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Some modified types of arrow domination

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Abstract

The aim of this paper is to introduce some new modified types of arrow domination by adding some conditions on the arrow dominating set or on its complement set. Co-independent arrow domination, restrained arrow domination, connected arrow domination, and complementary tree arrow domination are the main types of domination introduced here. More properties and bounds are discussed and applied to some graphs.

Keywords: Arrow domination, dominating set, domination number.

1. Introduction

Throughout this work, all graphs are finite, simple and undirected. A graph $G(V, E)$ has a vertex set $V(G)$ and edge set $E(G)$. For any vertex $u \in V(G)$, the set $N(u) = \{v \in V(G) : vu \in E(G)\}$ is an open neighborhood. For basics concepts of graph theory see [16, 26, 27]. A set $D \subseteq V$ is called dominating set of G if every vertex out of it is adjacent with one or more vertices from D. The order of D is called the domination number of G and denoted by $\gamma(G)$, for more details see [17, 18, 19]. There are several types of dominating parameters discussed more properties and applications, see for example [1, 12, 14, 15, 21, 23, 24, 28, 29, 30]. Some papers contains connection between domination and other branches of Mathematics such as [13, 20, 22, 25, 31].

Theorem 1.1. *Let G be a graph with an arrow dominating set, then:*

1. *Every arrow dominating set is a total dominating set.*
2. *G has no independent arrow dominating set.*
3. *G has no arrow bi-dominating set.*

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4. *Every arrow dominating set is not isolate dominating set.*

Proof .

1. *Since every $v \in D$ is adjacent with two vertices from D at least. Thus, D is total dominating set.*
2. *Let D be an arrow dominating set in G , then D has no isolated vertices. Thus, D is not independent dominating set.*
3. *Let D be a bi-dominating set , then every vertex in D dominates exactly two vertices in $V - D$, that is contradict definition of arrow dominating set.*
4. *Similar to case 2.*

□

2. Co- independent Arrow Domination

In this section, by adding new condition on the subgraph $G[\ V - D\ ]$.We get a new type of domination say co-independent arrow domination and its inverse. We study its properties and apply it on some graphs.

Definition 2.1. *A subset $D \subseteq V(G)$ is co – independent arrow dominating set of G, if D is an arrow dominating set and $G[\ V - D\ ]$ has no edges.*

Definition 2.2. *A set D is a minimal co – independent arrow dominating set if it has no co – independent arrow dominating subset. The minimum co – independent arrow dominating set is the smallest minimal co – independent arrow dominating set in G.*

Definition 2.3. *The co-independent arrow domination number $\gamma_{ar}^{coi}(G)$, is the cardinality of the minimum co-independent arrow dominating set of G. Such set is referred as $\gamma_{ar}^{coi}-$set.*

Definition 2.4. *Let G be a graph with $\gamma_{ar}^{coi}-$set . A subset $D^{-1} \subseteq V - D$ is an inverse co-independent arrow dominating set with respect to D, if D^{-1} is an arrow dominating set of G and $G[\ V - D^{-1}]$ has no edges.*

Definition 2.5. *A set D^{-1} is a minimal inverse co-independent arrow dominating set, if it has no inverse co-independent arrow dominating subset. The minimum inverse co-independent arrow dominating set is the smallest minimal inverse co-independent arrow dominating set of G.*

Definition 2.6. *The inverse co-independent arrow domination number $\gamma_{ar}^{-coi}(G)$, is the cardinality of the minimum inverse co-independent arrow dominating set of G. Such set is referred as $\gamma_{ar}^{-coi}-$set.*

Remark 2.7. *For any graph $G(n, m)$ with co-independent arrow dominating set D and $\gamma_{ar}^{-coi}(G)$, we have:*

1. $\deg\ (v)\ = 0\ \ ,\ \ \forall\ v \in G\ [V - D\ ]$.
2. $\gamma(G) \le \gamma_{ar}^{coi}(G)$.
3. $\gamma^{coi}(G) \le \gamma_{ar}^{coi}(G)$.

Theorem 2.8. *Let $G(n, m)$ be a graph with co-independent arrow dominating set, then: $2\ \gamma_{ar}^{coi}(G) \le m \le \frac{1}{2}[\gamma_{ar}^{coi}(G)]^2 + \gamma_{ar}^{coi}(G)]$.*

Proof . *Let D be $\gamma_{ar}^{coi}-$ set in G, then the number of edges between D and $G[\ V - D\ ]$ equals $|D| = \gamma_{ar}^{coi}(G)$ there are two cases proved as follows :*

Case 1: *To prove the lower bound , suppose that* $G\,[\,V-D\,]$ *is a null graph and since every* $v \in D$ *adjacent with at least two vertices in* D*, then deg* $(v) = 2$ *in* $G[\,D\,]$*. Thus,* $G[\,D\,]$ *is a cycle or a union of cycle graphs with order and size* $|D| = \gamma_{ar}\,(G)$*. Therefore,* $m \leq \gamma_{ar}(G) + \gamma_{ar}\,(G) = 2\gamma_{ar}(G)$.

Case 2 : *To prove the upper bound. Let* $G\,[D\,]$ *is a complete graph. Since* $G[\,V-D\,]$ *is a null graph , then the size of* $G[\,V-D\,]$ *equal to zero. Hence,* $m \geq \frac{|D||D-1|}{2} + \gamma_{ar}^{coi} = \frac{\gamma_{ar}^{coi}\left(\gamma_{ar}^{coi}-1\right)}{2} + \gamma_{ar}^{coi} = \frac{1}{2}[\gamma_{ar}^{coi2} + \gamma_{ar}^{coi}]$.

□

Remark 2.9. *There is no co-independent arrow domination in a path graph* P_n *and in a cycle graph* C_n *for all* n.

Proposition 2.10. *A complete graph* $K_n\,(n \geq 4)$ *, then* $\gamma_{ar}^{coi}\,(K_n) = \gamma_{ar}(K_n) = n-1$ *.*
Proof . *Since every vertex in* D *dominates exactly one vertex from* $V-D$ *and adjacent with at least two vertices from* D *,then* D *must be contains all vertices of* K_n *unless one vertex. See for example Figure 1.* □

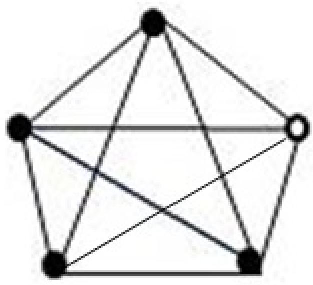

Figure 1: Co –independent Arrow Dominating Set in Complete Graph

Proposition 2.11. *A wheel graph* W_n $(n \geq 3)$*, then* $\gamma_{ar}^{coi}\,(W_n) = \gamma_{ar}\,(W_n) = n$.
Proof . *Since the wheel graph* W_n *is* $C_n + K_1$*. Let* v_1 *,* v_2*, ... ,*v_{n+1} *be the vertices of* W_n *such that* v_{n+1} *is the vertex of* K_1*. Since every vertex in* D *dominates exactly one vertex and adjacent with two or more vertices in* D *. Then ,* D *must be contians all vertices of* W_n *unless the vertex of* K_1*. If we delete any vertex from* D *, then there exist two vertices in* D *dominate two vertices and adjacent with only one vertex in* D *which is contradict arrow definition. Hence,* D *is minimum arrow dominating set. See for example Figure 2* □

Theorem 2.12. *A complete bipartite graph* $K_{n,m}$ $(n, m \geq 3)$ *has no co-independent arrow dominating set.*
Proof . *Let* μ_1*and* μ_2 *are two disjoint sets of* $K_{n,m}$ *such that* $|\mu_1| = n$ *and* $|\,\mu_2| = m$*. Since any arrow dominating set* D *contain* $n-1$ *vertices of* μ_1 *and* $m-1$*vertices of* μ_2*. Since there exist an edge between the two vertices in* $G[\,V-D\,]$*. Hence,* D *is not co-independent arrow dominating set.*
□

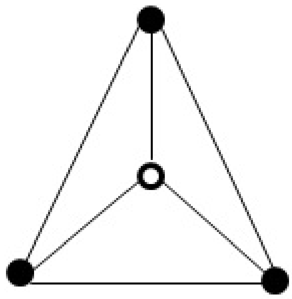

Figure 2: Co-independent Arrow Dominating Set in Wheel Graph

Proposition 2.13. *The Barbell graph $B_{n,n}$ $(n \geq 3)$ has co-independent arrow domination and $\gamma_{ar}^{coi}(B_{n,n}) = \gamma_{ar}(B_{n,n}) = 2n-2$ if and only if $n \geq 4$.*

Proof . *By Proposition 2.10. Since the Barbell graph have two copies of K_n , then γ_{ar} $(B_{n,n}) = 2n-2$. Where the edge that joined the two copies of K_n must be incident on two vertices both of.* □

Theorem 2.14. *A big helm graph $\mathcal{H}_n$ $(n \geq 3)$ has $\gamma_{ar}^{coi}(\mathcal{H}_n) = \gamma_{ar}(\mathcal{H}_n) = n+1$.*

Proof . *By Proposition 2.11, let D contain all vertices of W_n. Then, every vertex in D dominates one vertex and adjacent with three or more vertices from D. Hence, $\gamma_{ar}(\mathcal{H}_n) = n+1$. See for example Figure 3* □

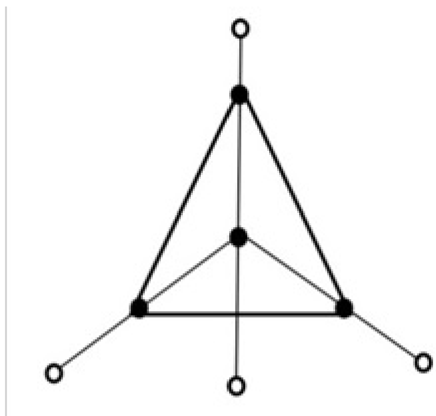

Figure 3: Co-independent Arrow Dominating Set in Big Helm Graph

Remark 2.15. *$\overline{P_n}$, $\overline{K_n}$and $\overline{W_n}$ graphs have no co-independent arrow dominating set.*

Proposition 2.16. *The complement of complete bipartite graph $\overline{K_{n,m}}$ has co-independent arrow dominating set if and only if $n, m \geq 4$ such that $\gamma_{ar}^{coi}(\overline{K_{n,m}}) = \gamma_{ar}\left(\overline{K_{n,m}}\right) = n+m-2$.*

Proof . *Since $\overline{K_{n,m}} = K_n \cup K_m$ then $\gamma_{ar}(\overline{K_{n,m}}) = n+m-2$ according to Proposition 2.10.* □

Proposition 2.17. *The complement cycle graph $\overline{C_n}$ has no co-independent arrow dominating set.*

Proof . *Since $\overline{C_n}$ has no arrow dominating set for all n unless $n = 6$. But , every vertex in $G[V-D]$ has degree two that means joined by two edges. Hence , $\overline{C_n}$ has no co-independent arrow dominating set.* □

Theorem 2.18. *Let G be a graph with co-independent arrow dominating set , then G has no inverse co-independent arrow dominating set.*

Proof . *Let G has inverse co-independent arrow dominating set D^{-1}. Since $D^{-1} = V - D$ by Corrollary 2.4.11, If a graph G has $\gamma_{ar}^{-1}(G)$, then $D^{-1} = V - D$. And $\gamma_{ar}(G) + \gamma_{ar}^{-1}(G) = n$, since $G[\ V - D\ ]$ has no edges , then every $v \in D^{-1}$ has no neighborhoods in D^{-1} which is contradict arrow dominating definition. Hence, G has no invers co-independent arrow dominating set.* □

3. Restrained Arrow Domination

In this section, by adding new condition on the subgraph $G[\ V - D\ ]$,we get restrained arrow domination. Inveres , bounds and properties are introduced for this type of domination.

Definition 3.1. *An arrow dominating set D of a graph G is a restrained arrow dominating set if $G[\ V - D\ ]$ has no isolated vertices. A restrained arrow dominating set of a graph G is said a minimal restrained arrow dominating set if it has no proper sub set as restrained arrow dominating set. The smaller restrained arrow dominating set is called minimum restrained arrow dominating set. The cardinality of the minimum restrained arrow dominating set is known as the restrained arrow domination number of G and denoted by $\gamma_{ar}^{r}(G)$. Such set is reffered as $\gamma_{ar}^{r}-$ set.*

Definition 3.2. *Let G be a graph with $\gamma_{ar}^{r}-$ set D, a subset $D^{-1} \subseteq V - D$ is an inverse restrained arrow dominating set , if D^{-1} is a restrained arrow dominating set of G and $G[\ V - D^{-1}\ ]$ has no isolated vertex.*

Theorem 3.3. *The size of any graph $G(n, m)$ having restrained arrow domination number $\gamma_{ar}^{r}(G)$ is $2\gamma_{ar}^{r} + \lceil \frac{n-\gamma_{ar}^{r}(G)}{2} \rceil \ \leq m \leq \binom{n}{2} + {\gamma_{ar}^{r}}^{2} + (1-n)\gamma_{ar}^{r}$*

Proof . *Let D be a $\gamma_{ar}^{r}-$set of G, since $v \in D$ dominates exactly one vertex from $V - D$, then the number of edges between D and $V - D$ equals to $|D| = \gamma_{ar}^{r}(G)$. Then , there are two cases are proved as follows:*

Case 1: *To prove the lower bound , suppose that $G[\ V - D\ ]$ contains as few edges as possible to be a graph with no isolated vertices. Thus, the number of edges in $G[V - D]$ equals $\lceil \frac{|V-D|}{2} \rceil = \lceil \frac{n-\gamma_{ar}^{r}(G)}{2} \rceil$. Since every vertex $v \in D$ adjacent with at least two vertices , then $\deg\deg\ (v) = 2$ in $G[\ D]$. Thus, $G[D]$ is a cycle or union of cycle graphs with order and size $|D| = \gamma_{ar}^{r}(G)$. Therefore , $m \geq 2\gamma_{ar}^{r} + \lceil \frac{n-\gamma_{ar}^{r}(G)}{2} \rceil$.*

Case 2: *To prove the upper bound. Let $G[D]$ and $G[V - D]$ are complete graphs. Then , let m_1 and m_2 be the number of edges of $G\ [\ D\ ]$ and $G[\ V - D\ ]$ respectively. Therefore , $m_1 = \frac{|\ D\ |\ \ |\ \ D-1|}{2} = \frac{\gamma_{ar}\ (\ \gamma_{ar}-1)}{2}$ and $m_2 = \frac{|\ \ V-D\ \ ||V-D-1\ |}{2} = \frac{(n-\gamma_{ar})\ (n-\gamma_{ar}-1)}{2}$. Hence , $m \geq m_1 + m_2 + \gamma_{ar}\ (G) = \gamma_{ar} + \frac{\gamma_{ar}\ (\gamma_{ar}-1)}{2} + \frac{(n-\gamma_{ar}\ )(n-\gamma_{ar}-1)}{2} = \binom{n}{2} + \ \gamma_{ar}^{2} + (1-n)\ \ \gamma_{ar}$.*

□

Proposition 3.4. *A complete graph K_n $(n \geq 4)$ has no restrained arrow dominating set.*

Proof . *Since every arrow dominating set in K_n has $n - 1$ vertices in D and one vertex in $-D$, then this vertex is isolated which is contradict restrained arrow dominating set.* □

Proposition 3.5. *A wheel graph $W_n(n \geq 4)$ has no restrained arrow dominating set.*

Proof . *Since the arrow dominating set in a wheel graph W_n has all vertices of the cycle , then $V - D$ has one isolated vertex which is contradict restrained arrow dominating set.* □

Proposition 3.6. *The complete bipartite graph $K_{n,m}$ has restrained arrow dominating set such that $\gamma_{ar}^{r}(K_{n,m}) = \gamma_{ar}(K_{n,m}) = n+m-2$ if and only if , $m \geq 3$.*

Proof . *By Theorem 2.3.6 there exist an edge in $G[\ V-D]$ from the remaining vertex of β_1 and the remaining vertex of β_2 which are belong to $V-D$, then $G[\ V-D\ ]$ has no isolated vertex , thus $K_{n,m}$ has a $\gamma_{ar}^{r}-$ set. See for example Figure 4.* □

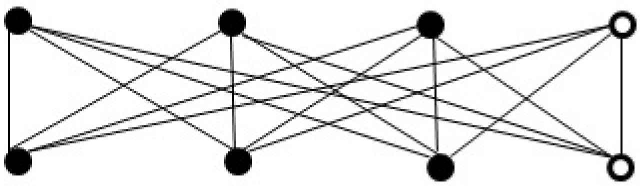

Figure 4: Restrained Arrow Dominating Set in Complete Bipartite Graph

Proposition 3.7. *A Barbell graph $B_{n,n}$ $(n \geq 3)$ has restrained arrow dominating set such that $\gamma_{ar}^{r}(B_{n,n}) = \gamma_{ar}(B_{n,n}) = 2n-2$ if and only if $n \geq 4$.*

Proof . *By Proposition 2.13. Where the edge that joined the two copies of K_n must be incident on two vertices of $V-D$. See Figure 5* □

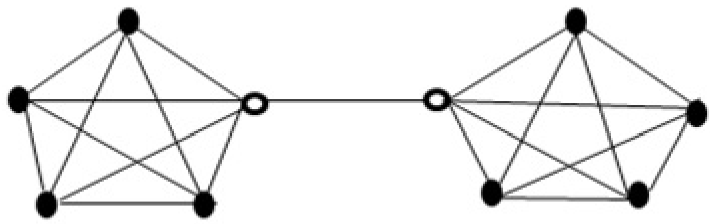

Figure 5: A restrained arrow dominating set in Barbell graph $B_{5,5}$.

Proposition 3.8. *A big helm graph $\mathcal{H}_n (n \geq 3)$ has no restrained arrow dominating set.*

Proof . *Since the arrow dominating set in $\mathcal{H}_n$ has all vertices of wheel graph, and other vertices in $V-D$ are isolated vertices. There is no restrained arrow domination.* □

Remark 3.9. $\overline{K_n}$, $\overline{W_n}$ *and* $\overline{K_{n,m}}$ *graphs have no restrained arrow dominating set.*

Proposition 3.10. *The complement path graph $\overline{P_n}$ has a restrained arrow dominating set if and only if $n=6$ where $\gamma_{ar}^{r}(\overline{P_6}) = \gamma_{ar}(\overline{P_6}) = 2$.*

Proof . *Since $\overline{P_6}$ has only two vertices and its joint by an edge, then $\gamma_{ar}^{r}(\overline{P_6}) = 2$.* □

Proposition 3.11. *The complement cycle graph $\overline{C_6}$ has a restrained arrow dominating set if and only if $n=6$ where $\gamma_{ar}^{r}(\overline{C_6}) = \gamma_{ar}(\overline{C_6}) = 3$.*

Proof . *It is clear, $\overline{C_3}$ is a null graph. $\Delta\left(\overline{C_4}\right) = 1$ and $\deg(v) = 2$ for every v in $\overline{C_5}$, then there is no arrow dominating set for $n \leq 5$. If $n=6$, let $D = \{v_1, v_3, v_5\}$, then D is a γ_{ar-}set of order three every vertex in it dominates only one vertex and adjacent with exactly two vertices. If ≥ 7 , then $\overline{C_n}$ has no an arrow dominating set for the same cause of $P_n\,(n \geq 6)$.* □

Proposition 3.12. *The complement cycle graph $\overline{C_n}$ $(n = 6)$ has inverse restrained arrow dominating set such that $\gamma_{ar}^{-r}\left(\overline{C_6}\right) = \gamma_{ar}^{-1}\left(\overline{C_6}\right) = 3$.*

Proof . *$\overline{C_3}$ it is a null graph. $\Delta\left(\overline{C_4}\right) = 1$ and $\deg\deg\ (v)\ = 2$ for every vertex in $\overline{C_5}$, then there is no inverse arrow dominating set for $n \leq 5$. If $n = 6$, let $D^{-1} = \{v_2, v_4, v_6\}$, then D^{-1} is a γ_{ar}^{-1}-set of order three every vertex in it dominate only one vertex and adjacent exactly two vertices , then $\gamma_{ar}^{-1}\left(\overline{C_6}\right) = 3$. If $n \geq 7$, then $\overline{C_n}$ has no inverse arrow dominating set.* □

Proposition 3.13. *There is no an inverse restrained arrow dominating set in a complement complete bipartite $\overline{K_{n,m}}$ graph.*

Proof . *Since $\overline{K_{n,m}} = K_n \cup K_m$. Since every arrow dominating set in K_n has $n-1$ vertices in D and one vertex in $-D$, then this vertex is isolated which is contradict restrained arrow dominating set.* □

4. Connected Arrow Domination

In this section, by adding new condition on the subgraph [D] , we get a new type of domination say connected arrow domination. We fined its inverse and study its properties and discussed it on some graphs.

Definition 4.1. *Let $D \subseteq V(G)$ is called connected arrow dominating set in G if D is an arrow dominating set such that $G[\ D\ ]$ is a connected induced sub graph. D is minimal connected arrow dominating set if it has no proper subset as connected arrow dominating set. The smallest connected arrow dominating set is called minimum connected arrow dominating set. The cardinality of minimum connected arrow dominating set is known as the connected arrow domination number and denoted by $\gamma_{ar}^{c}(G)$.*

Definition 4.2. *Let G be a graph with $\gamma_{ar}^{c}-$ set D a subset $D^{-1} \subseteq V - D$ is an inverse connected arrow dominating set , if D^{-1} is a connected arrow dominating set of G.*

Theorem 4.3. *The size of a graph $G(n, m)$ has connected arrow domination number $\gamma_{ar}^{c}(G)$ is $2\ \gamma_{ar}^{c}(G) \leq\ m \leq \binom{n}{2} + \gamma_{ar}^{c}(G) + (1-n)\gamma_{ar}^{c}(G)$.*

Proof . *Let D be a $\gamma_{ar}-$ set of G , since every $v \in D$ dominates exactly one vertex from $V - D$, then the number of edges between D and $V - D$ equal $|D| = \gamma_{ar}(G)$, then there are two cases proved as follows:*

Case 1: *To prove the lower bound suppose that $G[\ V - D\ ]$ is a null graph and since every $v \in D$ adjacent with at least two vertices in D, then $\deg\deg\ (v)\ = 2$ in $G[\ D]$. Thus, $G[\ D]$ is a cycle graph with order and size $|D| = \gamma_{ar}(G)$. Therefore, $\geq 2\gamma_{ar}(G)$.*

Case 2: *Similar the proof of Theorem 3.3 case (2).*

□

Proposition 4.4. *The complete graph K_n has connected arrow dominating set, such that $\gamma_{ar}^{c}(K_n) = \gamma_{ar}(K_n) = n - 1$.*

Proof . *Similar to proof of Proposition 3.4.* □

Proposition 4.5. *The wheel graph W_n has connected arrow dominating set, such that $\gamma_{ar}^{c}(W_n) = \gamma_{ar}(W_n) = n$.*

Proof . *Similar to proof of Proposition 3.5.* □

Proposition 4.6. *The complete bipartite graph $K_{n,m}$ has connected arrow domination set such that $\gamma^c_{ar}(K_{n,m}) = \gamma_{ar}(K_{n,m}) = n + m - 2$ if and only if $n, m \geq 3$.*
Proof . *Similar to proof of Proposition 3.6.* $\square$

Proposition 4.7. *The Barbell graph $B_{n,n}$ has connected arrow dominating set such that $\gamma^c_{ar}(B_{n,n}) = \gamma_{ar}(B_{n,n}) = 2n - 2$.*
Proof . *Similar to proof of Proposition 3.7.* $\square$

Proposition 4.8. *A big helm graph $\mathcal{H}_n$ has connected arrow dominating such that $\gamma^c_{ar}(\mathcal{H}_n) = \gamma_{ar}(\mathcal{H}_n) = n + 1$.*
Proof . *Similar to proof of Proposition 3.8.* $\square$

Proposition 4.9. *The complement path graph $\overline{P_n}$ has connected arrow domination such that $\gamma^c_{ar}\left(\overline{P_n}\right) = \gamma_{ar}(\overline{P_n}) = 4$ if and only if $n = 6$.*
Proof . *It is clear, $\Delta\left(\overline{P_n}\right) = 2$ for $n \leq 4$ and $\overline{P_n}$ without arrow domination. If $n = 5$ there are three vertices $v_2, v_3,\ v_4$ of degree two don't belong to D. So, if $D = \{v_1, v_5\}$, then every v_i of D has only one neighborhood in D and dominate two vertices. For $n = 6$, let $D = \{v_1, v_3, v_4, v_6\}$, since every $v_i \in D$ dominate exactly one vertex from $V - D$ and adjacent two or more vertices, then $\gamma_{ar}\left(\overline{P_6}\right) = 4$. If $n \geq 7$, every dominating set D has either a vertex that dominates two or more vertices or a vertex don't dominate any vertex. For example see Figure 6* $\square$

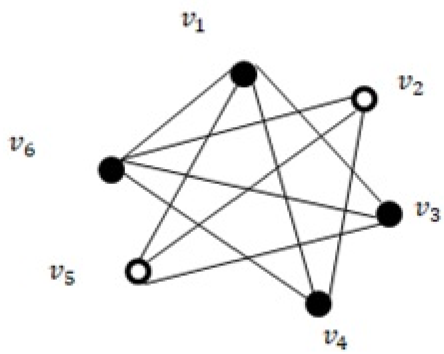

Figure 6: Connected Arrow Dominating Set in Complement Path Graph $\overline{P_6}$

Proposition 4.10. *The complement cycle graph $\overline{C_n}$ has connected arrow dominating set such that $\gamma^c_{ar}\left(\overline{C_n}\right) = \gamma_{ar}(\overline{C_n}) = 3$ if and only if $n = 6$.*
Proof . *Similar to proof of Proposition 3.11.* $\square$

Proposition 4.11. *The complement complete bipartite graph $\overline{K_{n,m}}$ has $\gamma^c_{ar}-$ set such that $\gamma^c_{ar}\left(\overline{K_{n,m}}\right) = \gamma_{ar}(\overline{K_{n,m}}) = n + m - 2$ if and only if $n, m \geq 4$.*
Proof . *Similar to proof of Proposition 3.4.* $\square$

Proposition 4.12. *The complement cycle graph $\overline{C_n}$ has an inverse connected arrow dominating set such that: $\gamma^{-c}_{ar}\left(\overline{C_n}\right) = \gamma^{-1}_{ar}(\overline{C_n}) = 3$ if and only if $n = 6$*
Proof . *Similar to proof of Proposition 3.12.* $\square$

Proposition 4.13. *The complement complete bipartite $\overline{K_{n,m}}$ graph has no inverse connected arrow domination set.*

5. Complementary tree arrow domination

In this section, by adding new condition on the subgraph $G[\ V-D\ ]$.We get a new type of arrow domination say a complementary tree arrow domination we fined its inverse and study its properties and discuss it on some graphs.

Definition 5.1. *The complementary tree arrow dominating set D is an arrow dominating set such that the induced subgraph $G[V-D]$ is a tree. D is minimal complementary tree arrow dominating set if it has no proper subset complementary tree arrow dominating set. The minimum complementary tree arrow dominating set is the smallest of complementary tree arrow dominating set. The cardinality of a complementary tree arrow domination number and denoted by $\gamma_{ar}^{ctd}(G)$.*

Definition 5.2. *Let G be a graph with a $\gamma_{ar}^{ctd}-$ set , subset $D^{-1} \subseteq V-D$ is an inverse complementary tree arrow dominating set if D^{-1} is a complementary tree arrow dominating set with respect to D and $G\left[V-D^{-1}\right]$ is a tree.*

Theorem 5.3. *The size of any graph $G(n,m)$ having complementary tree arrow domination number is $\gamma_{ar}^{ctd}(G)+(n-1) \leq m \leq \gamma_{ar}^{ctd}(G)+\frac{1}{2}[\gamma_{ar}^{ctd^2}(G)-\gamma_{ar}^{ctd}(G)]+(n-1)$.*
Proof . *Let D be a $\gamma_{ar}^{ctd}(G)-$ set of G, where every vertex in D dominates exactly one vertex from $V-D$, then the number of edges between D and $V-D$ equals to $|D|=\gamma_{ar}^{ctd}$. There are two cases proved as follows:*

Case 1: *To prove the lower bound , since the induced subgraph $G[\ V-D\ ]$ is a tree graph has the number of edges $m_1=|V-D|-1=n-\gamma_{ar}^{ctd}(G)-1$. Since $v \in D$ adjacent with at least two vertices from D, $G[\ D\ ]$ is a cycle graph or union of cycles with order and size $|D|=\gamma_{ar}^{ctd}(G)$ Therefore, $m \geq \gamma_{ar}^{ctd}(G)+(n-1)$.*

Case : *To prove the upper bound. Let $G[\ D\ ]$ is a complete graph where $G[\ V-D\ ]$ is a tree graph. Let m_1 and m_2 be the number of edges of $G[\ D\ ]$ and $G[\ V-D\ ]$ respectively. Therefore, $m_1=\frac{|D||\ D-1\ |}{2}=\frac{\gamma_{ar}^{ctd}(G)(\gamma_{ar}^{ctd}(G)-1)}{2}$, $m_2=n-\gamma_{ar}^{ctd}(G)-1$. Hence, $m \geq \gamma_{ar}^{ctd}(G)+\frac{\gamma_{ar}^{ctd}(G)\left(\gamma_{ar}^{ctd}(G)-1\right)}{2}+n-\gamma_{ar}^{ctd}(G)-1$.*

□

Proposition 5.4. *A complete graph K_n $(n \geq 4)$ has complementary tree arrow dominating set such that $\gamma_{ar}^{ctd}(K_n)=\gamma_{ar}(K_n)=n-1$.*
Proof . *Since K_n has only one vertex in $G[\ V-D]$ which is a tree. Thus, $\gamma_{ar}^{ctd}(K_n)=n-1$.* □

Proposition 5.5. *A wheel graph $W_n(n \geq 3)$ has complementary tree arrow dominating set such that $\gamma_{ar}^{ctd}(W_n)=\gamma_{ar}(W_n)=n$.*
Proof . *Since there is one vertex in $G[\ V-D]$ from W_n which is a tree , then $\gamma_{ar}^{ctd}(W_n)=n$.* □

Proposition 5.6. *The complete bipartite graph $K_{n,m}$ $(n,m \geq 3)$ has a $\gamma_{ar}^{ctd}-$set such that $\gamma_{ar}^{ctd}(K_{n,m})=\gamma_{ar}(K_{n,m})=n+m-2$.*
Proof . *Similar the proof of Proposition 3.6.* □

Proposition 5.7. *The Barbell graph $B_{n,n}$ $(n \geq 3)$ has complementary tree arrow dominating set such that $\gamma_{ar}^{ctd}(B_{n,n})=\gamma_{ar}(B_{n,n})=2n-2$.*
Proof . *By Proposition 2.13 , where the edge that joined the two copies of K_n must be incident on two vertices of $V-D$.* □

Proposition 5.8. *The big helm graph $\mathcal{H}_n$ has no $\gamma_{ar}^{ctd}-$ set.*
Proof . *Since $\mathcal{H}_n$ graph has only isolated vertices in $G\ [\ V-D]$. Then, it has no complementary tree arrow dominating set.* □

Proposition 5.9. *There is no complementary tree arrow dominating set in the complement cycle graph.*
Proof . *Since $\overline{C_n}$ has no arrow dominating set for all n unless when $=6$. But $G\,[V-D\ ]$ in $\overline{C_6}$ has cycle graph. Hence , $\overline{C_6}$ has no complementary tree arrow dominating set for all n.* □

Proposition 5.10. *The complement complete bipartite $\overline{K_{n,m}}$ graph has complementary tree arrow dominating set such that $\gamma_{ar}^{ctd}\left(\overline{K_{n,m}}\right)=n+m-2$.*
Proof . *Similar to proof of Proposition 5.4.* □

6. Conclusions

Several types of arrow domination are introduced here with some bounds and properties. These types are applied and discussed on more known graphs.

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