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Classes of certain analytic functions defining by subordinations

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Abstract

There are many results for analytic functions in the open unit disk U concerning subordinations. Two subclasses of analytic functions in U are introduced using subordinations in U. The object of the present paper is to discuss some properties of functions belonging to these two subclasses.

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1. Introduction

Let A be the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Let $f(z)$ and $g(z)$ be analytic in U. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in U satisfying $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$ and $f(z) = g(w(z))$. We denote this subordination by:

$$f(z) \prec g(z) \qquad (z \in U). \tag{1.1}$$

This subordination is applied for many papers for univalent function theary by Breaz, Owa and Breaz [1], Rogosinski [4], [5], and Singh and Gupta [6]. Let us consider a function $g(z)$ given by

$$g(z) = \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in U) \tag{1.2}$$

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for some real α $(0 < \alpha < 1)$. Then, $g(z)$ is analytic in U and $g(0) = 1$. If we write that

$$w = \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in U), \tag{1.3}$$

then

$$|z| = \left| \frac{\alpha(w-1)}{\alpha w - 1} \right| < 1. \tag{1.4}$$

This means that

$$Reg(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in U). \tag{1.5}$$

In view of the above, we say that $f(z) \in P(\alpha)$ if $f(z) \in A$ satisfies $f(z) \neq 0$ $(z \neq 0)$ and

$$\frac{z}{f(z)} \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in U) \tag{1.6}$$

for some real α $(0 < \alpha < 1)$. Further, we say that $f(z) \in Q(\alpha)$ if and only if $zf'(z) \in P(\alpha)$ for $f(z) \in A$.

2. Some properties

First, we derive

Theorem 1 *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} |a_n| \leqq \frac{1-\alpha}{1+\alpha} \tag{2.1}$$

for some real α $(0 < \alpha < 1)$, then $f(z) \in P(\alpha)$. The equality in (2.1) holds true for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\,\varepsilon}{n(n-1)(1+\alpha)}\, z^n \qquad (|\varepsilon| = 1). \tag{2.2}$$

Proof We note that if $f(z) \in A$ satisfies

$$\alpha \left| 1 - \frac{z}{f(z)} \right| < \left| 1 - \alpha\, \frac{z}{f(z)} \right| \qquad (z \in U) \tag{2.3}$$

for some real α $(0 < \alpha < 1)$, then $f(z) \in P(\alpha)$. The inequality (2.3) is equivalent to

$$|f(z) - z| < \left| \frac{1}{\alpha} f(z) - z \right| \qquad (z \in U). \tag{2.4}$$

This means that

$$\left| \sum_{n=2}^{\infty} a_n z^{n-1} \right| < \left| \left(\frac{1}{\alpha} - 1 \right) + \frac{1}{\alpha} \sum_{n=2}^{\infty} a_n z^{n-1} \right|. \tag{2.5}$$

Therefore, if $f(z)$ satisfies

$$\sum_{n=2}^{\infty} |a_n| \leqq \frac{1-\alpha}{\alpha} - \frac{1}{\alpha}\sum_{n=2}^{\infty} |a_n|, \tag{2.6}$$

that is, that

$$(1+\alpha)\sum_{n=2}^{\infty} |a_n| \leqq (1-\alpha), \tag{2.7}$$

then $f(z) \in P(\alpha)$. Further, if we consider $f(z)$ given by (2.2), then

$$\sum_{n=2}^{\infty} |a_n| = \frac{1-\alpha}{1+\alpha}\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \frac{1-\alpha}{1+\alpha}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1-\alpha}{1+\alpha}. \tag{2.8}$$

This completes the proof of the theorem.

Noting that $f(z) \in Q(\alpha)$ if and only if $zf'(z) \in P(\alpha)$, we have
Corollary 1 *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n\,|a_n| \leqq \frac{1-\alpha}{1+\alpha} \tag{2.9}$$

for some real α $(0 < \alpha < 1)$, then $f(z) \in Q(\alpha)$. The equality in (2.9) holds true for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\varepsilon}{n^2(n-1)(1+\alpha)}\, z^n \qquad (|\varepsilon| = 1). \tag{2.10}$$

To consider next properties for $f(z)$, we have to recall here the following lemma due to Miller and Mocanu [3] (also, due to Jack [2]).
Lemma 1 *Let $w(z)$ be analytic in U with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we have that*

$$z_0 w'(z_0) = k w(z_0), \tag{2.11}$$

where $k \geqq 1$.

Applying Lemma 1, we derive
Theorem 2 *If $f(z) \in A$ satisfies*

$$Re\left(1 + \frac{z}{f(z)}(1 - f'(z))\right) < \frac{1+3\alpha}{2\alpha(1+\alpha)} \qquad (z \in U) \tag{2.12}$$

for some real α $(0 < \alpha < 1)$, then $f(z) \in P(\alpha)$.
Proof Let us define a function $w(z)$ by

$$\frac{z}{f(z)} = \frac{\alpha - w(z)}{\alpha(1 - w(z))} \qquad (z \in U) \tag{2.13}$$

for some real α $(0 < \alpha < 1)$. Then $w(z)$ is analytic in U and $w(0) = 0$.

Since

$$1 - \frac{zf'(z)}{f(z)} = \frac{zw'(z)}{1 - w(z)} - \frac{zw'(z)}{\alpha - w(z)}, \tag{2.14}$$

we have that

$$1 + \frac{z}{f(z)}(1 - f'(z)) = \frac{\alpha - w(z)}{\alpha(1 - w(z))} + \frac{zw'(z)}{1 - w(z)} - \frac{zw'(z)}{\alpha - w(z)}. \tag{2.15}$$

If we assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1, \tag{2.16}$$

then Lemma 1 shows us that

$$z_0 w'(z_0) = k w(z_0) \qquad (k \geqq 1). \tag{2.17}$$

Letting that $w(z_0) = e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$, we have that

$$Re\left\{1 + \frac{z_0}{f(z_0)}(1 - f'(z_0))\right\} \tag{2.18}$$

$$= Re\left\{\frac{\alpha - w(z_0)}{\alpha(1 - w(z_0))} + \frac{z_0 w'(z_0)}{1 - w(z_0)} - \frac{z_0 w'(z_0)}{\alpha - w(z_0)}\right\}$$

$$= Re\left\{\frac{\alpha - e^{i\theta}}{\alpha(1 - e^{i\theta})} + \frac{ke^{i\theta}}{1 - e^{i\theta}} - \frac{ke^{i\theta}}{\alpha - e^{i\theta}}\right\}$$

$$= \frac{1 + \alpha}{2\alpha} + k\left(\frac{1 - \alpha\cos\theta}{1 + \alpha^2 - 2\,\alpha\cos\theta} - \frac{1}{2}\right).$$

Let

$$p(t) = \frac{1 - \alpha\, t}{1 + \alpha^2 - 2\,\alpha\, t} \qquad (t = \cos\theta). \tag{2.19}$$

Then

$$p'(t) = \frac{\alpha(1 - \alpha^2)}{1 + \alpha^2 - 2\alpha\, t} \qquad (-1 \leqq t \leqq 1) \tag{2.20}$$

satisfies $p'(t) > 0$ for $0 < \alpha < 1$. Thus, we obtain that

$$Re\left\{1 + \frac{z_0}{f(z_0)}(1 - f'(z_0))\right\} = \frac{1 + \alpha}{2\alpha} + k\left(\frac{1}{1 + \alpha} - \frac{1}{2}\right) \tag{2.21}$$

$$\geqq \frac{1 + 3\alpha}{2\alpha(1 + \alpha)} \tag{2.21}$$

for $0 < \alpha < 1$. This contradicts our condition (2.12). Therefore, there is no $z_0 \in U$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in U$. Thus we have that

$$|w(z)| = \left| \frac{\alpha\left(1 - \frac{z}{f(z)}\right)}{1 - \alpha\frac{z}{f(z)}} \right| < 1 \qquad (z \in U). \tag{2.22}$$

This gives us that

$$Re\left(\frac{z}{f(z)}\right) < \frac{1+\alpha}{2\,\alpha} \qquad (z \in U), \tag{2.23}$$

that is, that $f(z) \in P(\alpha)$.

For the class $Q(\alpha)$, we have
Corollary 2 *If $f(z) \in A$ satisfies*

$$Re\left\{\frac{1}{f'(z)}\left(1 - zf''(z)\right)\right\} < \frac{1+3\alpha}{2\alpha(1+\alpha)} \qquad (z \in U) \tag{2.24}$$

for some real α $(0 < \alpha < 1)$, then $f(z) \in Q(\alpha)$.

Example 1 If we consider $\alpha = \frac{1}{2}$ and

$$\frac{z}{f(z)} = \frac{1-2z}{1-z} \qquad (z \in U), \tag{2.25}$$

then we see that

$$Re\left(\frac{z}{f(z)}\right) < \frac{3}{2} \qquad (z \in U). \tag{2.26}$$

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