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Expanded Integral differential equations and their applications

Rusul Mohammed Hussein Al-Shmary[a,*]

[a]*Ministry of Education, Directorate of Education, Babylon, Iraq*

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Abstract

Differential equations can be used to examine patrials of higher rank with varying coefficients in various regions of the Cartesian coordinate plane. Meanwhile, the researchers and scientists have N. Rajabov, A.S. Star and F.A. Nasim Adeeb Haneen, and others. As a result, while the coefficients of partial differential equations differ from those of partial differential equations, this research examined the partial differential equation based on its rank (fourth rank). Conditions are established for the production of their coefficients within the context of that equation. In multiple different scenarios involving these coefficients, a single solution for that partial differential equation. These circumstances were summed up in five theories.

Keywords: Differential equations, partial differential equations, rank.

1. Introduction

Generally, A partial differential equation (or PDE) is a mathematical equation that contains two or more independent variables, an unknown function that is dependent on those variables, and the unknown function's partial derivatives with respect to those variables. The order of a partial differential equation is determined by the order of its highest derivative. When a partial differential equation is replaced into an equation, a solution (or a particular solution) is a function that solves the equation or, in other words, transforms it to an identity. A generic solution is one that encompasses all possible answers to the situation at hand.

It is usually preferred to use the term exact solution to a specific solution for second- and higher-order nonlinear PDEs (see also Preliminary remarks at Second-Order Partial Differential Equations).

*Corresponding author
Email address: rusull.alshmary89@gmail.com (Rusul Mohammed Hussein Al-Shmary)

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However, the usage of Partial differential equations is to solve physical and other problems that need functions of many variables, such as heat or sound propagation, fluid flow, elasticity, electrostatics, and electrodynamics.
While partial differential equations result in integral Volterra equations of type II, their solution is conditional on the existence of four potential consecutive transformers.
In our research division, fundamental methods were included for solving partial differential equations and their transformation into differential equations in terms of the performance of four first-rank partial differential effects, which simplifies the solution of the partial differential equation under consideration. Additionally, by solving the integral equations that result from solving the partial differential equation using the successive approximation method [6] and obtaining integral formulas, and then clarifying and incorporating the conditions in integral formulas, as well as their inclusion in theories of existence and singularity

2. Partial differential equations: Basic methods

first-order-partial-differential-equations Usually used for the processes of biological, social and economic. It is considered as the basic methods of partial differential equations. Depending on whether or not they contain partial derivatives, they are referred to as partial differential equations (pde) or ordinary differential equations (ode). However, differential equation's order is the highest order derivative that occurs. A differential equation of order n solution (or particular solution) consists of a function defined and n times differentiable on a domain D with the property that the functional ability to evaluate by replacing the component and its n derivatives into the differential equation retains for every point in D.

$$D = \{0 < x < \delta_1, \quad 0 < y < \delta_2\}$$

Whereas:

$$\Delta_1 = \{0 < x < \delta_1, y = 0\}, \qquad \Delta_2 = \{0 < y < \delta_2, x = 0\}$$

The following partial integral equation would be studied:

$$L^{c_1}_{a_1,b_1}\left(L^{c_2}_{a_2,b_2}V\right) = \frac{f(x,y)}{r^{\alpha+\beta}} \tag{2.1}$$

in which:

$$L^{c_j}_{a_j,b_j} \equiv \frac{\partial^2}{\partial x \partial y} + y\frac{a_j(x,y)}{r^{\alpha}}\frac{\partial}{\partial x} + x\frac{b_j(x,y)}{r^{\beta}}\frac{\partial}{\partial y} + \frac{c_j(x,y)}{r^{\alpha+\beta}}, \quad j = 1,2$$
$$r = \sqrt{x^2 + y^2}, \qquad \alpha, \beta \text{true facts.}$$

Therefore, the following symbols will be used $C\left(\overline{D}\right), C^1_x\left(\overline{D}\right), C^2_{xy}\left(\overline{D}\right), C^3_{xxy}\left(\overline{D}\right)$.
Herein, (D) represents the continuous successive inside the area D.
For the y and x transformers, $C^1_x\left(\overline{D}\right)$defines the row of partial derivatives continuous successive of the second rank In area D.
Similarly, $C^1_x\left(\overline{D}\right)$defines the row of partial derivatives continuous successive of the third rank as same as above.

2.1. Equations from Variational Problems

Alarge class of ordinary and partial differential equations arise from variational problems.

2.1.1. Ordinary differential equations

Set

$$E(v) = \int_a^b f(x, v(x), v^{'})\ dx$$

and for given $u_a, u_b\ inR$

$$V = \{v \in C^2[a, b] : v(a) = u_a,\ v(b) = u_b\},$$

where $-\infty < a < b < \infty$ and f is sufficiently regular. One of the basic problems in the calculus of variation is

$$(P) \qquad \min_{v \in V} E(v).$$

Euler equation. Let $u \in V$ be a solution of (P), then

$$\frac{d}{dx} f_{\acute{u}}(x, u(x), \acute{u}(x)) = f_u(x, u(x), \acute{u}(x)) \qquad in\ (a, b).$$

Lemma 2.1 (Basic Lemma in the calculus of variations). *Let $h \in C(a, b)$ and*

$$\int_a^b h(x)\phi(x)\ dx = 0$$

for all $\phi \in C_0^1(a, b)$. Then $h(x) \equiv 0$ on (a, b).

Proof . *Assume $h(x) > 0$ for an $x_0 \in (a, b)$, then there is a $\delta > 0$ such that $(x_0 - \delta,\ x_0 + \delta) \subset (a, b)$ and $h(x) \geq h(x_0)/2$ on $(x_0 - \delta,\ x_0 + \delta)$. Set*

$$\phi(x) = \begin{cases} (\delta^2 - |x - x_0|^2)^2 & if \quad x \in (x_0 - \delta,\ x_0 + \delta) \\ 0 & if \quad x \in (a, b) \backslash (x_0 - \delta,\ x_0 + \delta) \end{cases}$$

□

We assume that:

$$c_1^0\,(x, y) = x.ya_1\,(x, y)\,b_1\,(x, y) - c_1\,(x, y) + r^{\alpha+\beta}\frac{\partial}{\partial x}\left(y\frac{a_1\,(x, y)}{r^{\alpha}}\right)$$

From which, the equation can be formed as follows:

$$\left(\frac{\partial}{\partial x} + x\frac{b_1\,(x, y)}{r^{\beta}}\right)\left(\frac{\partial}{\partial y} + y\frac{a_1\,(x, y)}{r^{\alpha}}\right)U = \frac{f\,(x, y) + c_1^0\,(x, y)\,U}{r^{\alpha+\beta}} \equiv f_1\,(x, y) \qquad (2.2)$$

Assuming the following term, the other side will be:

$$\{b_2\,(x, y)\,, c_2\,(x, y)\} \in C\left(\overline{D}\right) a_2\,(x, y) \in C_{xxy}^3\left(\overline{D}\right)$$

Simplifying equation (2.2) leads to:

$$\left(\frac{\partial}{\partial x} + x\frac{b_2\,(x, y)}{r^{\beta}}\right)\left(\frac{\partial}{\partial y} + y\frac{a_2\,(x, y)}{r^{\alpha}}\right)V = c^{(2)}\,(x, y)\,V + U \qquad (2.3)$$

herein:

$$c^{(2)}(x,y)=\frac{c_2^0(x,y)}{r^{\alpha+\beta}}$$
$$c_2^0(x,y)=x.y.a_2(x,y)\,b_2(x,y)-c_2(x,y)+r^{\alpha+\beta}\frac{\partial}{\partial x}\left(y\frac{a_2(x,y)}{r^{\alpha}}\right)$$

The following formula can be represent for equation (2.2)

$$\left(\frac{\partial}{\partial x}+x\frac{b_2(x,y)}{r^{\beta}}\right)\left(\frac{\partial}{\partial y}+y\frac{a_2(x,y)}{r^{\alpha}}\right)V=\frac{c_2{}^0(x,y)}{r^{\alpha+\beta}}V+U \tag{2.4}$$

we assume the following term in order to determine a solution for (2.2):

$$\frac{\partial U}{\partial y}+y\frac{a_1(x,y)}{r^{\alpha}}U=U_1 \tag{2.5}$$

By substituting in (2.2), the following equation is found:

$$\frac{\partial U_1}{\partial x}+x\frac{b_1(x,y)-b_1(0,0)+b_1(0,0)}{r^{\beta}}U_1=f_1(x,y) \tag{2.6}$$

This represents the general solution of this first rank linear integral equation.

$$U_1{=}e^{-\Omega_{b_1}^{\beta}(x,y)+W_{b_1}^{\alpha}(x,y)}\left[\psi_1(y)+\int_0^x e^{\Omega_{b_1}^{\beta}(t,y)-W_{b_1}^{\beta}(t,y)}\left(t^2{+}y^2\right)^{-\left(\frac{\alpha+\beta}{2}\right)}\left(f(t,y)+c_1^0(t,y)\,U(t,y)\,dt\right)\right] \tag{2.7}$$

Similarly, the solution of the equation (2.5) can be found as follows:

$$U(x,y)=e^{-\Omega_{a_1}^{\alpha}(x,y)+W_{a_1}^{\alpha}(x,y)}\left[\varphi_1(x)+\int_0^y e^{\Omega_{a_1}^{\alpha}(x,s)-W_{a_1}^{\alpha}(x,s)}U_1(x,s)\,ds\right] \tag{2.8}$$

In which,

$$\Omega_{b_1}^{\beta}(x,y)=\int_0^x\frac{b_1(t,y)-b_1(0,0)}{(t^2+y^2)^{\frac{\beta}{2}}}tdt,\Omega_{a_1}^{\alpha}(x,y)=\int_0^y\frac{a_1(t,y)-a_1(0,0)}{(x^2+s^2)^{\frac{\alpha}{2}}}sds$$
$$W_{a_1}^{\alpha}(x,y)=a_1(0,0)(\alpha-2)^{-1}r^{2-\alpha},W_{b_1}^{\beta}(x,y)=b_1(0,0)(\beta-2)^{-1}r^{2-\beta}.$$

By substituting (2.8) for equation (2.7), the following equation is found:

$$\begin{aligned}&U(x,y)-\int_0^y ds_1\int_0^x e^{\Omega_{a_1}^{\alpha}(x,s)-W_{a_1}^{\alpha}(x,s_1)-\Omega_{b_1}^{\beta}(x,s_1)+W_{b_1}^{\beta}(x,s_1)+\Omega_{b_1}^{\beta}\left(t_1,s_1\right)-W_{b_1}^{\beta}(t_1,s_1)}\\&\left(t_1^2+s_1^2\right)^{-\left(\frac{\alpha+\beta}{2}\right)}c_1^0(t_1,s_1)\,U(t_1,s_1)\,dt_1\\&=e^{W_{a_1}^{\alpha}(x,y)-\Omega_{a_1}^{\alpha}(x,y)}\left[\phi_1(x)+\int_0^y e^{\Omega_{a_1}^{\alpha}(x,s_1)-W_{a_1}^{\alpha}(x,s_1)-\Omega_{b_1}^{\beta}(x,s_1)+W_{b_1}^{\beta}(x,s)}\psi_1(s_1)\right.\\&+\int_0^x e^{\Omega_{b_1}^{\beta}(t_1,s_1)-W_{b_1}^{\beta}(t_1,s_1)}.\left(t_1^2+s_1^2\right)^{-\frac{(\alpha+\beta)}{2}}f(t_1,s_1)\;\;dt_1)\;\;ds_1]\end{aligned} \tag{2.9}$$

It can also be found for the equation (2.4) in the same way, as follows

$$V(x,y)-\int_0^y ds_2\int_0^x e^{\Omega^{\alpha}_{a_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)+\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}}$$
$$(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}c^0_2(t_2,s_2)V(t_2,s_2)dt_2$$
$$=e^{W^{\alpha}_{a}(x,y)-\Omega^{\alpha}_{a_2}(x,y)}\left[\varphi_2(x)+\int_0^y e^{\Omega^{\alpha}_{a_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)}\right. \tag{2.10}$$
$$\left.\left(\psi_2(s_2)+\int_0^x e^{\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}\ U(t_2,s_2)\ dt_2\right)\ ds_2\right]$$

3. The result cases of differential equations

The first and fourth instances will be studied in this chapter; the first case is regarded the easiest, while the fourth case is the most difficult, and the second and third cases may be derived from the fourth case.
It can be found From (2.9) and (2.10) equations of four cases, as follows:
The first case $c^0_1(x,y)=0, c^0_2(x,y)=0$
The second case $c^0_1(x,y)\neq 0, c^0_2(x,y)=0$
The third case $c^0_1(x,y)=0, c^0_2(x,y)\neq 0$
The forth case $c^0_1(x,y)\neq 0, c^0_2(x,y)\neq 0$.
By assuming $c^0_1(x,y)=0, c^0_2(x,y)=0$ for the first case, equations (2.9) and (2.10) can be simplified as follows.

$$U(x,y)=e^{W^{\alpha}_{a_1}(x,y)-\Omega^{\alpha}_{a_1}(x,y)}\left[\varphi_1(x)+\int_0^y e^{\Omega^{\alpha}_{a_1}(x,s_1)-W^{\alpha}_{a_1}(x,s_1)-\Omega^{\beta}_{b_1}(x,s_1)+W^{\beta}_{b_1}(x,s_1)}\right. \tag{3.1}$$
$$\left.\left(\psi_1(s_1)+\int_0^x e^{\Omega^{\beta}_{b_1}(t_1,s_1)-W^{\beta}_{b_1}(t_1,s_1)}(t_1^2+s_1^2)^{-\frac{(\alpha+\beta)}{2}}\ f(t_1,s_1)\ dt_1\right)\ ds_1\right]$$

Example 3.1. *Differential equation example of order 4. 2, and 1 are set by the following terms*

$$V(x,y)=e^{W^{\alpha}_{a_2}(x,y)-\Omega^{\alpha}_{a_2}(x,y)}\left[\varphi_2(x)+\int_0^y e^{\Omega^{\alpha}_{a_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_1}(x,s_2)+W^{\beta}_{b_2}(x,s_2)}\right. \tag{3.2}$$
$$\left.\left(\psi_1(s_2)+\int_0^x e^{\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}\ f(t_2,s_2)\ dt_2\right)\ ds_2\right]$$

Substituting equation (3.2) *for* (3.1) *leads to*

$$V(x,y)=e^{-\Omega^{\alpha}_{a_2}(x,y)+W^{\alpha}_{a_2}(x,y)}\left\{\phi_2(x)+\int_0^y e^{\Omega^{\alpha}_{a_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)+W^{\beta}_{a_2}(x,s_2)}\right.$$
$$\left[\psi_2(s_2)+\int_0^x e^{\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)-\Omega^{\alpha}_{a_2}(t_2,s_2)+W^{\alpha}_{a_1}(t_2,s_2)}(t_2^2+s_2^2)^{\frac{(\alpha+\beta)}{2}}(\phi_1(t_2)\right.$$
$$+\int_0^{s_2} e^{\Omega^{\alpha}_{a_1}(t_2,s_1)-W^{\alpha}_{a_1}(t_2,s_1)-\Omega^{\beta}_{b_1}(t_2,s_1)+W^{\beta}_{b_1}(t_2,s_1)} \tag{3.3}$$
$$\left.\left.\left\langle\psi_1(s_1)+\int_0^{t_2} e^{\Omega^{\beta}_{b_1}(t_1,s_1)-W^{\beta}_{b_1}(t_1,s_1)}\left(t_1^2+s_1^2\right)^{\frac{(\alpha+\beta)}{2}}\ f(t_1,s_1)\,dt_1\right\rangle ds_1)\ dt_2\right]ds_2\right\}$$

In which,

$$\Omega_{a_j}^{\alpha}(x,y)=\int_0^y \frac{a_j(x,s_j)-a_j(0,0)}{\left(x^2+s_j^2\right)^{\frac{\alpha}{2}}}s_j ds_j,\qquad \Omega_{b_j}^{\beta}(x,y)=\int_0^x \frac{b_j(t_j,y)-b_j(0,0)}{\left(t_j^2+y^2\right)^{\frac{\beta}{2}}}t_j dt_j$$
$$W_{a_j}^{\alpha}(x,y)=a_j(0,0)(\alpha-2)^{-1}r^{2-\alpha},\qquad W_{b_1}^{\beta}(x,y)=b_j(0,0)(\beta-2)^{-1}r^{2-\beta},\quad j=1,2$$

In conclusion, the following equation was proved depending on the steps

Theorem 3.2. *The differential equation's coefficients is assumed* (2.1) *and included its right side to achieve the following conditions:*

(i) $a_2(x,y)\in C_{xxy}^3\left(\overline{D}\right),\{b_2(x,y),c_2(x,y)\}\in C\left(\overline{D}\right)$

(ii) $a_1(x,y)\in C_x^1\left(\overline{D}\right),\{b_2(x,y),c_2(x,y)\}\in C\left(\overline{D}\right)$

(iii) $|a_j(x,y)-a_j(0,0)|\le H_{a_j}r^{-\gamma_{j1}};0<\gamma_{j1}<2-\alpha,j=1,2$

(iv) $|b_j(x,y)-b_j(0,0)|\le H_{b_j}r^{-\gamma_{j2}};0<\gamma_{j2}<2-\beta,j=1,2$

(v) $r^{-(\alpha+\beta)}.f(x,y)=0\left(r^{-\delta}\right);0<\delta<1,for:r\to 0$

(vi) $c_j^0(x,y)=x.ya_j(x,y)b_j(x,y)-c_j(x,y)+r^{\alpha+\beta}\frac{\partial}{\partial x}\left(y\frac{a_j(x,y)}{r^{\alpha}}\right)\quad j=1,2,\alpha<2,\beta<2$

a single solution is exist for the differential equation (2.1) expressed by relation (3.3):

$$\phi_1(x),\phi_2(x),\psi_1(y),\psi_2(y)$$

optional successive achieve the following However, the complicated case will processed as follows:
Forth case $c_1^0(x,y)\neq 0,c_2^0(x,y)\neq 0$
By taking the relation (2.9). The following equation is found:

$$\begin{aligned}&U(x,y)-\int_0^y ds_1\int_0^x e^{\Omega_{a_1}^{\alpha}(x,s_1)-W_{a_1}^{\alpha}(x,s_1)-\Omega_{b_1}^{\beta}(x,s_1)+W_{b_1}^{\beta}(x,s_1)+\Omega_{b_1}^{\beta}(t_1,s_1)-W_{b_1}^{\beta}}\\&\left(t_1^2+s_1^2\right)^{-\frac{(\alpha+\beta)}{2}}c_1^0(t_1,s_1)V(t_1,s_1)dt_1=F_1[\phi_1(x),\psi_1(y),f(x,y)]\end{aligned}\tag{3.4}$$

Example 3.3 (Laplace equation). *Given that,*

$$E(v)=\frac{1}{2}\int_{\Omega}|\nabla v|^2\,dx-\int_{\partial\Omega}h(x)\,v\,ds$$

The associated boundary value problem is given as follows,

$$\begin{aligned}\Delta u&=0\qquad in\ \Omega\\ \frac{\partial u}{av}&=h\qquad on\ \partial\Omega\end{aligned}$$

The right side of the relation (2.9) is assumed to be equaled to:

$$[F_1\left[\phi_1\left(x\right),\psi_1\left(y\right),f\left(x,y\right)\right].$$

By taking the relation (2.10), it is found :

$$\begin{aligned}&V\left(x,y\right)-\int_0^y ds_2\int_0^x e^{\Omega_{a_2}^{\alpha}(x,s_2)-W_{a_2}^{\alpha}(x,s_2)-\Omega_{b_2}^{\beta}(x,s_2)+W_{b_2}^{\beta}(x,s_2)+\Omega_{b_2}^{\beta}(t_2,s_2)-W_{b_2}^{\beta}}\\&(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}c_2^0\left(t_2,s_2\right)V\left(t_2,s_2\right)dt_2=F_2\left[\phi_2(x),\psi_2(y)\right].\end{aligned}\tag{3.5}$$

The right side of (2.10) relation was assumed to match the following state:

$$F_2\left[\phi_2\left(x\right),\psi_2\left(y\right)\right]$$

Typical examples include

$$\begin{aligned}&\nabla^2 u\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left\{-D\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\right\}\qquad \textit{Laplace Eq.}\\&A=1,B=0,B^2-4AC=-4<0\qquad \textit{Poissin Eq.}\end{aligned}$$

The integrated equation (3.4) is a second-order integrated volterra equation, which we solve via successive approximation. To begin, we ensure that it is close to zero.

$$e^{\Omega_{a_2}^{\alpha}(x,s_2)-\Omega_{b_2}^{\beta}(x,s_2)-W_{a_2}^{\alpha}(x,s_2)+W_{b_2}^{\beta}(x,s_2)+\Omega_{b_2}^{\beta}(t_2,s_2)-W_{b_2}^{\beta}(t_2,s_2)}c_1^0(x,y)=0(r^{\gamma_1})$$

Then (2.2) achieving of the core of integral equation (3.4) is expressed by the following equation:

$$\begin{aligned}&I=\int_0^y ds_1\int_0^x\frac{\left(\sqrt{t_1^2+s_1^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta_1}}dt_1\Rightarrow\\&I_1=\int_0^x\frac{\left(\sqrt{t_1^2+s_1^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}dt_1\le\int_0^x\frac{(2t_1)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}dt_1\le 2^{\gamma_1}\int_0^x\frac{(t_1)^{\gamma_1-1}.t_1}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}dt_1\\&I_1\le\frac{2^{\gamma_1}}{\gamma_1-\alpha-\beta+1}\left(x^2+s_1^2\right)^{\frac{\gamma_1-\alpha-\beta+1}{2}}\Rightarrow\\&I\le\frac{2^{\gamma_1}}{\gamma_1-\alpha-\beta+1}\int_0^y\left(x^2+s_1^2\right)^{\frac{\gamma_1-\alpha-\beta+1}{2}}ds_1\le\frac{2^{\gamma_1}}{\gamma_1-\alpha-\beta+1}\int_0^y(2s_1)^{\gamma_1-\alpha-\beta+1}ds_1\\&I\le\frac{2^{\gamma_1-\alpha-\beta+1}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)}\left(x^2+y^2\right)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}\end{aligned}$$

Solving the other side can give:

$$\begin{aligned}|AU|&\le\frac{2^{\gamma_1-\alpha-\beta+1}\left\|U\right\|}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)}\left(x^2+y^2\right)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}\\|A2U|&=\left|\int_0^y ds_1\int_0^x\frac{c_1^0\left(t_1,s_1\right)}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}AU\left(t_1,s_1\right)dt_1\right|\\&=\frac{2^{\gamma_1-\alpha-\beta+1}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)}\int_0^y ds_1\int_0^x\frac{(t_1^2+s_1^2)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}}{(t_1^2+s_1^2)^{\frac{\alpha+\beta}{2}}}dt_1\Rightarrow\end{aligned}$$

$$|A2U| \leq \frac{2^{6\gamma_1-4\alpha-4\beta+6}\left(x^2+y^2\right)^{\frac{2\gamma_1-2\alpha-2\beta+4}{2}}\|U\|}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)\left(2\gamma_1-2\alpha-2\beta+3\right)\left(2\gamma_1-2\alpha-2\beta+4\right)}.$$

Therefore, it can be found that:

$$|AnU| \leq \frac{2^{(4n-2)\gamma_1-(3n-2)(\alpha+\beta)+5n-4}\|U\|\left(x^2+y^2\right)^{\frac{n(\gamma_1-\alpha-\beta+2)}{2}}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)\ldots n\left(\gamma_1-\alpha-\beta+1\right)}$$

The continuity of the effect A can be studied now as follows [7],

$$\begin{aligned}|AU_1-AU_2| &= \left|\int_0^y ds_1 \int_0^x \frac{\left(\sqrt{t_1^2+s_1^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}\left[U_2\left(t_1,s_1\right)-U_1\left(t_1,s_1\right)\right]dt_1\right| \\ &\leq \frac{2^{2\gamma_1-\alpha-\beta}\left(x^2+y^2\right)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)}\rho\left[U_2\left(x,y\right),U_1\left(x,y\right)\right]\end{aligned}$$

Also, it is chosen that $\varepsilon > 0$

$$\delta = \frac{\varepsilon\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)}{2^{\gamma_1-\alpha-\beta+1}\left(x^2+y^2\right)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}}$$

Next, the conditions is found as follows

$$\rho\left[U_2\left(x,y\right),U_1\left(x,y\right)\right] < \delta.$$

It is found that $\rho\left[AU_2\left(x,y\right),AU_1\left(x,y\right)\right] < \varepsilon$.
Resulting that A influence is continuous. Also, it could be concluded from the other side the following term:

$$|A^nU_2\left(x,y\right)-A^nU_1\left(x,y\right)| \leq \frac{2^{(4n-2)\gamma_1-(3n-2)(\alpha+\beta)+5n+4}\rho\left(U_2,U_1\right)\left(x^2+y^2\right)^{\frac{n(\gamma_1-\alpha-\beta+2)}{2}}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right)n\left[\gamma_1-\left(\alpha+\beta\right)+1\right]}$$

By assuming a big n sufficiently, it implies to:

$$\frac{2^{(4n-2)\gamma_1-(3n-2)(\alpha+\beta)+5n+4}\left(x^2+y^2\right)^{\frac{n(\gamma_1-\alpha-\beta+2)}{2}}}{\left(\gamma_1-\alpha-\beta+1\right)\left(\gamma_1-\alpha-\beta+2\right).....n\left[\gamma_1-\left(\alpha+\beta\right)+1\right]} < 1$$

Then, in accordance with the principle of pressing effects, we can say that A is a pressing effect for large n values. Because there is a single fixed point that is also the single fixed point of the A effect represented solution for the integrated equation (3.4), the successive approximation gives the following relation.:

$$U_{n+1}\left(x,y\right) = F_1\left[\phi_1\left(x\right),\psi_1\left(y\right),f\left(x,y\right)\right] + \int_0^y ds_1 \int_0^x K\left(t_1,s_1\right)U_n\left(t_1,s_1\right)dt_1 \tag{3.6}$$

Satisfied for

$$n = 0 \Rightarrow U_1\left(x,y\right) = F_1\left[\phi_1\left(x\right),\psi_1\left(y\right),f\left(x,y\right)\right]$$

For n=1 we substitute in we find:

$$U_2(x,y) = F_1 + \int_0^y ds_1 \int_0^x K(t_1,s_1)\, U_1(t_1,s_1)\, dt_1$$
$$U_2(x,y) = F_1 + \int_0^y ds_1 \int_0^x K(t_1,s_1)\, F_1(t_1,s_1)\, dt_1.$$

The following relation can be reached:

$$U_{n+1}(x,y) = F_1 + \int_0^y ds_1 \int_0^x K_1(t_1,s_1)\, F_1(t_1,s_1)\, dt_1 + \int_0^y ds_1 \int_0^x K_1(t_1,s_1)\, F_1(t_1,s_1)\, dt_1 + \ldots + \int_0^y ds_1 \int_0^x K_n(t_1,s_1)\, F_1(t_1,s_1)\, dt_1$$

in which,

$$K_1(t_1,s_1) = \frac{\left(\sqrt{t_1^2+s_1^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta}}$$

$$K_2(t_1,s_1) = \int_0^y ds_1 \int_0^x K_1(t_1,s_1) \left[\int_0^{s_1} ds \int_0^{t_1} \frac{\left(\sqrt{t_1^2+s^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s^2}\right)^{\alpha+\beta}} dt\right] dt_1$$

$$\vdots$$

$$K_n(t_1,s_1) = \int_0^y ds_1 \int_0^x K_{n-1}(t_1,s_1) \left[\int_0^{s_1} ds \int_0^{t_1} \frac{\left(\sqrt{t_1^2+s^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s^2}\right)^{\alpha+\beta}} dt\right] dt_1$$

Now, the approximation of the series [4] can be studied as follows,

$$\Gamma_1(t_1,s_1) = K_1(t_1,s_1) + K_2(t_1,s_1) + \ldots + K_n(t_1,s_1) \tag{3.7}$$

leading to:

$$|K_1(t_1,s_1)| = \left|\int_0^y ds_1 \int_0^x \frac{\left(\sqrt{t_1^2+s_1^2}\right)^{\gamma_1}}{\left(\sqrt{t_1^2+s_1^2}\right)^{\alpha+\beta_1}} dt_1\right| \le \frac{2^{\gamma_1-\alpha-\beta+1}\left(x^2+y^2\right)^{\frac{2\gamma_1-\alpha-\beta+2}{2}}}{(\gamma_1-\alpha-\beta+1)(\gamma_1-\alpha-\beta+2)}$$

$$|K_2(t_1,s_1)| \le \frac{2^{6\gamma_1-4\alpha-4\beta+6}\left(x^2+y^2\right)^{\frac{2\gamma_1-2\alpha-2\beta+4}{2}}}{(\gamma_1-\alpha-\beta+1)(\gamma_1-\alpha-\beta+2)(2\gamma_1-2\alpha-2\beta+3)(2\gamma_1-2\alpha-2\beta+4)}$$

$$\vdots$$

$$K_n(t_1,s_1) \le \frac{2^{(4n-2)\gamma_1-(3n-2)(\alpha+\beta)+5n-4}(x^2+y^2)^{\frac{n(\gamma_1-\alpha-\beta+2)}{2}}}{(\gamma_1-\alpha-\beta+1)(\gamma_1-\alpha-\beta+2)\ldots n(\gamma_1-\alpha-\beta+1)}$$
$$\le \frac{2^{(4n-2)\gamma_1-(3n-2)(\alpha+\beta)+5n-4}(\delta_1^2+\delta_2^2)^{\frac{n(\gamma_1-\alpha-\beta+2)}{2}}}{(\gamma_1-\alpha-\beta+1)(\gamma_1-\alpha-\beta+2)\ldots n(\gamma_1-\alpha-\beta+1)} = b_n$$

The approximation of numercal series for b_n. The last relation is represented as the general limit

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=0<1.$$

After that, we multiply the sides of (3.7) series according to the Firestrasse test in a systematic manner:

$$F_1=F_1\left[\phi_1(x),\psi_1(y),f(x,y)\right]$$

Next, the transformers y,x were integrated:

$$\begin{aligned}&\int_0^y ds_1\int_0^x \Gamma_1(t_1,s_1)\,F_1\left[\phi_1(t_1),\psi_1(s_1),f(t_1,s_1)\right]dt_1=\\&\int_0^Y ds_1\int_0^x K_1(t_1,s_1)\,F_1\left[\phi_1(t_1),\psi_1(s_1),f(t_1,s_1)\right]dt_1+\ldots\\&+\int_0^Y ds_1\int_0^X K_n(t_1,s_1)\,F_1\left[\phi_1(t_1),\psi_1(s_1),f(t_1,s_1)\right]dt_1\end{aligned}$$

Consequently

$$U_{n+1}(x,y)=F_1\left[\phi(x),\psi_1(y),f(x,y)\right]+\int_0^y ds_1\int_0^x \Gamma_1(t_1,s_1)\,F_1(x,y;t_1,s_1)\,dt_1$$

That is:

$$\begin{aligned}U(x,y)=&\,F_1\left[\phi_1(x),\psi_1(y),f(x,y)\right]\\&+\int_0^y ds_1\int_0^x \Gamma_1(x,y;t_1,s_1)\,F_1\left[\phi_1(t_1),\psi_1(s_1),f(t_1,s_1)\right]dt_1\end{aligned}\tag{3.8}$$

We can obtain integral Volterra equations with function of solution coefficient $\Gamma_1(x,y;t_1,s_1)$, [8], for the integrated equation by substituting (3.8) in the right side by (3.4)

$$\begin{aligned}&V(x,y)-\int_0^y ds_2\int_0^x e^{\Omega^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)+\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}.(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}\\&c_2^0(t_2,s_2)\,V(t_2,s_2)\,dt_2=e^{W^{\alpha}_{a_2}(x,y)-\Omega^{\alpha}_{a_2}(x,y)}\left\{\phi_2(x)+\int_0^y e^{\Omega^{\alpha}_{a_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)}\right.\\&\left[\psi_2(s_2)+\int_0^x e^{\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}F_1\left(\phi_1(t_2),\psi_1(s_2),f(t_2,s_2)\right)+\right.\\&\left.\left.\int_0^{s_2}ds_1\int_0^{t_2}\Gamma_1(x,y;t_1,s_1).F_1\left[\phi_1(t_1),\psi_1(s_1),f(t_1,s_1)\right]\ dt_1\ dt_2\right]\ ds_2\right\}\end{aligned}$$

The first side of equation is assumed to be equaled to :

$$F_3\left[\phi_1(x),\phi_2(x),\psi_1(y),\psi_2(y),\Gamma_1(x,y)\right]$$

The following equation can be found:

$$\begin{aligned}&V(x,y)-\int_0^y ds_2\int_0^x e^{\Omega^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)+\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}(t_2^2+s_2^2)^{-\frac{(\alpha+\beta)}{2}}\\&c_2^0(t_2,s_2)V(t_2,s_2)dt_2=F_3\left[\phi_1(x),\phi_2(x),\psi_1(y),\psi_2(y),\Gamma_1(x,y)\right]\end{aligned}\tag{3.9}$$

By assuming initially that approximate to zero, The integral equation is solved to achieve the following conditions:

$$e^{\Omega^{\alpha}_{a_2}(x,s_2)-\Omega^{\beta}_{b_2}(x,s_2)-W^{\alpha}_{a_2}(x,s_2)+W^{\beta}_{b_2}(x,s_2)+\Omega^{\beta}_{b_2}(t_2,s_2)-W^{\beta}_{b_2}(t_2,s_2)}c^0_2\left(x,y\right)=0\left(r^{\gamma_2}\right)$$

Similarly, by obtaining the solution for the integrated equation(19) using the relation given as follows is the a solution for the integral equation (3.4):

$$\begin{aligned} V\left(x,y\right)=F_3\left[\phi_1\left(x\right),\phi_2\left(x\right),\psi_1\left(y\right),\psi_2\left(y\right),f\left(x,y\right),\Gamma_1\left(x,y\right)\right]+\int_0^y ds_2\int_0^x \Gamma_2\left(x,y;t_2,s_2\right). \\ F_3\left[\phi_1\left(t_2\right),\psi_1\left(s_2\right),\phi_2\left(t_2\right),\psi_2\left(s_2\right),f\left(t_2,s_2\right),\Gamma_1\left(x,y;t_2,s_2\right)\right]dt_2\,(20) \end{aligned} \tag{3.10}$$

we the following theory is proved as a result of the above procedure:

Theorem 3.4. *Assumes that the coefficients of differential equation* (2.1)*, as well as the right side of the equation, meet the following criteria:*

(i) $a_2(x,y)\in C^3_{xxy}\left(\overline{D}\right),\left\{b_2\left(x,y\right),c_2\left(x,y\right)\right\}\in C\left(\overline{D}\right)$

(ii) $a_1(x,y)\in C^1_x\left(\overline{D}\right),\left\{b_2\left(x,y\right),c_2\left(x,y\right)\right\}\in C\left(\overline{D}\right)$

(iii) $\left|a_j\left(x,y\right)-a_j\left(0,0\right)\right|\le H_{a_j}r^{-\gamma_{j1}};0<\gamma_{j1}<2-\alpha,\quad j=1,2$

(iv) $\left|b_j\left(x,y\right)-b_j\left(0,0\right)\right|\le H_{b_j}r^{-\gamma_{j2}};0<\gamma_{j2}<2-\beta,\quad j=1,2$

(v) $r^{-(\alpha+\beta)}.f\left(x,y\right)=0\left(r^{-\delta}\right);0<\delta<1,\quad for:r\to 0$

(vi) $r^{-(\alpha+\beta)}c^0_j\left(x,y\right)=0\left(r^{-\gamma_j}\right);0<\gamma_j<1,\quad for:r\to 0$
$H_{a_j},H_{b_j}=const$

In which:

$$c^0_j\left(x,y\right)=x.ya_j\left(x,y\right)b_j\left(x,y\right)-c_j\left(x,y\right)+r^{\alpha+\beta}\frac{\partial}{\partial x}\left(y\frac{a_j\left(x,y\right)}{r^{\alpha}}\right)\qquad j=1,2,\alpha<2,\beta<2$$

We determine that there is a single solution for (2.1) equation $\Gamma_1\left(x,y\right),\Gamma_2\left(x,y\right)$coefficient to solve the two integrated equations (3.9) and (3.4), respectively, using relation(20), in addition to achieving the following:

$$\phi_1\left(x\right)\in C\left(\Delta_1\right),\phi_2\left(x\right)\in C^2\left(\Delta_1\right),\psi_1\left(y\right)\in C\left(\Delta_2\right),\psi_2\left(y\right)\in C^1\left(\Delta_2\right)$$

The previous result, which includes the two Theories 3.2 and 3.4, are for the following two states:

In terms of the other values for these two approximations, we found the same results, but under different circumstances from those given in the first and second theorems. The following theorem is found accordingly.

Theorem 3.5. *Assumes that the right side of differential equation* (2.1) *and its coefficients satisfy the following conditions:*

(i) $a_2\left(x,y\right)\in C^3_{xxy}\left(\overline{D}\right),\left\{b_2\left(x,y\right),c_2\left(x,y\right)\right\}\in C^2_{xy}\left(\overline{D}\right)$

(ii) $a_1\left(x,y\right)\in C^1_x\left(\overline{D}\right),\left\{b_1\left(x,y\right),c_1\left(x,y\right),f\left(x,y\right)\right\}\in C\left(\overline{D}\right)$

(iii) $|a_j(x,y) - a_j(0,0)| \leq H_{a_j} r^{-\gamma_{j1}}; 0 < \gamma_{j1} < 2 - \alpha, \quad j = 1,2$

(iv) $|b_j(x,y) - b_j(0,0)| \leq H_{b_j} r^{-\gamma_{j2}}; 0 < \gamma_{j2} < \beta - 2, \quad j = 1,2$
$H_{aj}, H_{bj} = const$

(v) $r^{-(\alpha+\beta)}.f(x,y) = 0\left(r^{-\delta}\right); 0 < \delta < 1, \quad for : r \to 0$

(vi) $e^{-W_{b_j}^{\beta}(x,y)}.r^{-(\alpha+\beta)}.c_j^0(x,y) = 0\left(r^{-\delta_j}\right); 0 < \delta_j < 1, \quad for : r \to 0, \quad j = 1,2$

It is found for the differential equation (2.1) *with relation* (3.10)*, there is single solution as follows :*

$$\phi_2(x) \in C^2(\Delta_1), \psi_2(y) \in C^1(\Delta_2), \phi_1(x) \in C(\Delta_1), \psi_1(y) \in C(\Delta_2)$$

following Theorem can be obtained For $\alpha > 2, \beta < 2$:

Theorem 3.6. *Aassumes the differential equitation* (2.1) *coefficients and its right side satisfy the following criteria:*

(i) $a_2(x,y) \in C_{xxy}^3\left(\overline{D}\right), \{b_2(x,y), c_2(x,y)\} \in C_{xy}^2\left(\overline{D}\right)$

(ii) $a_1(x,y) \in C_x^1\left(\overline{D}\right), \{b_1(x,y), c_1(x,y), f(x,y)\} \in C\left(\overline{D}\right)$

(iii) $|a_j(x,y) - a_j(0,0)| \leq H_{a_j} r^{-\gamma_{j1}}; 0 < \gamma_{j1} < \alpha - 2, \quad j = 1,2$

(iv) $|b_j(x,y) - b_j(0,0)| \leq H_{b_j} r^{-\gamma_{j2}}; 0 < \gamma_{j2} < 2 - \beta, \quad j = 1,2$
$H_{aj}, H_{bj} = const$

(v) $r^{-(\alpha+\beta)}.f(x,y) = 0\left(r^{-\delta}\right); 0 < \delta < 1, \quad for : r \to 0$

(vi) $e^{-W_{b_j}^{\beta}(x,y)}.r^{-(\alpha+\beta)}.c_j^0(x,y) = 0\left(r^{-\delta_j}\right); 0 < \delta_j < 1, \quad for : r \to 0, \quad j = 1,2$

For the differential equation (2.1) *gives with the relation* (3.10)*, there is single solution as follows:*

$$\phi_2(x) \in C^2(\Delta_1), \psi_2(y) \in C^1(\Delta_2), \phi_1(x) \in C(\Delta_1), \psi_1(y) \in C(\Delta_2)$$

The following Theorem can be obtained for $\alpha > 2, \beta > 2$

Theorem 3.7. *Aassumes that the differential equation* (2.1) *coefficients of, as well as the right side of the equation, satisfy the first, second, and fifth criteria, as well as the following condition:*

(i) $a_2(x,y) \in C_{xxy}^3\left(\overline{D}\right), \{b_2(x,y), c_2(x,y)\} \in C_{xy}^2\left(\overline{D}\right)$

(ii) $a_1(x,y) \in C_x^1\left(\overline{D}\right), \{b_1(x,y), c_1(x,y), f(x,y)\} \in C\left(\overline{D}\right)$

(iii) $|a_j(x,y) - a_j(0,0)| \leq H_{a_j} r^{-\gamma_{j1}}; 0 < \gamma_{j1} < \alpha - 2, \quad j = 1,2$

(iv) $|b_j(x,y) - b_j(0,0)| \leq H_{b_j} r^{-\gamma_{j2}}; 0 < \gamma_{j2} < \beta - 2, \quad j = 1,2,$
$H_{aj}, H_{bj} = const$

(v) $r^{-(\alpha+\beta)}.f(x,y) = 0\left(r^{-\delta}\right); 0 < \delta < 1, \quad for : r \to 0$

(vi) $e^{-W_{b_j}^{\beta}(x,y)}.r^{-(\alpha+\beta)}.c_j^0(x,y) = 0\left(r^{-\delta_{j1}}\right); 0 < \delta_{j1} < 1, \quad for : r \to 0, \quad j = 1,2$

(vii) $e^{-W_{a_j}^{\beta}(x,y)}.r^{-(\alpha+\beta)}.c_j^0(x,y) = 0\left(r^{-\delta_{j2}}\right); 0 < \delta_{j2} < 1, \quad for : r \to 0, \quad j = 1,2.$

For equation (2.1) *with relation* (3.10)*, there is single solution as follows:*

$$\phi_2(x) \in C^2(\Delta_1), \psi_2(y) \in C^1(\Delta_2), \phi_1(x) \in C(\Delta_1), \psi_1(y) \in C(\Delta_2)$$

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