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On studying bi-Γ-algebra and some related concepts

Mohaimen M. Abbood[a,*], Ali Al-Fayadh[b], Hassan H. Ebrahim[c]

[a]*Ministry of Education, General Directorate of Education in Diyala, Iraq*
[b] *Department of Mathematicss and Computer Applications, College of Science, Al-Nahraini University, Baghdad, Iraq*
[c]*Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Salahaddin, Iraq*

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Abstract

The aim of this research paper is to introduce the concept of bi-Γ-algebra space (bi-gamma algebra space). The concept of bi-μ-measurable set in a bi-Γ-algebra space is defined. With this concept, some properties of bi-Γ-algebra space are proved. We then define various separation axioms for bi-Γ-algebra space such as $M_0, M_1,\ M_2,\ M_3$, and M_4; then the relationships between them are studied. In addition, the concept of measurable function between two bi-measurable spaces is introduced and some results are discussed.

Keywords: algebra, σ-field, σ-algebra, Γ-algebra, measurable function.

1. Introduction

In 1972, Robert [6] studied the notion of σ–field to define measure and discussed many details about measure and proved several important results in measure theory. Many other authors studying the notion of measure due to its usefulness in the foundation of probability theory [5, 7, 3, 4, 2, 8].

The collection of all subsets of a set Ω, denoted by $\mathcal{P}(\Omega)$, and it is called a power set of Ω. We assume that the complement of a set Ω is the empty set $\varnothing$. A collection $\wp \subseteq \mathcal{P}(\Omega)$ is called σ–field if and only if $\Omega \in \wp$ and $\wp$ is closed under countable union and complementation. A measurable space is defined as a pair $(\Omega, \wp)$ where Ω be a nonempty set and $\wp$ is σ–field of Ω.

In [5], the concept of ring was studied, where a collection $\wp \subseteq \mathcal{P}(\Omega)$ is called ring if whenever $E,\ F \in \wp$, then $E \cup F \in \wp$ and $E - F \in \wp$, where $E \cup F$ denotes the union of E and F, and $E - F$ denotes the difference of E and F. Mohaimen M. et al [1] introduced the concept of Γ-algebra (Γ-field) and studied some related concepts.

In this paper we define the concept of bi-Γ-algebra space and then various separation axioms in bi-Γ-algebra spaces are considered as well as the relationships between them are studied. In addition, the bi-measurable function is introduced and some results are proved.

*Corresponding author
Email addresses: mohaimen.m.abbood35502@st.tu.edu.iq (Mohaimen M. Abbood), aalfayadh@yahoo.com (Ali Al-Fayadh), hassan1962pl@tu.edu.iq (Hassan H. Ebrahim)

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2. Basic Definitions

In this section, we review basic definitions relative to the work and then introduce the definition of bi-Γ-algebra. Various separation axioms such as bi-M_0 , bi-M_1 , bi-M_2 , bi-M_3 , and bi-M_4 are defined in the proposed bi-Γ-algebra space.

Definition 2.1 ([1]). *Let Ω be a nonempty set and $\wp \subseteq \mathcal{P}(\Omega)$. A nonempty collection $\wp$ of subsets of a set Ω is called Γ-algebra or (Γ-field) if the following conditions are satisfied:*

1. $\varnothing, \Omega \in \wp$.
2. *If $D \in \wp$ and there exist $\varnothing \neq E_i \subset D \subset \Omega$ then at least one of $E_i' s \in \wp$.*
3. *If $D_1, D_2, \ldots \in \wp$, then $\bigcup_{i=1}^{\infty} D_i \in \wp$.*

Definition 2.2 ([1]). *If $\wp$ is a Γ-algebra on a set Ω. A pair $(\Omega, \wp)$ is called measurable space relative to the Γ-algebra $\wp$ and the elements of $\wp$ are called the measurable sets.*

Definition 2.3. *A Γ-algebra on a set Ω is said to be discrete-Γ-algebra provided, if $A \subseteq \Omega$, then A is a measurable set.*

Definition 2.4. *Let Ω be a nonempty set and $P, Q \subseteq \mathcal{P}(\Omega)$ are two Γ-algebras on a set Ω such that $P \neq Q$. The ordered triple (Ω, P, Q) is said to be a bi-Γ-algebra.*

Example 2.5. *Let $\Omega = \{a, b, c\}$. Define P and Q as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$; *and* $Q = \{\varnothing, \{a\}, \{a, c\}, \Omega\}$.
Then (Ω, P, Q) is a bi-Γ-algebra of a set Ω since P and Q are two Γ-algebras of a set Ω.

Definition 2.6. *Let $P \neq Q$ are two Γ-algebras on a set Ω. A triple (Ω, P, Q) is called bi-measurable space relative to the Γ-algebras P and Q; and the elements of (Ω, P, Q) are called bi- measurable sets.*

Definition 2.7. *Let (Ω, P, Q) be a bi-Γ-algebra space. A subset U of Ω is said to be bi-μ-measurable set of (Ω, P, Q) if $U = \varnothing$ or there exist measurable sets $\varnothing \neq V \in P$ and $\varnothing \neq W \in Q$ such that $V, W \subseteq U$.*

Definition 2.8. *Let (Ω, P, Q) be a bi-Γ-algebra space. A subset C of Ω is said to be bi-μ^*-measurable set if its complement $\Omega - C$ (or C^c) is bi-μ-measurable set.*

Example 2.9. *Let $\Omega = \{a, b, c, d\}$. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \Omega\}$; *and*
$Q = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \Omega\}$.
Then (Ω, P, Q) is a bi-Γ-algebra on a set Ω since P and Q are two Γ-algebras of a set Ω.
If $U = \{a, b, d\} \subset \Omega$, then U is a bi-μ-measurable set since $\varnothing \neq \{a\} \in P$ and $\varnothing \neq \{b\} \in Q$ such that $\{a\} \subset U$ and $\{b\} \subset U$. But $U = \{d\} \subset \Omega$ is not a bi-μ-measurable set.

Definition 2.10. *A bi-Γ-algebra space (Ω, P, Q) is said to be bi-discrete provided, if $A \subset \Omega$, then A is bi-μ-measurable set.*

Theorem 2.11. *A bi-*Γ*-algebra space* (Ω, P, Q) *is bi-discrete if and only if each of* (Ω, P) *and* (Ω, Q) *is discrete.*

Proof . *Let* (Ω, P, Q) *is a bi-*Γ*-algebra on a set* Ω*. Suppose that* (Ω, P, Q) *is a bi-discrete. Let* $x \in \Omega$*. Since* (Ω, P, Q) *is a bi-discrete ,* $\{x\}$ *is a bi-*μ*-measurable set. There are thus measurable sets* $\varnothing \neq V \in P$ *and* $\varnothing \neq W \in Q$ *such that* $V, W \subseteq \{x\}$*. Therefore* $V = \{x\}$ *and* $W = \{x\}$*. Then* $\{x\} \in P$ *and* $\{x\} \in Q$*. So by definition of discrete,* (Ω, P) *and* (Ω, Q) *are discrete.*
Suppose (Ω, P) *and* (Ω, Q) *are discrete* Γ*-algebras on a set* Ω*, and let* $A \subset \Omega$*. Since* (Ω, P) *and* (Ω, Q) *are discrete then* $A \in P$ *and* $A \in Q$ *. Since* A *is a subset of* A *then* A *is a bi-*μ*-measurable set. Therefore* (Ω, P, Q) *is a bi-discrete bi-*Γ*-algebra space by Definition 2.10* □

Example 2.12. *Let* $\Omega = \{a, b, c\}$ *is a bi-*Γ*-algebra on a set* Ω*. Define the two collections* P *and* Q *as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \Omega\}$ *and* $Q = \{\varnothing, \{b\}, \{b, c\}, \Omega\}$*. Then* (Ω, P, Q) *is not bi-discrete. The subset* $U = \{a, c\}$ *of* Ω *is not a bi-*μ*-measurable set. There is no* $\varnothing \neq W \in Q$ *such that* $W \subseteq U$*.*
In the following we define various bi-separation axioms on bi-Γ*-algebra spaces and study the relationships between them.*

Definition 2.13. *A bi-*Γ*-algebra space* (Ω, P, Q) *is said to be bi-*M_0*, if for any two distinct points of a bi-*Γ*-algebra space* (Ω, P, Q)*, at least one of them has a bi-*μ*-measurable set which does not contain the other point.*

Example 2.14. *Let* $\Omega = \{a, b, c\}$ *is a bi-*Γ*-algebra on a set* Ω*. Define the two collections* P *and* Q *as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$ *and* $Q = \{\varnothing, \{a\}, \Omega\}$*. Then* (Ω, P, Q) *is a bi-*M_0*. The set of all bi-*μ*-measurable subsets of* Ω *is* $\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \Omega\}$*.*

Example 2.15. *Let* $\Omega = \{a, b, c\}$ *is a bi-*Γ*-algebra on a set* Ω*. Define the two collections* P *and* Q *as follows;*
$P = \{\varnothing, \{a\}, \{a, b\}, \Omega\}$ *and* $Q = \{\varnothing, \{b\}, \Omega\}$*. Then* (Ω, P, Q) *is Not a bi-*M_0*. The set of all bi-*μ*-measurable subsets of* Ω *is* $\{\varnothing, \{a, b\}, \Omega\}$*. There does not exist a bi-*μ*-measurable subsets of* Ω *which contains* a *and does not contain* b *or which contains* b *and does not contain* a*.*

Definition 2.16. *A bi-*Γ*-algebra space* (Ω, P, Q) *is said to be bi-*M_1*, if for any two distinct points of a bi-*Γ*-algebra space* (Ω, P, Q)*, each has a bi-*μ*-measurable set not containing the other point.*

Example 2.17. *Let* $\Omega = \{a, b, c\}$ *is a bi-*Γ*-algebra on a set* Ω*. Define the two collections* P *and* Q *as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \Omega\}$ *and* $Q = \{\varnothing, \{b\}, \{c\}, \{b, c\}, \Omega\}$*. Then* (Ω, P, Q) *is a bi-*M_1*. The set of all bi-*μ*-measurable subsets of* Ω *is* $\{\varnothing, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \Omega\}$*.*

Example 2.18. *Let* $\Omega = \{a, b, c\}$ *is a bi-*Γ*-algebra on a set* Ω*. Define the two collections* P *and* Q *as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$ *and* $Q = \{\varnothing, \{a\}, \Omega\}$*. Then* (Ω, P, Q) *is Not a bi-*M_1*. The set of all bi-*μ*-measurable subsets of* Ω *is* $\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \Omega\}$*. It is clear that there does not exist bi-*μ*-measurable subsets of* Ω *which contains* b *and does not contain* a*.*

Theorem 2.19. *If a bi-*Γ*-algebra space* (Ω, P, Q) *is a bi-*M_1*, then it is a bi-*M_0*.*

Proof . *Let* (Ω, P, Q) *be a bi-*M_1 *space. Suppose* $p \neq q \in \Omega$*, then by Definition 2.16, each of* p *and* q *has a bi-*μ*-measurable set not containing the other point. Thus* (Ω, P, Q) *be a bi-*M_0 *space.* □

Remark 2.20. *The converse of Theorem 2.19 need not be true as Examples 2.14 and 2.18 show.*

Theorem 2.21. *If a bi-Γ-algebra space (Ω, P, Q) is a bi-M_1, and $x \in \Omega$, then $\{x\}$ is a bi-μ^* measurable set.*

Proof . *Suppose (Ω, P, Q) is a bi-M_1, and $x \in \Omega$. Let $p \in \Omega - \{x\}$, then $p \neq x$. Since (Ω, P, Q) is a bi-M_1, there is a bi-μ-measurable subsets U_p of Ω such that $p \in U_p$ and $x \notin U_p$. Then $p \in U_p \subset \Omega - \{x\}$. Therefore, for each $p \in \Omega$, there exists a bi-μ-measurable subsets U_p of Ω which contains p and is contained in the complement of $\{x\}$. It remains to show that $\Omega - \{x\} = \cup\{U_p : \ p \in \Omega - \{x\}\}$. Let $y \in \Omega - \{x\}$. there is a bi-μ-measurable subsets U_y of Ω such that $y \in U_y$ which is contained in $\Omega - \{x\}$. Thus $y \in \cup\{U_p : \ p \in \Omega - \{x\}\}$, and hence $\Omega - \{x\} \subset \cup\{U_p : \ p \in \Omega - \{x\}\}$.*
Now, let $y \in \cup\{U_p : \ p \in \Omega - \{x\}\}$. Then, there is a U_p such that $y \in U_p \subset \Omega - \{x\}$, and hence $y \in \Omega - \{x\}$. Therefore, $\cup\{U_p : \ p \in \Omega - \{x\}\} \subset \Omega - \{x\}$. Hence $\Omega - \{x\} = \cup\{U_p : \ p \in \Omega - \{x\}\}$. Since U_p is a bi-μ-measurable subsets of Ω, then $\cup\{U_p : \ p \in \Omega - \{x\}\}$ is a bi-μ-measurable subsets of Ω. This implies that $\Omega - \{x\}$ is a bi-μ-measurable subsets of Ω. Then $\{x\}$ is a bi-μ^* measurable subsets of Ω. $\square$

Definition 2.22. *A bi-Γ-algebra space (Ω, P, Q) is said to be bi-M_2, if for any two distinct points of a bi-Γ-algebra space (Ω, P, Q), each has a bi-μ-measurable set which does not intersect the other.*

Example 2.23. *Let $\Omega = \{a, b, c, d\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \Omega\}$ *and* $Q = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \Omega\}$.
Then (Ω, P, Q) is a bi-M_2. The set of all bi-μ-measurable subsets of Ω is
$\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \Omega\}$.

Example 2.24. *Let $\Omega = \{a, b, c\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$ *and* $Q = \{\varnothing, \{a\}, \Omega\}$. *Then (Ω, P, Q) is Not a bi-M_2. The set of all bi-μ-measurable subsets of Ω is $\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \Omega\}$. For $a \neq b \in \Omega$, there are no two disjoint bi-μ-measurable subsets U and V of Ω such that $a \in U$ and $b \in V$.*

Theorem 2.25. *If a bi-Γ-algebra space (Ω, P, Q) is a bi-M_2, then it is a bi-M_1.*

Proof . *Suppose (Ω, P, Q) is a bi-M_2, and $p \neq q \in \Omega$. Since (Ω, P, Q) is a bi-M_2 space, there are two disjoint bi-μ-measurable subsets U and V of Ω such that $p \in U$ and $q \in V$. But $U \cap V = \varnothing$ implies that $q \notin U$. Thus there is a bi-μ-measurable subsets U such that $p \in U$ and $q \notin U$. Thus (Ω, P, Q) is a bi-M_1 space.* $\square$

Remark 2.26. *The converse of Theorem 2.25 need not be true as the following example shows.*

Example 2.27. *Let $\Omega = \{a, b, c\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \Omega\}$ *and* $Q = \{\varnothing, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \Omega\}$.
Then (Ω, P, Q) is a bi-M_1 but it is Not a bi-M_2 space. The set of all bi-μ-measurable subsets of Ω is $\{\varnothing, \{a, b\}, \{b, c\}, \{c\}, \{a, c\}, \Omega\}$. For $a \neq b \in \Omega$, there are no two disjoint bi-μ-measurable subsets U and V of Ω such that $a \in U$ and $b \in V$.

Definition 2.28. *A bi-Γ-algebra space (Ω, P, Q) is said to be bi-$\mathbf{M_3}$, if (Ω, P, Q) is a bi-M_1 space and such that if $p \in \Omega$ and F is a bi-μ^* measurable subset of Ω with $p \notin F$, then there exists two disjoint bi-μ-measurable subsets U and V of Ω with $p \in U$ and $F \subseteq V$.*

Note that, a bi-Γ-algebra space (Ω, P, Q) in Example 2.23 is a bi-M_3 space, while a bi-Γ-algebra space (Ω, P, Q) in Example 2.27 is Not a bi-M_3 space since $\{a\}$ is a bi-μ^* measurable subset of Ω with $b \notin \{a\}$ but there are no two disjoint bi-μ-measurable subsets U and V of Ω such that $b \in U$ and $\{a\} \subseteq V$.

Theorem 2.29. *If a bi-Γ-algebra space (Ω, P, Q) is a bi-M_3, then it is a bi-M_2 space.*
Proof . *Suppose (Ω, P, Q) is a bi-M_3 , and $p \neq q \in \Omega$. Since (Ω, P, Q) is a bi-M_3 space, then (Ω, P, Q) is a bi-M_1 space. Thus by Theorem 2.21, $\{p\}$ is a bi-μ^* measurable subset of Ω. Therefore, there are two disjoint bi-μ-measurable subsets U and V of Ω such that $q \in U$ and $\{p\} \subseteq V$. Since $\{p\} \subseteq V$, $p \in V$. So $q \in U$, $p \in V$ and $U \cap V = \varnothing$. Therefore, (Ω, P, Q) is a bi-M_2 space.* $\square$

Remark 2.30. *The converse of Theorem 2.29 need Not be true as the following example shows.*

Example 2.31. *Let $\Omega = \{1, 2, 3, 4, 5\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{1\}, \{5\}, \{1,5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,5\}, \{1,2,3\}, \{2,3,5\}, \{1,3,5\}, \{1,2,3,5\}, \Omega\}$, *and* $Q = \{\varnothing, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}, \Omega\}$. *Then (Ω, P, Q) is a bi-M_2.*
The set of all bi-μ-measurable subsets of Ω is,

$$\left\{\begin{array}{c}\varnothing, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,5\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \\ \{2,3,4\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{2,3,4,5\}, \{1,3,4,5\}, \\ \{1,2,4,5\}, \Omega\end{array}\right\}$$

The set of all bi-μ^* measurable subsets of Ω is.

$$\left\{\begin{array}{c}\Omega, \{3,4,5\}, \{2,4,5\}, \{2,3,5\}, \{1,4,5\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3\}, \{4,5\}, \{3,5\}, \{3,4\}, \{2,5\}, \\ \{2,4\}, \{1,5\}, \{1,4\}, \{2,3\}, \{1,3\}, \{1,2\}, \{5\}, \{4\}, \{1\}, \{2\}, \{3\}, \varnothing\end{array}\right\}$$

Then (Ω, P, Q) is a bi-M_2 but it is Not a bi-M_3 since $4 \in \Omega$, $4 \notin \{1,5\}$ and there are no two disjoint bi-μ-measurable subsets U, V of Ω such that $4 \in U$ and $\{1,5\} \subseteq V$.

Definition 2.32. *A bi-Γ-algebra space (Ω, P, Q) is said to be bi-M_4, if (Ω, P, Q) is a bi-M_1 space and such that for any two disjoint bi-μ^* measurable subsets F and G of Ω , then there exists two disjoint bi-μ-measurable subsets U and V of Ω with $F \subseteq U$ and $G \subseteq V$.*

The following example is a bi-M_4.

Example 2.33. *Let $\Omega = \{1, 2, 3, 4\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$
and $Q = \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{2,3,4\}, \Omega\}$. *Then (Ω, P, Q) is a bi-M_4. The set of all bi-μ-measurable subsets of Ω is*
$\{\varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}, \Omega\}$.
The set of all bi-μ^* measurable subsets of Ω is.
$\{\Omega, \{2,3,4\}, \{1,3,4\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,4\}, \{1,3\}, \{4\}, \{3\}, \{1\}, \{2\}, \varnothing\}$.

Remark 2.34. *Note that, a bi-Γ-algebra space (Ω, P, Q) in Example 2.31 is Not a bi-M_4 space because if we take two disjoint bi-μ^* measurable subsets $F = \{2,4\}$ and $G = \{1,5\}$ of Ω there are no two disjoint bi-μ-measurable subsets U and V of Ω with $F \subseteq U$ and $G \subseteq V$.*

Theorem 2.35. *If a bi-Γ-algebra space (Ω, P, Q) is a bi-M_4, then it is a bi-M_3 space.*

Proof . *Suppose (Ω, P, Q) is a bi-M_4 , and $a \in \Omega$. Let G be a bi-μ^* measurable subset of Ω such that $a \notin G$. Since (Ω, P, Q) is a bi-M_4 space, then (Ω, P, Q) is a bi-M_1 space, and $\{a\}$ is a bi-μ^* measurable set by Theorem 2.21. Then there exist two disjoint bi-μ-measurable subsets U and V of Ω such that $\{a\} \subseteq U$ and $G \subseteq V$. But $\{a\} \subseteq U$ implies that $a \in U$. Thus $a \in U$ and $G \subseteq V$ and $U \cap V = \varnothing$. Hence by Definition 2.28, (Ω, P, Q) is a bi-M_3 space.* $\square$

Remark 2.36. *The converse of Theorem 2.34 need Not be true as the following example shows.*

Example 2.37. *Let $\Omega = \{1, 2, 3, 4, 5\}$ is a bi-Γ-algebra on a set Ω. Define the two collections P and Q as follows;*
$P = \{\varnothing, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}, \{3, 4\}, \Omega\}$,
and $Q = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{5\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}, \Omega\}$.
The set of all bi-μ-measurable subsets of Ω is,

$$\left\{\begin{array}{c}\varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \\ \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \\ \{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \Omega\end{array}\right\}$$

The set of all bi-μ^* measurable subsets of Ω is,

$$\left\{\begin{array}{c}\Omega, \{2,3,4,5\}, \{3,4,5\}, \{2,4,5\}, \{2,3,5\}, \{2,3,4\}, \{1,4,5\}, \{1,3,5\}, \{1,2,4\}, \{1,2,3\}, \{4,5\}, \\ \{3,5\}, \{3,4\}, \{2,5\}, \{2,4\}, \{1,5\}, \{1,4\}, \{2,3\}, \{1,3\}, \{1,2\}, \{5\}, \{4\}, \{1\}, \{2\}, \{3\}, \varnothing\end{array}\right\}$$

Then (Ω, P, Q) is a bi-M_3 but it is Not a bi-M_4 since $\{3, 4\}$ and $\{2, 5\}$ are two disjoint bi-μ^* measurable subsets of Ω and there are no two disjoint bi-μ-measurable subsets U, V of Ω such that $\{3, 4\} \subseteq U$ and $\{2, 5\} \subseteq V$.

3. Measurable Functions on bi-Γ-algebra space

Definition 3.1. *Let (X, P, Q) and (Y, W, Z) are two bi-Γ-algebra spaces and $f : X \to Y$ be a function. Then f is said to be a bi-measurable function provided, if U is a bi-μ-measurable subset of Y, then $f^{-1}(U)$ is a bi-μ-measurable subset of X .*

Example 3.2. *Let $X = \{a, b, c\}$. Define the two collections P and Q on X as follows;*
$p = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, *and* $Q = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$.
Let $Y = \{r, s, t\}$. Define the two collections C and D on Y as follows;
$W = \{\varnothing, Y, \{r\}, \{r, s\}\}$, *and* $Z = \{\varnothing, Y, \{r\}, \{t\}, \{r, t\}\}$. *Define $f : X \to Y$ as the following;*
$f(a) = r; f(b) = s; f(c) = t$. *Then f is a bi-measurable function.*

Proof . *The set of all bi-μ-measurable subsets of X is $\{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, and the set of all bi-$\mu$-measurable subsets of Y is $\{\varnothing, Y, \{r\}, \{r, s\}, \{r, t\}\}$. Then;*
$f^{-1}(\varnothing) = \varnothing; f^{-1}(Y) = X;\ f^{-1}(\{r\}) = \{a\};\ f^{-1}(\{r, s\}) = \{a, b\};\ f^{-1}(\{r, t\}) = \{a, c\}$.
Note that for every bi-μ-measurable subset U of Y, then $f^{-1}(U)$ is a bi-μ-measurable subset of X. Therefore f is a bi-measurable function. $\square$

Example 3.3. *Let $X = \{a, b, c\}$. Define the two collections P and Q on X as follows;*
$p = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ *and* $Q = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$. *Then the set of all bi-μ-measurable subsets of X is $\{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $Y = \{r, s, t\}$. Define the two collections W and Z on Y as follows;*

$W = \{\varnothing, Y, \{r\}, \{r,s\}\}$, and $Z = \{\varnothing, Y, \{r\}, \{t\}, \{r,t\}\}$. Then the set of all bi-μ-measurable subsets of Y is $\{\varnothing,\ Y, \{r\}, \{r,s\}, \{r,t\}\}$. Define $g: Y \to X$ as the following;
$g(r) = b; g(s) = a; g(t) = c$. Then g is Not a bi-measurable function.

Proof . *By definition of inverse*
$g^{-1}(\varnothing) = \varnothing; g^{-1}(X) = Y;\ g^{-1}(\{a\}) = \{s\};\ g^{-1}(\{a,b\}) = \{r,s\};\ g^{-1}(\{a,c\}) = \{s,t\}; g^{-1}(\{b,c\}) = \{r,t\}$. Note that $\{a\}$ is a bi-μ-measurable subset of X but $g^{-1}(\{a\}) = \{s\}$ is not a bi-μ-measurable subset of Y. Therefore g is Not a bi-measurable function. □

Definition 3.4. *Let $(X,\ P,\ Q)$ and (Y, W, Z) are two bi-Γ-algebra spaces. Let $f: X \to Y$ be a mapping and $x \in X$. Then f is said to be bi-measurable function at* **x** *provided, if given any bi-μ-measurable subset V of Y, $f(x) \in V$, then there exists a bi-μ-measurable subset U of X such that $x \in U$ and $f(U) \subset V$.*

Example 3.5. *Let $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ be defined as in Example 3.3. Define g, mapping Y into X, by $g(r) = b; g(s) = a; g(t) = c$. Then g is a bi-measurable function at a point and is Not a bi-measurable function.*

Proof . *Since $\{r\}$ is bi-μ-measurable subset of Y, and each bi-μ-measurable subset of X which contains $g(r) = b$ also contains $g(\{r\}) = \{b\}$, then g is a bi-measurable function at $r \in Y$. It was shown, however, that g is Not a bi-measurable function.* □

Theorem 3.6. *If $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ are two bi- Γ–algebra spaces, then f, mapping X into Y, is a bi-measurable if and only if f is a bi-measurable function at each point of X.*

Proof . *Suppose $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ are two bi- Γ–algebra spaces. Let f map X into Y. Suppose that f is a bi-measurable. Let $x \in X$. Let V be a bi-μ-measurable subset of Y such that $f(x) \in V$. Since $f(x) \in V$, $x \in f^{-1}(V)$. But f is a bi-measurable, therefore $f^{-1}(V)$ is a bi-μ-measurable subset of X. Thus $x \in f^{-1}(V)$ and $f(f^{-1}(V)) \subset V$. Therefore f is a bi-measurable at x. Suppose that f is a bi-measurable function at each point $x \in X$. Let U be a bi-μ-measurable subset of Y. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. By Definition 3.1, there is a bi-μ-measurable subset $G_x \subset X$ such that $x \in G_x$, and $G_x \subset f^{-1}(U)$. Such a bi-μ-measurable subset can be found for each x in X. It remains to show that $f^{-1}(U) = \cup\{G_x\ : x \in f^{-1}(U)\}$. Let $y \in f^{-1}(U)$. Then $f(y) \in U$. So $y \in G_y \subset\ \cup\{G_x\ : x \in f^{-1}(U)\}$. Thus $f^{-1}(U) \subset\ \cup\{G_x\ : x \in f^{-1}(U)\}$. Let $y \in\ \cup\{G_x\ : x \in f^{-1}(U)\}$. Then there is an $x \in X$ such that $y \in G_x \subset f^{-1}(U)$. Therefore $y \in f^{-1}(U)$. Hence $\cup\{G_x\ : x \in f^{-1}(U)\} \subset f^{-1}(U)$. So $f^{-1}(U) =\ \cup\{G_x\ : x \in f^{-1}(U)\}$.*
Since each G_x is a bi-μ-measurable subset, $f^{-1}(U)$, which is a union of bi-μ-measurable subsets, is a bi-μ-measurable subset. Thus f is bi-measurable. □

Theorem 3.7. *If $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ are two bi- Γ–algebra spaces and f, mapping X into Y, is bi-measurable, and F is a bi-μ^* measurable subset of Y, then $f^{-1}(F)$ is a bi-μ^* measurable subset of X.*

Proof . *Let $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ are two bi- Γ–algebra spaces. Let f, mapping X into Y, be bi-measurable. Let F be a bi-μ^* measurable subset of Y. Since F is a bi-μ^* measurable subset, then $Y\ -\ F$ is a bi-μ-measurable subset of Y. Thus $f^{-1}(Y - F)$ is a bi-μ-measurable subset of X, and $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$. Since $X - f^{-1}(F) = f^{-1}(Y - F)$, then $X - f^{-1}(F)$ is a bi-μ-measurable subset of X. But then $f^{-1}(F) = X - (X - f^{-1}(F))$ is a bi-μ^* measurable subset of X.* □

Theorem 3.8. *If $(X,\ P,\ Q)$ and $(Y,\ W,\ Z)$ are two bi- Γ–algebra spaces, and f, mapping X into Y, is such that, if F is a bi-μ^* measurable subset of Y, then $f^{-1}(F)$ is a bi-μ^* measurable subset of X, then f is a bi-measurable function.*

Proof . *Let* $(X,\ P,\ Q)$ *and* $(Y,\ W,\ Z)$ *are two bi-* $\Gamma-algebra$ *spaces. Let* f*, mapping* X *into* Y*, be such that, if* F *is a bi-*μ^* *measurable subsets of* Y*, then* $f^{-1}(F)$ *is a bi-*μ^* *measurable subsets of* X*. Let* U *be a bi-measurable subset of* Y*. Then* $U = Y - (Y - U)$ *and* $(Y - U)$ *is a bi-*μ^* *measurable subset of* Y*. So* $f^{-1}(U) = f^{-1}(Y - (Y - U)) = f^{-1}(Y) - f^{-1}(Y - U) = X - f^{-1}(Y - U)$*. Since* $(Y - U)$ *is a bi-*μ^* *measurable subset of* Y*,* $f^{-1}(Y - U)$ *is a bi-*μ^* *measurable subset of* X*. Then* $f^{-1}(U) = X - f^{-1}(Y - U)$*, which is a bi-*$\mu$*-measurable subset of* X*.*
Therefore f *is a bi-measurable.* □

4. Conclusion

In this study, we introduce the concept of "bi-Γ-algebra space (bi-gamma algebra space)" and then generalized some basic properties of classic Γ-algebra. We then define various separation axioms for bi-Γ-algebra space and study the relationships between them. We proved that each bi- M_i axiom is bi-M_{i-1} $(i = 1, 2, 3, 4)$ and the converse need not true. Finally, the concept of bi-measurable function was introduced.

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