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$H(.,.,.,.)$-φ-η-cocoercive operator with an application to variational inclusions

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Abstract

This paper deals with the generalized $H(.,.,.,.)$-φ-η-cocoercive operator and use its application via resolvent equation approach to solve the variational-like inclusion involving infinite family of set-valued mappings in semi-inner product spaces. Applying the generalized resolvent operator technique involving generalized $H(.,.,.,.)$-φ-η-cocoercive operator, an equivalence between the set-valued variational-like inclusion problem and fixed point problem is established. A relationship between the set-valued variational-like inclusion problem and resolvent equation is also established. Using this equivalent formulation an iterative algorithm is developed that approximate the unique solution of the resolvent equation.

Keywords: Variational-like inclusions, cocorecive operator, Semi-inner product spaces, Resolvent operator, Lipschitz continuity
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1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. Some of the most interesting and important problems in the theory of variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. Variational Inequality theory is very important due to its large application in various problem for example, partial differential equations and optimization problems, see [3]. Monotonicity play a very important role in the study of variational inclusions. In recent past, monotone mappings

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have a large number of applications, especially in differential equations, integral equations, mathematical economics, optimal control, and so forth. There are several kinds of generalized monotonicity such as pseudomonotone, quasimonotone, paramonotone, maximal monotone mapping, relaxed monotone mapping, H-monotone mapping, A-monotone mapping etc., see for example, [4, 5, 6, 9, 10, 14, 17, 18, 29, 31, 33, 35, 36, 38, 39]. The cocoercive mappings were studied by Tseng [37], Magnanti and Perakis [32], and Zhu and Marcotte [43] which are also the generalized forms of monotone mappings.

The resolvent operator techniques are very crucial to study the existence of solutions and to create iterative schemes for various kinds of variational inequalities and their generalizations, which gives mathematical models to certain problems appearing in optimization and control theory, engineering sciences and economics. In order to study several equilibrium problem, variational inequalities and variational inclusions, Farajzadeh et al. [13] introduced various kinds of vector equilibrium problems which are combinations of a vector equilibrium problem and generalized vector variational inequality problem, Farajzadeh et al. [12] established an iterative process for a hybrid pair of a finite family of generalized I-asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings, Fang and Huang, Kazmi and Khan, and Lan et al. investigated various generalized operators such as H-monotone [9], H-accretive [11], (P,η)-proximal point [23], (P,η)-accretive [22], (H,η)-monotone [11], and (A,η)-accretive mappings [28]. Zou and Huang [44] introduced and studied $H(.,.)$-accretive operators, Kazmi et al. [24, 25, 26] introduced and studied generalized $H(.,.)$-accretive operators and $H(.,.)$-η-proximal point mapping, Xu and Wang [41] introduced and studied $(H(.,.),\eta)$-monotone operators, Ahmad et al. [1] introduced and studied $H(.,.)$-cocoercive operators and Husain and Gupta [19, 20] introduced and studied $H(.,.)$-mixed operator and generalized $H(.,.,.)$-η-cocoercive operators.

Recently, Sahu et al. [35] showed the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [36]. Luo and Huang [31], introduced and studied (H,φ)-η-monotone mapping in Banach spaces which gives a unifying framework for certain classes of monotone mapping. Most recently, Bhat and Zahoor [5, 6], introduced and studied (H,ϕ)-η-monotone mapping in semi-inner product space and examined the convergence analysis of proposed iterative schemes for various classes of variational inclusion via generalized resolvent operator. For the applications view point of discussed operators in variational inequalities and variational inclusion, see [9, 10, 19, 20, 21, 25, 29, 31, 33, 35, 36, 38, 39, 41, 44]. Khan et al. [27] by making the use of resolvent operator technique of maximal monotone mapping and the property of fixed-point set of set-valued contractive mapping study the behavior and sensitivity analysis of a solution set for a parametric generalized mixed multi-valued implicit quasi-variational inclusion problem involving a map $N : H \times H \times H \times \Omega \to H$ in Hilbert space H where $\Omega \subset H$ is nonempty and open, which generalize the results presented by Ram [34] for parametric generalized nonlinear quasi-variational inclusion problem involving a map $N : H \times H \times \Omega \to H$.

Very recently Gupta and Singh [16] examine the notion generalized $H(.,.,.)$-φ-η-cocoercive operator which is the generalization of of $H(.,.,.)$-η-cocoercive operator [20, 21] and use its application via the resolvent equation approach to find the solution of variational-like inclusion involving involving infinite family of set valued mappings in semi-inner product spaces. Furthermore, they constructed an iterative algorithm by using the equivalent formulation of the set-valued variational-like inclusion problem and the resolvent equations to solve the resolvent equation.

Inspired and motivated by the above research work going on in this direction and [27, 34], we consider the notion of generalized $H(.,.,.,.)$-φ-η-cocoercive operator a natural generalization of $H(.,.,.)$-φ-η-cocoercive operator [16]. Next, we consider the variational inclusion problem involving infinite family of set-valued mappings. We find an equivalence between the variational-like inclusion and the resolvent operator equation containing generalized $H(.,.,.,.)$-φ-η-cocoercive operator and also find a relation between variational-like inclusion and fixed point problem. These equivalent fixed resolvent equation formulation and the fixed point problem give us a sketch to obtain an iterative algorithm. As an application, we attempt to solve the variational-like inclusions involving infinite family of set-valued mappings in 2-uniformly smooth Banach space. The results presented in this paper can be viewed as an extension and generalization of the existing results in the literature, see [1, 2, 5, 6, 16, 19, 25, 31, 44].

First of all we recall the following definitions and important concepts which are needed throughout the paper.

Definition 1.1. *(see [30, 35]) Let us consider the vector space X over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following conditions:*

(i) $[x_1 + x_2, y_1] = [x_1, y_1] + [x_2, y_1], \ \forall\ x_1, x_2, y_1 \in X$

(ii) $[\alpha x_1, y_1] = \alpha[x_1, y_1], \ \forall \alpha \in F, \ x_1, y_1 \in X$

(iii) $[x_1, x_1] \geq 0, \ \textit{for } x_1 \neq 0$

(iv) $|[x_1, y_1]|^2 \leq [x_1, x_1][y_1, y_1], \ \forall\ x_1, y_1 \in X$

The pair $(X, [.,.])$ is called a semi-inner product space.

We note that $\|x_1\| = [x_1, x_1]^{\frac{1}{2}}$ is a norm on X and we can say every semi-inner product space is a normed linear space. On the other hand, every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [15] had proven that if the underlying space X is a uniformly convex smooth Banach space then it is feasible to define a semi-inner product uniquely.

Remark 1.2. *(see [5]) This unique semi-inner product has the following nice properties:*

(i) $[x_1, y_1] = 0$ if and only if y_1 is orthogonal to x_1, that is if and only if $\|y_1\| \leq \|y_1 + \alpha x_1\|$, for all scalars α.

(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y_1 \in X$ such that $f(x_1) = [x_1, y_1]$, for all $x_1 \in X$.

(iii) The semi-inner product is continuous, that is for each $x_1, y_1 \in X$, we have $Re[y_1, x_1 + \alpha y_1] \to Re[y_1, x_1]$ as $\alpha \to 0$.

The sequence space $l^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces. Therefore one can define semi-inner product on these spaces, uniquely.

Example 1.3. *(see [35]) The real sequence space l^p, for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_j x_i y_i |y|^{p-2}, \ x, y \in l^p.$$

Example 1.4. *(see [15, 35]) The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x))d\mu, \;\; f, g \in L^p.$$

Definition 1.5. *(see [35, 40]) Let X be a real Banach space. The modulus of smoothness ρ_X : $[0, +\infty) \to [0, +\infty)$ of X is defined as*

$$\rho_X(t) = \sup \left\{ \frac{\|x_1 + y_1\| + \|x_1 - y_1\|}{2} - 1 : \|x_1\| = 1, \|y_1\| = t, \;\; t > 0 \right\}.$$

X is said to be uniformly smooth if $\lim\limits_{t \to 0} \dfrac{\rho_X(t)}{t} = 0$.
X is said to b p-uniformly smooth for $p > 1$, if there exists a real constant $c > 0$ such that $\rho_X(t) \leq ct^p$.
X is said to be 2-uniformly smooth if there exists a real constant $c > 0$ such that $\rho_X(t) \leq ct^2$.

Lemma 1.6. *(see [35, 40]) Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is 2-uniformly smooth.

(ii) There is a constant $c > 0$, such that for every $x_1, y_1 \in X$, the following inequality holds

$$\|x_1 + y_1\|^2 \leq \|x_1\|^2 + 2\langle y_1, f_{x_1} \rangle + c\|y_1\|^2, \tag{1.1}$$

where $f_{x_1} \in J(x_1)$ and
$J(x_1) = \left\{x_1{}^* \in X^* : \langle x_1, x_1{}^* \rangle = \|x_1\|^2 \text{ and } \|x_1{}^*\| = \|x_1\|\right\}$ is the normalized duality mapping, where X^* denotes the dual space of X and $\langle x_1, x_1{}^* \rangle$ denotes the value of the functional $x_1{}^*$ at x_1, that is, $x_1{}^*(x_1)$.

Remark 1.7. *(see [35]) Every normed linear space is a semi-inner product space (see Lumer [30]). In fact by Hahn Banach theorem, for each $x_1 \in X$, there exists at least one functional $f_{x_1} \in X^*$ such that $\langle x_1, f_{x_1} \rangle = \|x_1\|^2$. Given any such mapping f from X into X^*, we can verify that $[y_1, x_1] = \langle y_1, f_{x_1} \rangle$ defines a semi-inner product. Hence we can write the inequality (1.1) as*

$$\|x_1 + y_1\|^2 \leq \|x_1\|^2 + 2[y_1, x_1] + c\|y_1\|^2, \;\; \forall \; x_1, y_1 \in X. \tag{1.2}$$

The constant c is chosen with best possible minimum value. We call c, as the constant of smoothness of X.

Example 1.8. *(see [35]) The functions space L^p is 2-uniformly smooth for $p \geq 2$, and is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x_1, y_1 \in L^p$,*

$$\|x_1 + y_1\|^2 \leq \|x_1\|^2 + 2[y_1, x_1] + (p-1)\|y_1\|^2,$$

where $(p - 1)$ is the constant of smoothness.

The rest part of paper is organized as follows:

In section 2, we present some definitions which will be used later. In section 3, we present some definitions and assumptions to prove some result. As an application, we prove Lemmas and developed algorithm to prove strongly convergence and uniqueness of the solutions of the resolvent equation corresponding to set-valued variational-like inclusion problem.

2. Preliminaries

Let X be a 2-uniformly smooth Banach space. Its norm and topological dual space is given by $\|.\|$ and X^*, respectively. The semi-inner product $[.,.]$ signify the dual pair among Y and Y^*.

Definition 2.1. *(see [31, 35]) Let X be real a real 2-uniformly smooth Banach space, $\eta : X \times X \to X$ and $P : X \to X$ be single valued mappings. Then P is said to be:*

(i) P is (r, η)-strongly monotone if there exists a constant $r > 0$ such that

$$[P(x) - p(y), \eta(x, y)] \geq r\|x - y\|^2, \ \forall \ x, y \in X,$$

(ii) P is (s, η)-cocoercive if there exists a constant $s > 0$ such that

$$[P(x) - p(y), \eta(x, y)] \geq s\|P(x) - P(y)\|^2, \ \forall \ x, y \in X,$$

(iii) P is (s', η)-relaxed cocoercive if there exists a constant $s' > 0$ such that

$$[P(x) - p(y), \eta(x, y)] \geq -s'\|P(x) - P(y)\|^2, \ \forall \ x, y \in X,$$

(iv) P is α-expansive if there exists constant $\alpha > 0$ such that

$$\|P(x) - P(y)\| \geq \alpha \|x - y\|, \ \forall x, y \in X,$$

(v) η is said to be τ-Lipschitz continuous if there exists constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \ \forall x, y \in X.$$

Definition 2.2. *(see [20, 21]) Let $P, Q, R, S : X \to X$, $\eta : X \times X \to X$, $H : X \times X \times X \times X \to X$ are single-valued mappings, then*

(i) $H(P, ., ., .)$ is (μ_1, η)-cocoercive with respect to P if there exists constant $\mu_1 > 0$ such that

$$[H(Px_1, u, u, u) - H(Px_2, u, u, u), \eta(x_1, x_2)] \geq \mu_1\|Px_1 - Px_2\|^2, \ \forall \ u, x_1, x_2 \in X,$$

(ii) $H(., Q, ., .)$ is (μ_2, η)-cocoercive with respect to Q if there exists constant $\mu_2 > 0$ such that

$$[H(u, Qx_1, u, u) - H(u, Qx_2, u, u), \eta(x_1, x_2)] \geq \mu_2\|Qx_1 - Qx_2\|^2, \ \forall \ u, x_1, x_2 \in X,$$

(iii) $H(., ., R, .)$ is (γ, η)-relaxed cocoercive with respect to R if there exists constant $\gamma > 0$ such that

$$[H(u, u, Rx_1, u) - H(u, u, Rx_2, u), \eta(x_1, x_2)] \geq -\gamma\|Rx_1 - Rx_2\|^2, \ \forall \ u, x_1, x_2 \in X,$$

(iv) $H(., ., ., S)$ is (δ, η)-strongly monotone with respect to S if there exists constant $\delta > 0$ such that

$$[H(u, u, u, Sx_1) - H(u, u, u, Sx_2), \eta(x_1, x_2)] \geq \delta \|x_1 - x_2\|, \ \forall \ u, x_1, x_2 \in X,$$

(v) $H(P, ., ., .)$ is κ_1-Lipschitz continuous with respect to P if there exists constant $\kappa_1 > 0$ such that

$$\|H(Px_1, u, u, u) - H(Px_2, u, u, u)\| \leq \kappa_1 \|x_1 - x_2\|, \ \forall \ u, x_1, x_2 \in X.$$

Similarly we can define the Lipschitz continuity for $H(.,.,.,.)$ with respect to other components.

Let $M : X \to 2^X$ be a set-valued mapping, the graph of M is given by $\mathcal{G}(M) = \{(x, y) : y \in M(x)\}$. The domain of M is given by

$$D(M) = \{x \in X : \exists\ y \in X : (x, y) \in \mathcal{G}(M)\}.$$

The Range of M is given by

$$R(M) = \{y \in X : \exists\ x \in X : (x, y) \in \mathcal{G}(M)\}.$$

The inverse of M is given by

$$M^{-1} = \{(y, x) : (x, y) \in \mathcal{G}(M)\}.$$

For any two set-valued mappings N and M, and any real number ρ, we define

$$N + M = \{(x, y_1 + y_2) : (x, y_1) \in \mathcal{G}(N),\ (x, y_2) \in \mathcal{G}(M)\},$$

$$\rho M = \{(x, \rho y) : (x, y) \in \mathcal{G}(M)\}.$$

For any mapping A and a set-valued mapping $M : X \to 2^X$, we define
$A + M = \{(x, y_1 + y_2) : Ax = y_1,\ (x, y_2) \in \mathcal{G}(M)\}$.

Definition 2.3. *(see [31, 35]) A set valued mapping $M : X \to 2^X$ is said to be (m, η)-relaxed monotone if there exists a constant $m > 0$ such that*

$$[x^* - y^*, \eta(x, y)] \geq -m\|x - y\|^2,\ \forall\ x, y \in X,\ x^* \in M(x),\ y^* \in M(y).$$

Definition 2.4. *Let $G : X^\infty = X \times X \times X \times X... \to X$ be a mapping. Then G is α_i-Lipschitz continuous with respect to i^{th} component if there exists a constant $\alpha_i > 0$ such that*

$$\|G(.,.,., x_i, ...) - G(.,.,., y_i, ...)\| \leq \alpha_i \|x_i - y_i\|,\ \forall\ x_i, y_i \in X.$$

Definition 2.5. *The Hausdorff metric $\mathcal{H}(.,.)$ on $CB(X)$, is defined by*

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\right\},\ A, B \in CB(X),$$

where $d(.,.)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 2.6. *(see [16]) A set-valued mapping $S : X \to CB(X)$ is called $\mathcal{H}$-Lipschitz continuous with constant $\lambda_S > 0$, if*

$$\mathcal{H}(Sx, Sy) \leq \lambda_S \|x - y\|,\ \forall\ x, y \in X$$

3. Generalized $H(.,.,.,.)$-φ-η-cocoercive operator

In this section, we present some definitions, assumptions to prove the main results related with generalized $H(.,.,.,.)$-φ-η-cocoercive operator.

Let X be 2-uniformly smooth Banach space. Suppose that $\eta : X\times X \to X,\ H : X\times X\times X\times X \to X$ and $\varphi, P, Q, R, S : X \to X$ be the single-valued mappings and $M : X \to 2^X$ be a set-valued mapping.

Definition 3.1. *Let $H(.,.,.,.)$ is (μ_1,η)-cocoercive with respect to P with non-negative constant μ_1, (μ_2,η)-cocoercive with respect to Q with non-negative constant μ_2, (γ,η)-relaxed cocoercive with respect to R with non-negative constant γ and (δ,η)-strongly monotone with respect to S with non-negative constant δ, then M is called generalized $H(.,.,.,.)$-φ-η-cocoercive with respect to P, Q, R and S if*

(i) $\varphi\circ M$ *is (m,η)-relaxed monotone,*

(ii) $(H(.,.,.,.)+\lambda\varphi\circ M(X)=X,\ \lambda>0.$

Now we need the following assumptions:

A1: Let $H(.,.,.,.)$ is (μ_1,η)-cocoercive with respect to P with non-negative constant μ_1, (μ_2,η)-cocoercive with respect to Q with non-negative constant μ_2, (γ,η)-relaxed cocoercive with respect to R with non-negative constant γ and (δ,η)-strongly monotone with respect to S with non-negative constant δ with $\mu_1,\mu_2>\gamma$.
A2: Let P is α_1-expansive, Q is α_2-expansive and R is β-Lipschitz continuous with $\alpha_1,\alpha_2>\beta$.
A3: Let η is τ-Lipschitz continuous
A4: Let M is generalized $H(.,.,.,.)$-φ-η-cocoercive operator with respect to P, Q, R and S.

Theorem 3.2. *Suppose assumptions A1, A2 and A3 hold good with $\ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta>m\lambda$, then $(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}$ is single valued.*

Proof . Let $v,w\in(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}(u)$ for any given $u\in X$. It is obvious that

$$\begin{cases} -H(Pv,Qv,Rv,Sv)+u\in\lambda\varphi\circ M(v)\\ -H(Pw,Qw,Rw,Sw)+u\in\lambda\varphi\circ M(w)\end{cases}$$

Since $\varphi\circ M$ is (m,η)-relaxed monotone in the first argument, we have

$$\begin{aligned}
-m\lambda\|v-w\|^2 &\leq [-H(Pv,Qv,Rv,Sv)+u-(-H(Pw,Qw,Rw,Sw)+u),\eta(v,w)]\\
&= [-H(Pv,Qv,Rv,Sv)+H(Pw,Qw,Rw,Sw),\eta(v,w)]\\
&= -[H(Pv,Qv,Rv,Sv)-H(Pw,Qv,Rv,Sv),\eta(v,w)]\\
&\quad -[H(Pw,Qv,Rv,Sv)-H(Pw,Qw,Rv,Sv),\eta(v,w)]\\
&\quad -[H(Pw,Qw,Rv,Sv)-H(Pw,Qw,Rw,Sv),\eta(v,w)]\\
&\quad -[H(Pw,Qw,Rw,Sv)-H(Pw,Qw,Rw,Sw),\eta(v,w)].
\end{aligned}$$

Since assumption A1 holds, we have

$$-m\lambda\|v-w\|^2\leq-\mu_1\|Pv-Pw\|^2-\mu_2\|Qv-Qw\|^2+\gamma\|Rv-Rw\|^2-\delta\|v-w\|^2.$$

Since assumption A2 holds, we have

$$\begin{aligned} -m\lambda\|v-w\|^2 &\le -\mu_1{\alpha_1}^2\|v-w\|^2-\mu_2{\alpha_2}^2\|v-w\|^2+\gamma\beta^2\|v-w\|^2-\delta\|v-w\|^2 \\ &= -(\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta)\|v-w\|^2 \\ 0 &\le -(\ell-m\lambda)\|v-w\|^2\le 0,\ \text{where } \ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta. \end{aligned}$$

Since $\mu_1,\mu_2>\gamma,\ \alpha_1,\alpha_2>\beta,\ \delta>0$, it follows that $\|v-w\|\le 0$ and hence $v=w$. Therefore $(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}$ is single-valued. □

Definition 3.3. *Let assumptions A1, A2 and A4 hold good with* $\ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta>m\lambda$, *then the resolvent operator* $R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}:X\to X$ *is given as*

$$R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(x)=(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}(x),\ \forall\ x\in X. \tag{3.1}$$

Theorem 3.4. *Suppose assumptions A1-A4 hold good with* $\ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta.>m\lambda$, *and* η *is* τ*-Lipschitz then* $R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}:X\to X$ *is* $\frac{\tau}{\ell-m\lambda}$*-Lipschitz continuous, that is,*

$$\left\|R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(v)-R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(w)\right\|\le\frac{\tau}{\ell-m\lambda}\|v-w\|,\ \forall\ v,w\in X.$$

Proof . Suppose $v,w\in X$ be any given points, then from (3.1), we have

$$R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(v)=(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}(v),$$

$$R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(w)=(H(P,Q,R,S)+\lambda\varphi\circ M)^{-1}(w).$$

Let $x_0=R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(v)$ and $x_1=R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(w)$.

$$\begin{cases} \lambda^{-1}\left(v-H\left(P(x_0),Q(x_0),R(x_0),S(x_0)\right)\right)\in\varphi\circ M(x_0) \\ \lambda^{-1}\left(w-H\left(P(x_1),Q(x_1),R(x_1),S(x_1)\right)\right)\in\varphi\circ M(x_1). \end{cases}$$

Since $\varphi\circ M$ is (m,η)-relaxed monotone in the first arguments, we have
$[(v-H\left(P(x_0),Q(x_0),R(x_0),S(x_0)\right))-(w-H\left(P(x_1),Q(x_1),R(x_1),S(x_1)\right)),$
$\eta(x_0,x_1)]\ge -m\lambda\|x_0-x_1\|^2$,
which implies

$$\begin{aligned} [v-w,\eta(x_0,x_1)] \ \ge\ & [H\left(P(x_0),Q(x_0),R(x_0),S(x_0)\right)- \\ & H\left(P(x_1),Q(x_1),R(x_1),S(x_1)\right),\eta(x_0,x_1)]-m\lambda\|x_0-x_1\|^2. \end{aligned}$$

Now, we have

$$\begin{aligned} &\|v-w\|\,\|\eta(x_0,x_1)\|\ge[v-w,\eta(x_0,x_1)] \\ &\ge[H\left(P(x_0),Q(x_0),R(x_0),S(x_0)\right)-H\left(P(x_1),Q(x_1),R(x_1),S(x_1)\right),\eta(x_0,x_1)] \\ &-m\lambda\|x_0-x_1\|^2. \end{aligned}$$

Since assumption A1, A2, A3 hold and η is τ-Lipschitz continuous, we have

$$\|v-w\|\,\tau\,\|x_0-x_1\|\ge(\ell-m\lambda)\|x_0-x_1\|^2$$

$$\text{or}\ \left\|R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(v)-R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(w)\right\|\le\frac{\tau}{\ell-m\lambda}\|v-w\|,\ \forall\ v,w\in X.$$

Hence, we get the required result. □

4. Application

Now we shall show that generalized $H(.,.,.,.)$-φ-η-cocoercive operator can be used as an effective tool to solve variational inclusion problems under suitable assumptions.

Let X be 2-uniformly smooth Banach space. Let $V, T_i : X \to CB(X),\ i = 1,2,3,...$ be the infinite family of set-valued mappings and $P, Q, R, S, h, k, \varphi : X \to X$ be the single-valued mappings. Let $\eta : X \times X \to X, H : X \times X \times X \times X \to X$ and $G : X^\infty = X \times X \times X... \to X$ be the mappings. Suppose that set-valued mapping $M : X \to 2^X$ be a generalized $H(.,.,.,.)$-φ-η-cocoercive operator with respect to P, Q, R and S. We consider the following variational like inclusion problem involving infinite family of set-valued mappings to find $y \in X,\ a \in V(y)$ and $y_i \in T_i(y),\ i = 1,2,...$ such that

$$0 \in G(y_1, y_2, y_3, ...) + k(a) + M(h(y) - k(y)). \tag{4.1}$$

Variational inclusion problem of type (4.1), studied by Ahmed and Dilshad [1] and Wang [42] in the setting of real Banach space.

Lemma 4.1. *Let us consider the mapping $\varphi : X \to X$ such that $\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$ and $Ker(\varphi) = \{0\}$ where $Ker(\varphi) = \{y_1 \in X : \varphi(y_1) = 0\}$. If $(y, a, (y_1, y_2, ...))$, where $y \in X,\ a \in V(y)$ and $y_i \in T_i(y),\ i = 1,2,3,...$ is a solution of problem (4.1) if and only if $(y, a, (y_1, y_2, ...))$ satisfies the following relation:*

$$\begin{aligned} h(y) \;&=\; k(a) + R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}[H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ &\quad S(h(y) - k(a))) - \lambda\{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}]. \end{aligned} \tag{4.2}$$

The resolvent equation corresponding to set-valued variational-like inclusion problem (4.1) is

$$\varphi \circ G(y_1, y_2, y_3, ...) + k(a) + \lambda^{-1} J^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q) = 0, \tag{4.3}$$

where $\lambda > 0$.

$$\begin{aligned} J^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q) \;&=\; [I - H(P(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), Q(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), \\ &\quad R(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), S(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)))], \end{aligned}$$

I is the identity mapping and

$$\begin{aligned} H(P, Q, R, S)\left[R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)\right] \;&=\; H(P(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), Q(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), \\ &\quad R(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), S(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q))). \end{aligned}$$

Next we show the solution of variational-like inclusion problem (4.1) is equivalent to the resolvent equation (4.3) in the following lemma.

Lemma 4.2. *If $(y, a, (y_1, y_2, y_3, ...))$ with $y \in X,\ a \in V(y)$ and $y_i \in T_i(y),\ i = 1,2,3,...$ is a solution of problem (4.1) if and only if the resolvent equation (4.3) has a solution $(q, y, a, (y_1, y_2, y_3, ...))$ with $y, q \in X,\ a \in V(y)$ and $y_i \in T_i(y,\ i = 1,2,3,...,)$ where*

$$h(y) - k(a) = R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q), \tag{4.4}$$

and $q = H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) - \lambda\{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}.$

Proof . Let $(y, a, (y_1, y_2, y_3, ...))$ be a solution of problem (4.1), and using Lemma 4.1 we have

$$\begin{aligned} J_{M,\lambda,\varphi}^{H(.,.,.,.)-\eta} &= [I - H(P(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), Q(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), R(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), \\ &\quad S(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}))], \end{aligned}$$

$$\begin{aligned} & J_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}(q) = J_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}[H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ & S(h(y) - k(a))) - \lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ = \; & [I - H(P(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), Q(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), R(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), \\ & S(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}))] \\ & [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ = \; & [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ & -H(P(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), Q(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), R(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), S(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta})) \\ & [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ = \; & [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ & -H(P(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta})(H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ & S(h(y) - k(a))) - \lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}), \\ & Q(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta})(H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ & S(h(y) - k(a))) - \lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}), \\ & R(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta})(H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ & S(h(y) - k(a))) - \lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}), \\ & S(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta})(H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ & S(h(y) - k(a))) - \lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\})) \\ = \; & [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}] \\ & -H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) \\ = \; & -\lambda \{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}. \end{aligned}$$

This implies that

$$\varphi \circ G(y_1, y_2, y_3, ...) + k(a) + \lambda^{-1} J_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}(q) = 0. \tag{4.5}$$

Conversely, let $(q, y, a, (y_1, y_2, y_3, ...))$ is a solution of resolvent equation (4.3), then $J_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}(q) = -\lambda[\varphi \circ G(y_1, y_2, y_3, ...) + k(a)]$

$$[H(P(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), Q(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), R(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}), S(R_{M,\lambda,\varphi}^{H(.,.,.,.,.)-\eta}))](q) = -\lambda[\varphi \circ G(y_1, y_2, y_3, ...) + k(a)]$$

$$q - H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) = -\lambda[\varphi \circ G(y_1, y_2, y_3, ...) + k(a)].$$

It follows that

$$q = H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), S(h(y) - k(a))) - \lambda[\varphi \circ G(y_1, y_2, y_3, ...) + k(a)].$$

Hence $(y, a, (y_1, y_2, y_3, ...))$ is a solution of variational inclusion problem (4.1). □

Lemma 4.1 and Lemma 4.2 are very important from the numerical point of view. They permit us to suggest the following iterative scheme for finding the approximate solution of (4.3).

Algorithm 1 For any given $(q_0, y_0, a_0, (y_1^0, y_2^0, y_3^0, ...))$, we can choose $q_0,\ y_0 \in X,\ a_0 \in V(y_0)$ and $y_i^0 \in T_i(y_0),\ i = 1, 2, 3, ...$ and $0 < \epsilon < 1$ such that sequences $\{q_n\},\ \{y_n\},\ \{a_n\}$ and $\{y_i^n\}$ satisfy

$$\begin{cases} h(y_n) = k(a_n) + R_{M,\lambda,\varphi}^{H(.,.,.,.)-\eta}(q_n), \\ a_n \in V(a_n),\ \|a_n - a_{n+1}\| \leq \mathcal{H}(V(y_n), V(y_{n+1})) + \epsilon^{n+1}\|y_n + y_{n+1}\|, \\ for\ each\ i,\ y_i^n \in T_i(y_n), \left\|y_i^n - y_i^{n+1}\right\| \leq \mathcal{H}(T_i(y_n), T_i(y_{n+1})) \\ +\epsilon^{n+1}\|y_n + y_{n+1}\|, \\ q_{n+1} = H(P(h(y_n) - k(a_n)), Q(h(y_n) - k(a_n)), R(h(y_n) - k(a_n)), \\ S(h(y_n) - k(a_n))) - \lambda\{\varphi \circ G(y_n, z_n) + k(a_n)\}, \end{cases}$$

where $\lambda > 0, n \geq 0$, and $\mathcal{H}$ is the Hausdorff metric on $CB(X)$.

Next, we find the convergence of the iterative algorithm for the resolvent equation (4.3) corresponding to set-valued variational inclusion problem (4.1) and the unique solution (t, x, y, z) of the resolvent equation (4.3).

Theorem 4.3. *Let us consider the problem (4.1) with assumptions A1-A4 hold good and $\varphi : X \to X$ be a single valued mapping with $\varphi(y_1+y_2) = \varphi(y_1)+\varphi(y_2)$ and $Ker(\varphi) = \{0\}$. Let set-valued mappings $V,\ T_i : X \to CB(X),\ i = 1, 2, 3, ...$ be $\lambda_V,\ \beta_i$-$\mathcal{H}$-Lipschitz continuous respectively. Let single valued mapping $h : X \to X$ be r-strongly monotone and λ_h-Lipschitz continuous, and $k : X \to X$ be λ_k-Lipschitz continuous. Let mapping $H : X \times X \times X \times X \to X$ be $\kappa_1, \kappa_2, \kappa_3$ and κ_4-Lipschitz continuous with respect to P, Q, R and S, respectively. Let $\varphi \circ G$ be α_i-Lipschitz continuous with respect to i^{th} component, $i = 1, 2, 3, ...$ Suppose that the following condition holds*

$$\begin{aligned} 0 &< (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)\{\lambda_h + \lambda_k\lambda_V\} + \lambda\sum_{i=1}^{\infty}\alpha_i\beta_i + \lambda\lambda_k\lambda_V \\ &< \frac{(\ell - m\lambda)\left\{1 - \sqrt{1 - 2r + {\lambda_h}^2} - \lambda_k\lambda_V\right\}}{\tau}. \end{aligned}$$

Then there exist $q, y \in X,\ a \in V(y)$ and $y_i \in T_i(y)$ that satisfy the resolvent equation (4.3). The iterative sequence $\{q_n\}, \{y_n\}, \{a_n\}$ and $\{y_i^n\},\ i = 1, 2, 3, ...$ and $n = 1, 2, 3, ...$, generated by Algorithm 1 converges strongly to the unique solution q, y, a and y_i respectively.

Proof . Using Algorithm 1 and $\lambda_V,\ \beta_i$-$\mathcal{H}$ Lipschitz continuity of $V,\ T_i$, respectively, we have

$$\|a_n - a_{n-1}\| \leq \mathcal{H}(V(y_n), V(y_{n-1})) + \epsilon^n\|y_n - y_{n-1}\| \leq \{\lambda_V + \epsilon^n\}\|y_n - y_{n-1}\| \tag{4.6}$$

$$\left\|y_i^n - y_i^{n-1}\right\| \leq \mathcal{H}(T_i(y_n), T_i(y_{n-1})) + \epsilon^n\|y_n - y_{n-1}\| \leq \{\beta_i + \epsilon^n\}\|y_n - y_{n-1}\|, \tag{4.7}$$

where n=1,2,3,... .
Now, we compute

$$\begin{aligned}
\|q_{n+1}-q_n\| &= ||H(P(h(y_n)-k(a_n)),Q(h(y_n)-k(a_n)),R(h(y_n)-k(a_n)),\\
&\quad S(h(y_n)-k(a_n))))-H(P(h(y_{n-1})-k(a_{n-1})),\\
&\quad Q(h(y_{n-1})-k(a_{n-1})),R(h(y_{n-1})-k(a_{n-1})),\\
&\quad S(h(y_{n-1})-k(a_{n-1})))-\lambda\{\varphi\circ G(y_1^n,y_2^n,y_3^n,...)+k(a_n)\\
&\quad -\varphi\circ G(y_1^{n-1},y_2^{n-1},y_3^{n-1},...)+k(a_{n-1})\}||\\
&\le ||H(P(h(y_n)-k(a_n)),Q(h(y_n)-k(a_n)),R(h(y_n)-k(a_n)),\\
&\quad S(h(y_n)-k(a_n)))-H(P(h(y_{n-1})-k(a_{n-1})),Q(h(y_{n-1})-k(a_{n-1})),\\
&\quad R(h(y_{n-1})-k(a_{n-1})),S(h(y_{n-1})-k(a_{n-1})))||\\
&\quad +\lambda\left\|\varphi\circ G(y_1^n,y_2^n,y_3^n,...)-\varphi\circ G(y_1^{n-1},y_2^{n-1},y_3^{n-1},...)\right\|\\
&\quad +\lambda\left\|k(a_n)-k(a_{n-1})\right\|. \qquad (4.8)
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\|h(y_n)-k(a_n)-h(y_{n-1})-k(a_{n-1})\| &\le \|h(y_n)-h(y_{n-1})\|+\|k(a_n)-k(a_{n-1})\|\\
&\le \lambda_h\|y_n-y_{n-1}\|+\lambda_k\|a_n-a_{n-1}\|\\
&\le \lambda_h\|y_n-y_{n-1}\|+\lambda_k(\lambda_V+\epsilon^n)\|y_n-y_{n-1}\|\\
&\le \{\lambda_h+\lambda_k(\lambda_V+\epsilon^n)\}\|y_n-y_{n-1}\|. \qquad (4.9)
\end{aligned}$$

Since, $H(P,Q,R,S)$ is $\kappa_1,\kappa_2,\kappa_3,\kappa_4$-Lipschitz continuous with respect to P,Q,R and S respectively, we have

$$\begin{aligned}
&||H(P(h(y_n)-k(a_n)),Q(h(y_n)-k(a_n)),R(h(y_n)-k(a_n)),S(h(y_n)-k(a_n))))\\
&-H(P(h(y_{n-1})-k(a_{n-1})),Q(h(y_{n-1})-k(a_{n-1})),\\
&R(h(y_{n-1})-k(a_{n-1})),S(h(y_{n-1})-k(a_{n-1})))||\\
\le\;& (\kappa_1+\kappa_2+\kappa_3+\kappa_4)\|(h(y_n)-k(a_n))-(h(y_{n-1})-k(a_{n-1}))\|\\
\le\;& (\kappa_1+\kappa_2+\kappa_3+\kappa_4)\{\lambda_h+\lambda_k(\lambda_V+\epsilon^n)\}\|y_n-y_{n-1}\|. \qquad (4.10)
\end{aligned}$$

Since $\varphi\circ G_i,\ i=1,2,3,...$ are α_i-Lipschitz continuous and $T_i's$ are β_i-$\mathcal{H}$-Lipschitz continuous, we have

$$\begin{aligned}
&\left\|\varphi\circ G(y_1^n,y_2^n,...)-\varphi\circ G(y_1^{n-1},y_2^{n-1},...)\right\|\\
=\;& \left\|\varphi\circ G(y_1^n,y_2^n,...)-\varphi\circ G(y_1^{n-1},y_2^n,...)+\varphi\circ G(y_1^{n-1},y_2^n,...)+...\right\|\\
\le\;& \left\|\varphi\circ G(y_1^n,y_2^n,...)-\varphi\circ G(y_1^{n-1},y_2^n,...)\right\|\\
&+\left\|\varphi\circ G(y_1^{n-1},y_2^n,...)-\varphi\circ G(y_1^{n-1},y_2^{n-1},...)\right\|+...\\
\le\;& \alpha_1\left\|y_1^n-y_1^{n-1}\right\|+\alpha_2\left\|y_2^n-y_2^{n-1}\right\|+...\\
\le\;& \alpha_1(\beta_1+\epsilon^n)\|y_n-y_{n-1}\|+\alpha_2(\beta_2+\epsilon^n)\|y_n-y_{n-1}\|+...\\
\le\;& \sum_{i=1}^{\infty}\alpha_i(\beta_i+\epsilon^n)\|y_n-y_{n-1}\|. \qquad (4.11)
\end{aligned}$$

Using (4.6),(4.10) and (4.11) in (4.8), we have

$$\begin{aligned}\|q_{n+1}-q_n\| &\leq (\kappa_1+\kappa_2+\kappa_3+\kappa_4)\{\lambda_h+\lambda_k(\lambda_V+\epsilon^n)\}\|y_n-y_{n-1}\| \\ &\quad +\lambda\sum_{i=1}^{\infty}\alpha_i(\beta_i+\epsilon^n)\|y_n-y_{n-1}\|+\lambda\lambda_k(\lambda_V+\epsilon^n)\|y_n-y_{n-1}\| \\ &= \{(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\{\lambda_h+\lambda_k(\lambda_V+\epsilon^n)\}+\lambda\sum_{i=1}^{\infty}\alpha_i(\beta_i+\epsilon^n) \\ &\quad +\lambda\lambda_k(\lambda_V+\epsilon^n)\}\|y_n-y_{n-1}\|. \end{aligned} \tag{4.12}$$

By condition (4.6) and Lipschitz continuity of resolvent operator, we have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|\{y_n-y_{n-1}-(h(y_n)-h(y_{n-1}))\}+\{k(a_n)-k(a_{n-1})\} \\ &\quad +R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q_{n-1})-R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q_n)\| \\ &\leq \|y_n-y_{n-1}-(h(y_n)-h(y_{n-1}))\| \\ &\quad +\|R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q_{n-1})-R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q_n)\|+\|k(a_n)-k(a_{n-1})\| \\ &\leq \|y_n-y_{n-1}-(h(y_n)-h(y_{n-1}))\|+\frac{\tau}{\ell-m\lambda}\|q_n-q_{n-1}\| \\ &\quad +\lambda_k(\lambda_V+\epsilon^n)\|y_n-y_{n-1}\|. \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\text{Now,}\quad \|y_n-y_{n-1}-(h(y_n)-h(y_{n-1}))\|^2 \\ &= \|y_n-y_{n-1}\|^2-2[h(y_n)-h(y_{n-1}),y_n-y_{n-1}]+\|h(y_n)-h(y_{n-1})\|^2 \\ &\leq \|y_n-y_{n-1}\|^2-2r\|y_n-y_{n-1}\|^2+{\lambda_h}^2\|y_n-y_{n-1}\|^2 \\ &\leq (1-2r-{\lambda_h}^2)\|y_n-y_{n-1}\|^2. \end{aligned} \tag{4.14}$$

Using (4.14) in (4.13), we have

$$\begin{aligned}\|y_n-y_{n-1}\| &\leq \sqrt{1-2r+{\lambda_h}^2}\|y_n-y_{n-1}\|+\frac{\tau}{(\ell-m\lambda)}\|q_n-q_{n-1}\| \\ &\quad +\lambda_k(\lambda_V+\epsilon^n)\|y_n-y_{n-1}\|.\end{aligned}$$

$$\|y_n-y_{n-1}\|\leq\frac{\tau}{\left\{1-\{\sqrt{1-2r+{\lambda_h}^2}+\lambda_k(\lambda_V+\epsilon^n)\}\right\}(\ell-m\lambda)}\|q_n-q_{n-1}\|. \tag{4.15}$$

Using (4.15) in (4.12), we have

$$\|q_{n+1}-q_n\|\leq\Psi(\epsilon^n)\|q_k-q_{k-1}\|, \tag{4.16}$$

where

$$\Psi(\epsilon^n)=\frac{\tau\left\{(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\{\lambda_h+\lambda_k(\lambda_V+\epsilon^n)\}+\lambda\sum_{i=1}^{\infty}\alpha_i(\beta_i+\epsilon^n)+\lambda\lambda_k(\lambda_V+\epsilon^n)\right\}}{\left\{1-\{\sqrt{1-2r+{\lambda_h}^2}+\lambda_k(\lambda_V+\epsilon^n)\}\right\}(\ell-m\lambda)}.$$

Since $0<\epsilon<1$, this implies that $\Psi(\epsilon^n)\to\Psi$ as $n\to\infty$, where

$$\Psi=\frac{\tau\left\{(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\{\lambda_h+\lambda_k\lambda_V\}+\lambda\sum_{i=1}^{\infty}\alpha_i\beta_i+\lambda\lambda_k\lambda_V\right\}}{\left\{1-\{\sqrt{1-2r+{\lambda_h}^2}+\lambda_k\lambda_V+\}\right\}(\ell-m\lambda)}.$$

Clearly, $\{q_n\}$ is a Cauchy sequence in Banach space X, because it is given that $\Psi < 1$, then there exists $q \in X$ such that $q_n \to q$ as $n \to \infty$. From (4.15), it is clear that y_n is also a Cauchy sequence in Banach space X, then there exists y in X such that $y_n \to y$ as $n \to \infty$.
From Algorithm 1 and equations (4.6) and (4.7), the sequences $\{y_i^n\}$ and $\{a_n\}$ are also Cauchy sequences in X. Thus there exists y_i and a such that $y_i^n \to y_i$ and $a_n \to a$ as $n \to \infty$. Further, we will prove that $y_i \in T_i(y)$. Since $y_i^n \in T_i(y)$, then

$$\begin{aligned} d(y_i, T_i(y)) &\leq \|y_i - y_i^n\| + d(y_i^n, T_i(y)) \\ &\leq \|y_i - y_i^n\| + \mathcal{H}(T_i(y_n), T_i(y)) \\ &\leq \|y_i - y_i^n\| + \beta_i \|y_n - y\| \to 0,\ n \to \infty, \end{aligned}$$

implies that $d(y_i, T_i(y)) = 0$.
Since $T_i(y) \in CB(X)$, we have $y_i \in T_i(y),\ i = 1, 2, 3, ...$

Similarly we can easily show that $a \in V(y)$.
By Algorithm 1 and continuity of of $R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}$, P, Q, R, S, V, T_i, $\varphi \circ G$, k, h, η and M, we know that $(q, y, a, (y_1, y_2, y_3, ...))$ satisfy

$$\begin{aligned} q_{n+1} &= [H(P(h(y_n) - k(a_n)), Q(h(y_n) - k(a_n)), R(h(y_n) - k(a_n)), \\ &\quad S(h(y_n) - k(a_n))) - \lambda\{\varphi \circ G(y_1^n, y_2^n, y_3^n, ...) + k(a_n)\}], \\ \to q &= [H(P(h(y) - k(a)), Q(h(y) - k(a)), R(h(y) - k(a)), \\ &\quad S(h(y) - k(a))) - \lambda\{\varphi \circ G(y_1, y_2, y_3, ...) + k(a)\}],\ \text{as } n \to \infty \end{aligned}$$

$$R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q_n) = h(y_n) - k(a_n) \to h(y) - k(a) = R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q) \text{ as } n \to \infty.$$

By using Lemma 4.2, we have

$$\begin{aligned} &\varphi \circ G(y_1, y_2, y_3, ...) + \lambda^{-1}(q - H(P(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), Q(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), \\ &R(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)), S(R^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q)))) = 0. \end{aligned}$$

Thus we have

$$\varphi \circ G(y_1, y_2, y_3, ...) + \lambda^{-1} J^{H(.,.,.,.)-\eta}_{M,\lambda,\varphi}(q) = 0. \tag{4.17}$$

Hence $(q, y, a, (y_1, y_2, y_3, ...))$ is a unique solution of the problem (4.3). □

Example 4.4. *Let $X = \mathbb{R}^2$ with usual inner product*
Let $Vx = T_i x = \left\{ \begin{pmatrix} \frac{1}{n} x_1 \\ \frac{1}{n} x_2 \end{pmatrix} : \ \forall\, n \in \ \mathbb{N}, \quad x = (x_1, x_2) \in \mathbb{R}^2 \right\}$.
Then it is easy to check that V is $\frac{1}{10}$-Lipschitz continuous for $n = 10$ and T_i is 1-$\mathcal{H}$-Lipschitz continuous for $n = 1,\ i = 1, 2, 3, ...$
Let $h, k, P, Q, R, S, \varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$h(x) = \begin{pmatrix} \frac{1}{20} x_1 \\ \frac{1}{20} x_2 \end{pmatrix},\ k(x) = \begin{pmatrix} \frac{1}{5} x_1 \\ \frac{1}{5} x_2 \end{pmatrix},\ P(x) = \begin{pmatrix} \frac{1}{16} x_1 \\ \frac{1}{16} x_2 \end{pmatrix},\ Q(x) = \begin{pmatrix} \frac{1}{17} x_1 \\ \frac{1}{17} x_2 \end{pmatrix},$$
$$R(x) = \begin{pmatrix} \frac{1}{18} x_1 \\ \frac{1}{18} x_2 \end{pmatrix},\ S(x) = \begin{pmatrix} \frac{1}{19} x_1 \\ \frac{1}{19} x_2 \end{pmatrix},\ \text{and } \varphi(x) = \begin{pmatrix} \frac{1}{3} x_1 \\ \frac{1}{3} x_2 \end{pmatrix}$$

Clearly h is $\frac{1}{20}$-Lipschitz continuous and $\frac{1}{20}$-strongly monotone, k is $\frac{1}{5}$-Lipschitz continuous

Now define $G : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times ... \to \mathbb{R}^2$ by

$$G(x^1, x^2, x^3, ...) = \begin{pmatrix} \frac{1}{2} x_1^i \\ \frac{1}{2} x_2^i \end{pmatrix},\ i = 1, 2, 3, ... \ .$$

Then for $i = 1$, $\varphi \circ G$ is $\frac{1}{6}$-Lipschitz continuous.

Suppose that $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *is defined by*

$$H(Px, Qx, Rx, Sx) = Px + Qx + Rx + Sx, \ \forall \ x \in \mathbb{R}^2.$$

Then it is obvious that H *is* $\frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}$*-Lipschitz continuous with respect to* P, Q, R *and* S *respectively.*

Then it is easy to check that

$$\begin{aligned} 0 &< (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)\{\lambda_h + \lambda_k \lambda_V\} + \lambda \sum_{i=1}^{\infty} \alpha_i \beta_i + \lambda \lambda_k \lambda_V \\ &< \frac{(\ell - m\lambda)\left\{1 - \sqrt{1 - 2r + {\lambda_h}^2} - \lambda_k \lambda_V\right\}}{\tau} \end{aligned}$$

for $i = 1$.
Therefore for the constants
$l = 3, \ m = 2, \ \lambda = 1, \ \lambda_V = \frac{1}{10}, \ \beta_1 = 1, \ \lambda_h = r = \frac{1}{20}, \ \alpha_1 = \frac{1}{6}, \ \lambda_k = \frac{1}{5}, \ \kappa_1 = \frac{1}{16}, \ \kappa_2 = \frac{1}{17}, \ \kappa_3 = \frac{1}{18}$ *and* $\kappa_4 = \frac{1}{19}$
obtained above. All the condition of the Theorem 4.3 are satisfied for $\tau = 0.02$.

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