n+(m-1)} + hz'^{p}_{n+(m-1)} + \frac{h^2}{2!}z^{(4)p}_{n+(m-1)} + \frac{h^3}{3!}\psi^c_{n+(m-1)}, \\
z'^{p}_{n+m} &= z'^{p}_{n+(m-1)} + hz''^{p}_{n+(m-1)} + \frac{h^2}{2!}z'''^{p}_{n+(m-1)} + \frac{h^3}{3!}z^{(4)p}_{n+(m-1)} + \frac{h^4}{4!}\psi^c_{n+(m-1)}, \\
z^{p}_{n+m} &= z^{p}_{n+(m-1)} + hz'^{p}_{n+(m-1)} + \frac{h^2}{2!}z''^{p}_{n+(m-1)} + \frac{h^3}{3!}z'''^{p}_{n+(m-1)} + \frac{h^4}{4!}z^{(4)p}_{n+(m-1)} \\
&\quad + \frac{h^5}{5!}\psi^c_{n+(m-1)}; \quad m = 1, 2, \\
\psi^p_{n+m} &= \psi(\zeta_{n+m}, z^p_{n+m}, z'^{p}_{n+m}, z''^{p}_{n+m}, z'''^{p}_{n+m}, z^{(4)p}_{n+m}), \\
g^p_{n+m} &= \psi'(\zeta_{n+m}, z^p_{n+m}, z'^{p}_{n+m}, z''^{p}_{n+m}, z'''^{p}_{n+m}, z^{(4)p}_{n+m}).
\end{aligned}$$

The corrector equations are

$$\begin{aligned}
z^{(4)c}_{n+1} &= \triangle_{11} + \frac{h}{240}(101\psi^c_n + 128\psi^p_{n+1} + 11\psi^p_{n+2} + h(13g^c_n - 40g^p_{n+1} - 3g^p_{n+2})). \\
z'''^{c}_{n+1} &= \triangle_{12} + \frac{h^2}{1680}(520\psi^c_n + 7\psi^p_{n+1} + \psi^p_{n+2} + h(59g^c_n - 128g^p_{n+1} - 11g^p_{n+2})). \\
z''^{c}_{n+1} &= \triangle_{13} + \frac{h^3}{6720}(817\psi^c_n + 256\psi^p_{n+1} + 47\psi^p_{n+2} + h(83g^c_n - 140g^p_{n+1} - 13g^p_{n+2})). \\
z'^{c}_{n+1} &= \triangle 14 + \frac{h^4}{60480}(30(67\psi^c_n + 14\psi^p_{n+1} + 3\psi^p_{n+2}) + h(185g^c_n - 256g^p_{n+1} - 25g^p_{n+2})). \\
z^{c}_{n+1} &= \triangle_{15} + \frac{h^5}{241920}(1699\psi^c_n + 256\psi^p_{n+1} + 61\psi^p_{n+2} + h(143g^c_n - 168g^p_{n+1} - 17g^p_{n+2})). \\
z^{(4)}_{n+2} &= \triangle_{16} + \frac{h}{240}(11\psi^c_n + 128\psi^p_{n+1} + 101\psi^p_{n+2} + h(3g^c_n + 40g^p_{n+1} - 13g^p_{n+2})). \\
z'''_{n+2} &= \triangle_{17} + \frac{h^2}{1680}(37\psi^c_n + 616\psi^p_{n+1} + 187\psi^p_{n+2} + 2h(5g^c_n + 76g^p_{n+1} - 16g^p_{n+2})). \\
z''_{n+2} &= \triangle_{18} + \frac{h^3}{6720}(41\psi^c_n + 928\psi^p_{n+1} + 151\psi^p_{n+2} + h(11g^c_n + 188g^p_{n+1} - 29g^p_{n+2})). \\
z'_{n+2} &= \triangle_{19} + \frac{h^4}{30240}(350\psi^c_n + 148\psi^p_{n+1} + 15\psi^p_{n+2} + h(10g^c_n + 190g^p_{n+1} - 23g^p_{n+2})). \\
y_{n+2} &= \triangle_{20} + \frac{h^5}{241920}(49\psi^c_n + 1840\psi^p_{n+1} + 127\psi^p_{n+2} + h(13g^c_n + 272g^p_{n+1} - 27g^p_{n+2})).
\end{aligned}$$

where

$$
\begin{aligned}
\triangle_{11} &= z^{(4)c}(\zeta_n)\\
\triangle_{12} &= z^{'''c}(\zeta_n) + hz^{(4)c}(\zeta_n)\\
\triangle_{13} &= z^{''c}(\zeta_n) + hz^{'''c}(\zeta_n) + \frac{h^2}{2}z^{(4)c}(\zeta_n)\\
\\
\triangle_{14} &= z^{'c}(\zeta_n) + hz^{''c}(\zeta_n) + \frac{h^2}{2}z^{'''c}(\zeta_n) + \frac{h^3}{6}z^{(4)c}(\zeta_n)\\
\triangle_{15} &= z^{c}(\zeta_n) + hz^{'c}(\zeta_n) + \frac{h^2}{2}z^{''c}(\zeta_n) + \frac{h^3}{6}z^{'''c}(\zeta_n) + \frac{h^4}{24}z^{(4)c}(\zeta_n)\\
\triangle_{16} &= z^{(4)c}(\zeta_{n+1})\\
\triangle_{17} &= z^{'''c}(\zeta_{n+1}) + hz^{(4)c}(\zeta_{n+1})\\
\triangle_{18} &= z^{''c}(\zeta_{n+1}) + hz^{'''c}(\zeta_{n+1}) + \frac{h^2}{2}z^{(4)c}(\zeta_{n+1})\\
\triangle_{19} &= z^{'c}(\zeta_{n+1}) + hz^{''c}(\zeta_{n+1}) + \frac{h^2}{2}z^{'''c}(\zeta_{n+1}) + \frac{h^3}{6}z^{(4)c}(\zeta_{n+1})\\
\triangle_{20} &= z^{c}(\zeta_{n+1}) + hz^{'c}(\zeta_{n+1}) + \frac{h^2}{2}z^{''c}(\zeta_{n+1}) + \frac{h^3}{6}z^{'''c}(\zeta_{n+1}) + \frac{h^4}{24}z^{(4)c}(\zeta_{n+1}).
\end{aligned}
$$

And the next corrector equations will be taken as follows:

$$
\begin{aligned}
z^{(4)c}_{n+1} &= \triangle_{11} + \frac{h}{240}(101\psi^c_n + 128\psi^c_{n+1} + 11\psi^c_{n+2} + h(13g^c_n - 40g^c_{n+1} - 3g^c_{n+2})).\\
z^{'''c}_{n+1} &= \triangle_{12} + \frac{h^2}{1680}(40(13\psi^c_n + 7\psi^c_{n+1} + \psi^c_{n+2}) + \frac{h^3}{1680}(59g^c_n - 128g^c_{n+1} - 11g^c_{n+2})).\\
z^{''c}_{n+1} &= \triangle_{13} + \frac{h^3}{6720}(817\psi^c_n + 256\psi^c_{n+1} + 47\psi^c_{n+2} + h(83g^c_n - 140g^c_{n+1} - 13g^c_{n+2})).\\
z^{'c}_{n+1} &= \triangle_{14} + \frac{h^4}{60480}(30(67\psi^c_n + 14\psi^c_{n+1} + 3\psi^c_{n+2}) + h(185g^c_n - 256g^c_{n+1} - 25g^c_{n+2}).\\
z^{c}_{n+1} &= \triangle_{15} + \frac{h^5}{241920}(1699\psi^c_n + 256\psi^c_{n+1} + 61\psi^c_{n+2} + h(143g^c_n - 168g^c_{n+1} - 17g^c_{n+2})).\\
z^{(4)}_{n+2} &= \triangle_{16} + \frac{h}{240}(11\psi^c_n + 128\psi^c_{n+1} + 101\psi^c_{n+2}) + h(3g^c_n + 40g^c_{n+1} - 13g^c_{n+2})).\\
z^{'''}_{n+2} &= \triangle_{17} + \frac{h^2}{1680}(37\psi^c_n + 616\psi^c_{n+1} + 187\psi^c_{n+2} + 2h(5g^c_n + 76g^c_{n+1} - 16g^c_{n+2})).\\
z^{''}_{n+2} &= \triangle_{18} + \frac{h^3}{6720}(41\psi^c_n + 928\psi^c_{n+1} + 151\psi^c_{n+2}) + h(11g^c_n + 188g^c_{n+1} - 29g^c_{n+2})).\\
z^{'}_{n+2} &= \triangle_{19} + \frac{h^4}{30240}(70(5\psi^c_n + 148\psi^c_{n+1} + 15\psi^c_{n+2}) + h(10g^c_n + 190g^c_{n+1} - 23g^c_{n+2})).\\
z_{n+2} &= \triangle_{20} + \frac{h^5}{241920}(49\psi^c_n + 1840\psi^c_{n+1} + 127\psi^c_{n+2} + h(13g^c_n + 272g^c_{n+1} - 27g^c_{n+2})).\\
\psi^c_{n+m} &= \psi(\zeta_{n+m}, z^c_{n+m}, z'^c_{n+m}, z''^c_{n+m}, z'''^c_{n+m}, z^{(4)c}_{n+m}),\\
g^c_{n+m} &= \psi'(\zeta_{n+m}, z^c_{n+m}, z'^c_{n+m}, z''^c_{n+m}, z'''^c_{n+m}, z^{(4)c}_{n+m}),\ m = 1, 2.
\end{aligned}
$$

4. Numerical Results

In this section, a set of quasi linear ODEs of fifth-order is solved by using the proposed GIBM and GRKM methods respectively. The numerical results of them are compared in Figure 1 to indicate the identical of two numerical solutions. Some notations are used as follows:

- **Step**: Step-size used.
- **GIBM**: Direct proposed method.
- **GRKM**: General RKM method.

4.1. Problems Tested of ODEs

Example 4.1. *(Linear)*

$$z^{(5)}(\zeta) = cos(\zeta);, \qquad 0 < \zeta \leq b.$$

with ICs,

$$z^{(j)}(0) = 0, z'(0) = -z'''(0) = 1; j = 0, 2, 4;.$$

Exact solution: $z(\zeta) = \sin(\zeta), \quad b = \pi.$

Example 4.2. *(Non constant coefficients)*

$$z^{(5)}(\zeta) = (-32\zeta^5 + 16\zeta^4 - 8\zeta^3 + 4\zeta^2 - 2\zeta)y(\zeta), \qquad 0 < \zeta \leq b.$$

with ICs,

$$z(0) = 1, z^{(j)}(0) = 0; j = 1, 2, 3, 4.$$

Exact solution: $z(\zeta) = e^{-\zeta^2}, \quad b = 1.$

Example 4.3. *(Non linear)*

$$z^{(5)}(\zeta) = -120z^6(\zeta), \qquad 0 < \zeta \leq b.$$

with ICs

$$z'(0) = (-1)^j j!; j = 0, 1, 2, 3, 4.$$

Exact solution: $z(\zeta) = \frac{1}{1+\zeta}, \quad b = 1.$

Example 4.4. *(Linear System)*

$$\begin{aligned} z_1^{(5)}(\zeta) &= -212z_1(\zeta) - 180z_2(\zeta) - 211z_3(\zeta), \\ z_2^{(5)}(\zeta) &= 211z_1(\zeta) + 179z_2(\zeta) + 211z_3(\zeta), \\ z_3^{(5)}(\zeta) &= -242z_1(\zeta) - 242z_2(\zeta) - 243z_3(\zeta). \end{aligned} \tag{4.1}$$

with ICs

$$\begin{array}{lllll} z_1(0) = 1, & z_1'(0) = -2, & z_1''(0) = 6, & z_1'''(0) = -20, & z_1^{(4)}(0) = 66; \\ z_2(0) = 0, & z_2'(0) = 1, & z_2''(0) = -5, & z_2'''(0) = 19, & z_2^{(4)}(0) = 65; \\ z_3(0) = 0, & z_3'(0) = -2, & z_3''(0) = 8, & z_3'''(0) = -26, & z_3^{(4)}(0) = 80. \end{array}$$

The system is integrated over the interval $[0, 2]$

Exact solution:

$$\begin{aligned} z_1(\zeta) &= e^{-\zeta} - e^{-2\zeta} + e^{-3\zeta}, \\ z_2(\zeta) &= e^{-2\zeta} - e^{-3\zeta}, \\ z_3(\zeta) &= e^{-3\zeta} - e^{-\zeta} \end{aligned}$$

Example 4.5. *(Homogenous ODE)*

$$z^{(5)}(\zeta) = z(\zeta) + z'(\zeta) + z''(\zeta);, \qquad 0 < \zeta \leq b.$$

with ICs,

$$z^{'}(0) = -z^{'''}(0) = 1, z^{(j)}(0) = 0; j = 0, 2, 4.$$

Exact solution: $z(\zeta) = \sin(\zeta)$, $\quad b = 0.1$

Example 4.6. *(Nonlinear ODE)*

$$z^{(5)}(\zeta) = z^6(\zeta) + z'^3(\zeta) - 30z''^2(\zeta), \qquad 0 < \zeta \leq b.$$

with ICs,

$$y^{'}(0) = (-1)^j j!; j = 0, 1, 2, 3, 4.$$

Exact solution: $z(\zeta) = \frac{1}{1+\zeta}$, $\quad b = 10$

5. Conclusion and Discussion

In this paper, general implicit block method (GIBM) for solving general class of ODEs of fifth-order has been derived using the approach of Hermite approximation. It named as GIBM method. The aim of this article, is to derive direct-implicit block method for solving general class of ODEs of fifth-order. Numerical results of proposed GIBM method have compared with the results which obtained using GRKM method of the same order. From this comparison, we can conclude that the new GIBM method is more efficient than existing GRKM method in term of number of evaluation. In view the results in the implementation, we can conclude that, the proposed method is powerful method in computation and meanwhile require less function-evaluations and more cost-effective, in terms of time of computation.

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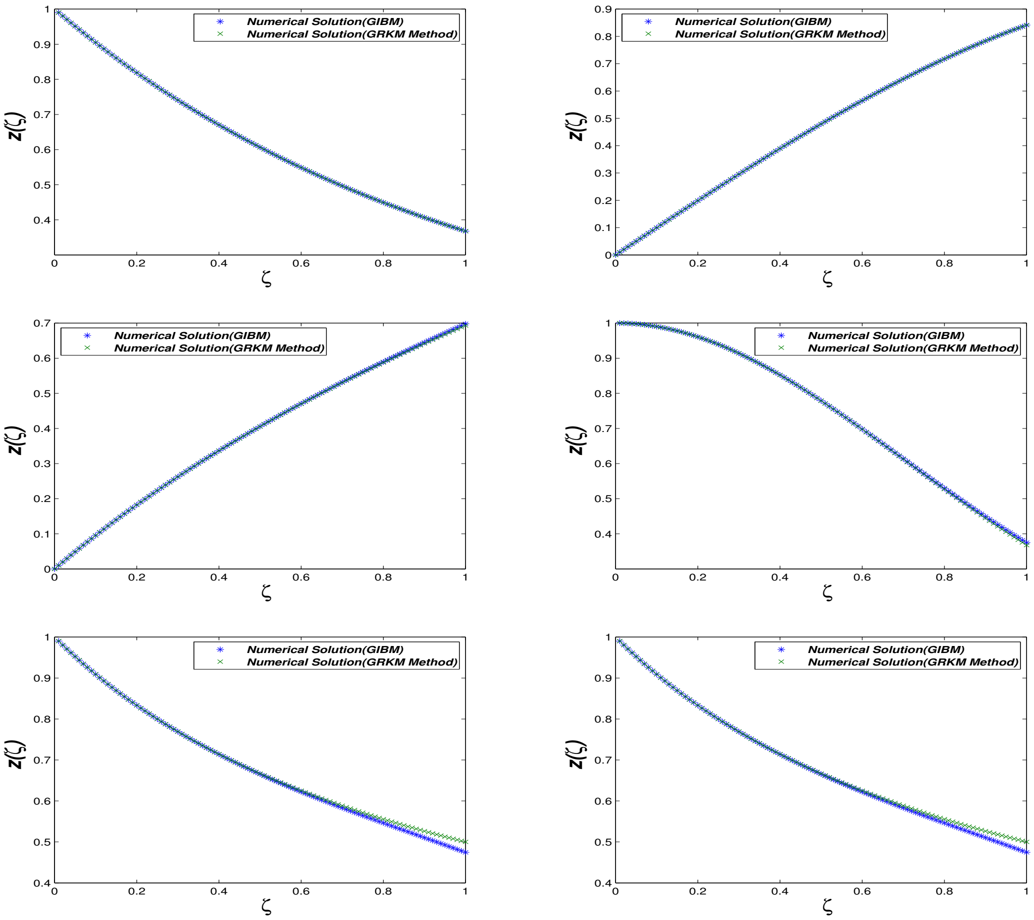

Figure 2: Numerical Solutions of Proposed GIBM Method Versus Numerical Solutions of GRKM Method in Examples 1,2,3,4,5 and 6

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