Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 1291-1301
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.19923.2111

Comparison between Sinc approximation and differential transform methods for nonlinear Hammerstein integral equations

GHasem Kazemi Gelian[a,*], Rezvan Ghoochani Shirvan[a], Mohammad Ali Fariborzi Araghi[b]

[a]*Department of Mathematics, Shirvan Branch, Islamic Azad University, Shirvan, Iran*
[b]*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

(Communicated by Choonkil Park)

Abstract

Here, the comparison between Sinc method in combination with double exponential transformations (DE) and approximation by means of differential transform method (DTM) for nonlinear Hammerstein integral equations is considered. Convergence analysis is presented. Detection of effectiveness from various aspects such as run time, different norms, condition number are highlighted and plotted graphically. Results of two schemes are practically well, but in manner of separable kernel, DTM solution is more accurate and so fast.

Keywords: Volterra integral equations, Sinc collocation method, double exponential transformation, differential transform method.
2010 MSC: 14F05,18E30,35C08

1. Introduction

We consider usual form of the nonlinear Hammerstein integral equation as :

$$
\begin{aligned}
y(x) &= g(x) + \int_a^x \omega(x,t,y(t))dt, \qquad a \le x \le b, \\
\omega(x,t,y(t)) &= K(x,t)H(t,y(t)).
\end{aligned}
\tag{1.1}
$$

*Corresponding author
Email addresses: kazemigelian@yahoo.com (GHasem Kazemi Gelian), ghoochani@yahoo.com (Rezvan Ghoochani Shirvan), fariborzi.araghi@gmail.com (Mohammad Ali Fariborzi Araghi)

Received: February 2020 *Accepted:* April 2020

for known real constants a, b and given functions $g(x), K(x,t)$. Modeling of natural phenomena such as fluid dynamics, electro-magnetic leads to applicable and usual form of Eq. (1). More information and other applications can be found in [5].

Final goal in Eq. (1) is finding $y(x)$ in analytical or numerical methods. For this purpose different methods are applied. In [5, 14] and other references, successive approximation, RBF method, Adomian decomposition, block-pulse functions, Chebyshev and Taylor collocation, polynomials approximation, wavelets, etc are mentioned.

Comparison between computational components has great rule in numerical mathematics. The main focus of the present research is comparison, especially, between sinc approximation and differential transform method for nonlinear Hammerstein integrals by considering calculation remarks such as run time of program, Condition number of obtained system, error norms.

Firstly, Sinc approximations which are powerful means in numerical analysis. Special property of Sinc method in comparison with other methods is order of errors, which is $O(\exp(-k\sqrt{N}))$ and is said as exponentially. For more discussions in this manner see [16, 18, 21].
In the recent years sinc method is combined with other transformation which is known as double exponential transformation and abbreviated as **DE** to give better results[23, 24]. In this manner error bound has order $O(\exp(-cN/\log N))$. This approximation are applied in several problems in numerical analysis [17, 19].

Secondly, Differential Transform method (DTM) as a meshless and semi analytical method is a fast an accurate method to solve different problems in linear and nonlinear initial value problems, differential and integral equations [15, 25].
Although the base of DTM is Taylor series expansion, but new additional works in this method such as reduced differential transform method follows it practically well and more user friendly. For fully discussion see references [2, 3, 6, 13].
This is organised as: Main definitions and important tools for sinc approximation DTM are described in sections 2 and 3. Error bound for two mentioned methods is presented in section 4. Finally, in section 5, two numerical examples and computational remarks are listed in different tables and plotted graphically.

2. Sinc method to nonlinear Hammerstein integral equations

Whittaker cardinal for a function g on real axis can be defined as

$$C(g,h)(x) = \sum_{i=-\infty}^{\infty} g(ih)S(i,h)(x), \tag{2.1}$$

whenever this series is convergent, and

$$S(i,h)(x) = Si(\frac{x-ih}{h}), \quad i = 0, \pm 1, \pm 2, ..., \tag{2.2}$$

Also

$$Si\,(x) = \begin{cases} 1 & x = 0, \\ \frac{\sin(\pi x)}{\pi x} & x \neq 0. \end{cases}$$

Definition 2.1. *We say function g has decay DE property respect to map ψ if for real constant α, β we have*

$$|g(\psi(t))\psi'(t)| \leq \beta \exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty).$$

The following main theorem states sinc indefinite integration based on DE transformation. Here, the order of error is double exponentially.

Theorem 2.2. *[11, 21] The DE formula for indefinite integration is:*

$$\begin{aligned}\int_a^s f(x)dx = h \sum_{j=-N}^{N} f(\psi(jh))\psi'(jh) \left(\frac{1}{2} + \frac{1}{\pi} si(\frac{\psi^{-1}(s)}{h} - j\pi)\right) \\ + O\left(\frac{\log N}{N} \exp(-\frac{\pi d N}{\log(\pi d N/\alpha)})\right),\end{aligned} \tag{2.3}$$

with the following assumptions:

$$\psi(t) = \frac{b-a}{2} \tanh(\frac{\pi}{2} \sinh t) + \frac{a+b}{2}, \tag{2.4}$$

$$\psi'(t) = \frac{b-a}{2} \frac{\pi/2 \cosh(t)}{\cosh^2(\pi/2 \sinh(t))}, \tag{2.5}$$

and

$$h = \frac{1}{N} \log(\pi d N/\alpha). \tag{2.6}$$

3. Sinc approximation to nonlinear Hammerstein integral equations

To apply sinc approximation to Eq. (1), from Theorem 1, integral term can be replaced by the expansion series. It follows:

$$\begin{aligned}\int_a^x K(x,t)F(t,y(t))dt \simeq h \sum_{i=-N}^{N} K(x,\psi(ih))\psi'(ih) \left(\frac{1}{2} + \frac{1}{\pi} si(\frac{\pi\psi^{-1}(x)}{h} - j\pi)\right) F_j, \\ F_j = F(t_j, y(t_j)), j = -N...N.\end{aligned} \tag{3.1}$$

Substituting Eq. (8) in Eq. (1), it concludes that

$$y(x) - h \sum_{i=-N}^{N} K(x,\psi(ih))\psi'(ih) \left(\frac{1}{2} + \frac{1}{\pi} si(\frac{\pi\psi^{-1}(x)}{h} - j\pi)\right) F_j \simeq g(x). \tag{3.2}$$

Sinc collocation points x_k as $x_k = \psi(kh), k = -N...N$ are imported to obtain unknown F_j. These imbedding points lead to a nonlinear system of equation that must be solved with an appropriate method and useful package in software. In the Sinc approximation based on the mentioned algorithm, the size of obtained system is $(2N+1)(2N+1)$ with $2N+1$ variables y_j.

$$\begin{aligned}y(x_k) - h \sum_{j=-N}^{N} K(x_k,\psi(jh))\psi'(jh) \left(\frac{1}{2} + \frac{1}{\pi} si(\pi(k-j))\right) F_j \simeq g(x_k), \\ k, j = -N..N.\end{aligned} \tag{3.3}$$

By using the notations:

$$\begin{aligned}\mathbf{C} &= [hK(x_k,\psi(jh))\psi'(jh)\left(\frac{1}{2}+\frac{1}{\pi}si(\pi(k-j))\right)],\\ \mathbf{Y} &= (y_{-N},...,y_N)^t \quad ,\mathbf{g}=(g(x_{-N}),...,g(x_N))^t,\\ \mathbf{F} &= (F_{-N},...,F_N)^t.\end{aligned} \tag{3.4}$$

Matrix notation (9) is

$$\mathbf{Y}-\mathbf{CF}=\mathbf{g}. \tag{3.5}$$

From obtained approximate solution y_j, the following interpolation form can be used to get new approximate solution in all arbitrary points:

$$y_N(x)=g(x)+h\sum_{i=-N}^{N}K(x,\psi(ih))\psi'(ih)\left(\frac{1}{2}+\frac{1}{\pi}si(\frac{\pi\phi^{-1}(x)}{h}-j\pi))\right)F_j. \tag{3.6}$$

4. Differential transform method to nonlinear Hammerstein integral equations

In this section, brief description of differential transform method is presented. For more discussions see references [1, 4, 10].

Definition 4.1. *Based on derivative of function f from order $k-th$, transformation is defined as*

$$F(k)=\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}, \tag{4.1}$$

equivalently, the differential inverse transform of $f(x)$ can be defined as

$$f(x)=\sum_{k=0}^{\infty}F(k)(x-x_0)^k. \tag{4.2}$$

Taking together Eqs. (14) and (15), it follows

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}(x-x_0)^k. \tag{4.3}$$

The basis operations which are used in the transformation analysis are shown in Table 1.

Main function	Transformed function
$y(x)=\alpha u(x)\pm\beta v(x)$	$Y(k)=\alpha U(k)\pm\beta V(k)$
$y(x)=\frac{d^n v(x)}{dx^n}$	$Y(k)=\frac{(k+n)!}{k!}V(k+n)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{i=0}^{k}U(i)V(k-i)$
$y(x)=x^n$	$Y(k)=\delta(k-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$
$y(x)=\int_{x_0}^{x}u(t)dt$	$Y(k)=\frac{U(k-1)}{k},\quad k\geq 1,\ U(0)=0$
$y(x)=\int_{x_0}^{x}u(t)v(t)dt$	$Y(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}U(k_1)V(k-k_1-1)$
$y(x)=u(x)\int_{x_0}^{x}v(t)dt$	$Y(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}U(k-k_1)V(k_1-1)\quad k\geq 1$

Table 1. Operations of differential transform method.

5. Error bounds

To show accuracy and powerfully in these methods, error analysis of methods are described. By mentioned relation in previous section, two main formula for convergence analysis and error bounds are stated:

Theorem 5.1. *[8] Suppose $y(x)$ and $y_N(x)$ are exact and approximate solution of Eq. (1), respectively, for constants α and C in the strip region $D_d, d > 0$ we have*

$$\sup_{x\in(a,b)} |y(x) - y_N(x)| \leq O\left(\frac{\log N}{N}\exp(-\frac{\pi d N}{\log(\pi d N/\alpha)})\right). \tag{5.1}$$

Theorem 5.2. *Based on Taylor expansion of $y(x)$ suppose $y(x) = \sum_{k=0}^{\infty}\phi_k(x)$ with $\phi_k(x) = Y(k)(x-x_0)^k$ be the series solution of Eq. (1). Also $y_m(x) = \sum_{k=0}^{m}\phi_k(x)$ be the truncated and approximate solution, for constant $0 < \xi < 1$ we have*

$$\sup_{x\in(a,b)} |y(x) - y_m(x)| \leq \frac{1}{1-\xi}\xi^{m+1}\|\phi_0\|. \tag{5.2}$$

Proof

Set $S_n = \phi_0 + \phi_1 + \cdots + \phi_n$. It can be easily showed that sequence $\{S_n\}$ is a Cauchy sequence in the Banach space. Since,

$$\|S_{n+1} - S_n\| = \|\varphi_{n+1}\| \leq \xi\|\phi_n\| \leq \cdots \leq \xi^{n+1}\phi_0. \tag{5.3}$$

For $n, l \in N, n \geq l$, it follows that

$$\begin{aligned}\|S_n - S_l\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{l+1} - S_l)\|\\ &\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + ... + \|(S_{l+1} - S_l)\|\\ &\leq \frac{1}{1-\xi}\xi^{n-l}\|\phi_0\|.\end{aligned}$$

So, for large values of n, l this deviation vanishes.

$$\lim_{n,l\to\infty} \|S_n - S_l\| = 0. \tag{5.4}$$

Also, inequality $1 - \xi^{n-l} < 1$, gives

$$\|S_n - S_l\| \leq \frac{1}{1-\xi}\xi^{l+1}\|\phi_0\|. \tag{5.5}$$

Similarly, for large value on n sequens S_n tends to $y(x)$.

6. Numerical Experiments

To show authority and capability of these methods in numerical computations and possibility of comparison we use two examples. All the examples are programmed with Maple 20 software. Moreover, to have meaningful comparison, beside errors measures some useful remarks such as Condition number and run time are calculated and presented. Moreover the following infinity norm

$$\|.\|_\infty = \max |y(x_j) - y_M(x_j)|, \qquad -N \leq j \leq N. \tag{6.1}$$

other similar and important norms are defined for accuracy in details such as L_2 and the root mean square (RMS):

$$\|.\|_2 = \sqrt{\sum_{j=-N}^{N} [y(x_j) - y_N(x_j)]^{\frac{1}{2}}},$$

$$RMS = \sqrt{\frac{1}{N} \sum_{j=-N}^{N} [y(x_j) - y_N(x_j)]^{\frac{1}{2}}}.$$

Example 1. The first nonlinear Hammerstein integral equation example is selected from [8] with the exact solution $y(x) = \cos(x)$.

$$y(x) = 1 + \sin^2(x) + \int_0^x -3\sin(x-t)y^2(t), \quad 0 \leq x \leq 1. \tag{6.2}$$

Similar to the source of [8], all the parameters in calculation with software are listed below:

$$\begin{aligned} &a = 0, b = 1, d = \frac{\pi}{2}, \alpha = 1, \\ &x_k = \psi(kh), k = -N...N, \quad h = \frac{1}{N}\log(\pi d N/\alpha). \end{aligned} \tag{6.3}$$

As an important factor in comparison, Sinc points are selected as collocation points for two methods to have better results. Outputs from runed program are presented in Table 2.
In Table 2, column N, shows numbers of Sinc basis functions. Also, run time T(second), norms and condition number are showed in different columns in Table 2. Decrease in error measure is seen by addtion in the values of N. Detection of error improving can be obtanid easily from the extracted results. As a good criteria in calculation for Sinc approximation is Condition number column. This property comes from the structure of systems of equations which means sparsity in cofficient matrix.

N	$T(s)$	$\|.\|_\infty$	$\|.\|_2$	RMS	$Cond$
5	34.60	9.3E-004	1.9E-002	5.9E-004	3.8E+000
8	480.80	8.7E-005	1.2E-004	5.5E-005	5.5E+000
11	100.1	9.1E-006	2.7E-005	5.7E-006	7.1E+000

Table 2. Outputs for Example 1 by Sinc approximation.

Difference between the exact solution and the approximate solution for $N = 2$ in the Sinc method is plotted in Figure 1.

Figure 1: Comparison between the exact and approximate solution of Example 1 by sinc method

In DTM solution, by taking differential transform from Eq. (23), based on Table 1, we get

$$
\begin{aligned}
y(x) &= 1+\sin^2(x)+\int_0^x \sin x\cos t y^2(t)dt-\int_0^x \sin t\cos x y^2(t)dt,\\
Y(k) &= \delta(k)+\sum_{i=0}^{k}\frac{1}{i!}\sin\big(\frac{i\pi}{2}\big)\frac{1}{(k-i)!}\sin\big((k-i)\frac{\pi}{2}\big)\\
&\quad -3\sum_{k_3=1}^{k}\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2}\frac{1}{k_3}\cdot\frac{1}{(k-k_3)!}\cdot\frac{1}{(k_1-1)!}\\
&\quad \{\sin\big((k-k_3)\frac{\pi}{2}\big)\cos\big((k_1-1)\frac{\pi}{2}\big)Y(k_2-k_1)Y(k_3-k_2)\\
&\quad -\frac{1}{k_3}\cdot\frac{1}{(k-k_3)!}\cdot\frac{1}{(k_1-1)!}\cos\big((k-k_3)\frac{\pi}{2}\big)\}\\
&\quad \sin\big((k_1-1)\frac{\pi}{2}\big)Y(k_3-k_1)Y(k_3-k_2).
\end{aligned}
$$

Figure 2: Comparison between the exact and approximate solution of Example 1 in DTM method with $N = 5$.

So, for $k = 1, 2, 3, ...$ with initial condition $Y(0) = 1$ it follows

$$\begin{aligned}
Y(1) &= 0, \\
Y(2) &= -\frac{1}{2}, \\
Y(3) &= 0, \\
Y(4) &= \frac{1}{4!}, \\
Y(x) &= \sum_{k=0}^{\infty} Y(k)x^k = Y(0) + Y(1)x + Y(2)x^2 + \cdots \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \cos(x).
\end{aligned}$$

Table 3, presents results of Example 1 by DTM solution. As seen, By decreasing the number of basis functions the errors have been improved. Also run time and created error for different values of N are more remarkable compared to sinc collocation method. Also note that choice $N = 5$ in the Sinc method we have 11 basis functions, while in DTM approximation we must take also 11 basis functions to have meaningful comparison.

N	$T(s)$	$\|.\|_\infty$	$\|.\|_2$
5	0.03	1.00E-10	6.32E-003
8	0.06	1.00E-11	4.47E-003

Table 3. Results for Example 1 by DTM solution.

Figure 2, shows difference between the exact solution and approximate solution for $N = 5$.

As seen in Figure 3, different plots for $N = 2, 3, 4$ and exact solution are plotted. Accuracy is

Figure 3: Comparison between the exact and the approximate solution of Example 1 in DTM method with $N = 2, 3, 4$.

decreased as N increased.

Example 2. Consider the following nonlinear Hammerstein integral equation

$$y(t) = -\frac{15}{56}t^8 + \frac{13}{14}t^7 - \frac{11}{10}t^6 + \frac{9}{20}t^5 + t^2 - t + \int_0^t (t+s)y^3(s)ds, \tag{6.4}$$

with the exact solution $y(t) = t^2 - t$.

Results of Sinc approximation with collocation points x_k for Example 2 are showed in Table 4. Similar to the results of previous example large values in N give accurate and better advantages in outputs.

N	$T(s)$	$\|.\|_\infty$	$\|.\|_2$	RMS	$Cond$
5	22.5	7.4E-004	2.5E-002	4.9E-003	2.2E+000
8	280.5	7.7E-006	1.3E-003	2.5E-006	4.8E+000
11	90.13	3.1E-006	5.4E-006	5.4E-006	1.1E+000

Table 4. Output results of Example 2 by Sinc approximation.

By taking differential transform based on Table 1, from Eq. (25) we get

$$\begin{aligned} Y(k) = & -\frac{15}{56}\delta(k-8) + \frac{13}{14}\delta(k-7) - \frac{11}{10}\delta(k-6) + \frac{9}{20}\delta(k-5)\delta(k-2) - \delta(k-1) \\ & + \sum_{k_3=1}^{k}\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} \frac{1}{k_3} Y(k_1-1)Y(k_2-k_1)Y(k_3-k_2)\delta(k-k_3-1) \\ & + \frac{1}{k}\sum_{k_3=0}^{k-1}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} \delta(k_1-1)Y(k_2-k_1)Y(k_3-k_2)Y(k-k_3-1). \end{aligned}$$

Figure 4: Behavior of error plot from Example 2 by Sinc method.

So, for $k = 1, 2, 3, ...$ with initial condition $Y(0) = 0$, it follows

$$\begin{aligned}
&Y(0) = 0,\\
&Y(1) = -1,\\
&Y(2) = 1,\\
&Y(k) = 0 \quad k \geq 3,\\
&y(x) = \sum_{k=0}^{\infty} Y(k)t^k = Y_0 + Y_1 t + Y_2 t^2 + ... = -t + t^2.
\end{aligned}$$

Because of polynomial expansion in DTM approximation, here the exact solution is attained.

N	$T(s)$	$\|.\|_\infty$
1	0.01	0.0029
2	0.01	0.0000
3	0.01	0.0000

Table 5. Results for Example 2 by DTM approximation.

Numerical results which are presented in Ttable 5, show that for $N = 2, 3, ...$ error is vanished. Run time for program also is very small in comparison with Sinc method. However, results show that two methods are practically well, but DTM approximation gives better accuracy than the Sinc collocation at the expense of more computational effort.

7. Conclusion

Comparing two methods in numerical analysis in various areas is more important and so applicable. Memory of computer, errors and time of programs are main factors in comparison. Here, we use Sinc method with DE transformation and DTM solution to nonlinear Hammerstein integral equations. Based on works [7, 9, 12, 14], comparing Sinc collocation method to DTM approximation have the following advantages:

1-The numerical methods demonstrate the good accuracy of two schemes but DTM outputs in a same manner have better results.

2-By increasing the value of M_t, Sinc points, RMS and $\|.\|$ columns are closely similar but run time and condition number columns are different.

3-Determining the parameter α in Sinc collocation method is still computationally intensive. Finding optima value will improve accuracy.

4-In DTM approximation for 4 nested loops, order of complexity is $O(n^4)$ while in sinc method is $O(n^2)$. This characteristic makes most importance in computational cost.
5- In some problems, DTM solution gives exact solution in closed form such Example 2.
6-Based on results, although the implementation and coding are very easy in sinc method, but DTM approximation gives higher accuracy at the calculation and error of approximation.
7-For non separable kernel in Eq. (1) calculations will be more complicated.

References

[1] I.H. Abdel-Halim Hassan, *Differential transformation technique for solving higher-order initial value problems*, Appl. Math. Comput. 154 (2004) 299–311.

[2] A. Arikoglu and I. Ozkol, Solution of boundary value problems for integro-differential equations by using differential transform method Appl. Math. Comput. 168 (2005) 1145–1158.

[3] N. Bildik, A. Konuralp, F. Bek and *S. Kucukarslan, Solution of different type of the partial differential equation by differential transform method and Adomian's decomposition method*, Appl. Math. Comput. 127 (2006) 551–567.

[4] C.K. Chen and S.H. Ho, *Application of differential transformation to eigenvalue problems*, Appl. Math. Comput. 79 (1996) 173–188.

[5] H.T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, 1962.

[6] F. Ayaz, *Application of differential transform method to differential-algebraic equations*, Appl. Math. Comput. 152 (2004) 649–657.

[7] M.A. Fariborzi Araghi and Gh. Kazemi Gelian, *Numerical solution of integro differential equations based on double exponential transformation in the sinc-collocation method*, App. Math. and Comp. Intel. 1 (2012) 48–55.

[8] M.A. Fariborzi Araghi and Gh. Kazemi Gelian, *Numerical solution of nonlinear Hammerstien integral equations via Sinc collocation method based on double exponential transformation*, Math. Sci. 30 (2013).

[9] M.A. Fariborzi Araghi and Gh. Kazemi Gelian, *Solving fuzzy Fredholm linear integral equations using Sinc method and double exponential transformation*, Soft Comput. 19(4) (2015) 1063–1070.

[10] R. Ghoochani-Shirvan, J. Saberi-Nadjafi and M. Gachpazan, *An analytical and approximate solution for non-linear Volterra partial integro-differential equations with a weakly singular kernel using the fractional differential transform method*, Int. J. Diff. Equ. 2018 (2018).

[11] S. Haber, *Two formulas for numerical indefinite integration*, Math. Comp. 60 (1993) 279–296.

[12] M. Hadizadeh and Gh. Kazemi Gelian,*Error estimate in the Sinc collocation method for Volterra-Feredholm integral equations based on DE transformations*, ETNA 30 (2008) 75–87.

[13] I.H. Hassan, *Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems* Chaos Solitons Fract. 36(1) (2008) 53–65.

[14] Gh. Kazemi Gelian and M. A. Fariborzi Araghi, *Numerical solution of the Burgers' equation based on Sinc method*, Theory Approx. Appl. 13(1) (2019) 27–42.

[15] H. Liu and Y. Song, *Differential transform method applied to high index differential-algebraic equations*, Appl. Math. Comput. 184 (2) (2007) 748–753.

[16] J. Lund and k. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[17] M. Mori and M. Sugihara, *The double exponential transformation in numerical analysis*, J. Comput. Appl. Math. 127 (2001) 287–296.

[18] M. Muhammad, A. Nurmuhammad, M. Mori and M. Sugihara, *Numerical solution of integral equations by means of the Sinc colocation based on the double exponential transformation,* J. Comput. Appl. Math. 177(2) (2005) 269–286.

[19] M. Muhammad and M. Mori, *Double exponetial formulas for numerical indefinite integration*, J. Comput. Appl. Math. 161 (2003) 431–448.

[20] A. Nurmuhammad, M. Muhammad and M. Mori, *Double exponential transformation in the Sinc collocation method for a boundry value problem*, J. Comput. Appl. Math. Appl. 38 (1999) 1–8.

[21] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer, 1993.

[22] M. Sugihara, *Optimality of the double exponential formula - functional analysis approach*, Numer. Math. 75 (1997) 379–395.

[23] M. Sugihara and T. Matsuo, *Recent development of the Sinc numerical methods*, J. Comput. Appl. Math. 164 (2004) 673–689.

[24] H. Takahasi and M. Mori, *Double exponetial formulas for numerical integration*, Publ. Res. Inst. Math. Sci. 9 (1974) 721–741.

[25] J. K. Zhou, *Differential Transformation and its Application for Electrical Circuits*, Huarjung University Press, Wuhan, China, 1986.