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On the maximum likelihood, Bayes and expansion estimation for the reliability function of Kumaraswamy distribution under different loos function

Mohammed A. Mahmoud[a,*], Sudad K. Abraheem[a], Amal A. Mohammed[b]

[a] *Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq*
[b] *Department of Ways and Transportation, College of Engineering, Mustansiriyah University, Baghdad, Iraq*

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Abstract

This work deals with Kumaraswamy distribution. Maximum likelihood, Bayes and expansion methods of estimation are used to estimate the reliability function. A symmetric Loss function (De-groot and NLINEX) are used to find the reliability function based on four types of informative prior three double priors and one single prior. In addition expansion methods (Bernstein polynomials and Power function) are applied to find reliability function numerically. Simulation research is conducted for the comparison of the effectiveness of the proposed estimators. Matlab (2015) will be used to obtain the numerical results.

Keywords: Kumaraswamy distribution, Maximum likelihood, Bayes, expansion methods.

1. Introduction

Poondi Kumaraswamy (1980) [12] proposed a new probability distribution for variables that are lower and upper bounded. In probability and statistic, the Kumarasway's double bound distribution is a family of continuous probability distributions defined on the interval (0,1) differing in the values of their two non-negative shape parameters, $\varnothing$ and ν. In reliability and life testing experiments,

*Corresponding author
Email addresses: mahommedadnan1993@gmail.com (Mohammed A. Mahmoud), sudad1968@uomustansiriyah.edu.iq (Sudad K. Abraheem), dr.amal70@uomustansiriyah.edu.iq (Amal A. Mohammed)

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many times the data are modelled by finite range distribution. Many authors studied and developed the generalizations of Kumaraswamy distribution (KD) such as exponential KW distribution studies by Lemonte et al. (2013) [13]. Eugene et al. (2002) [7] and Jones (2004) [11] constructed a new class of Kumaraswamy generalized distribution (kw-G distribution) on the interval (0, 1). Bantan et al. (2019) [4] introduced truncated inverted KD.

Mohammed, A.A et al. (2018) [16]: Al-Bayyati's loss function, and suggest loss function are used to find the reliability function. In addition find the numerical estimator by expanded the reliability function in terms of a set of power function. Sultana et al (2018) [23] discussed and estimated parameters of the KD with hybrid censoring scheme. Abraheem S.k. et al. (2020) [1] Reliability function are obtained using asymmetric loss function by using three types of informative priors two single prior and one double priors. Also, numerical solution of reliability function is found using expansion method.

In this work, deriving some estimators of the reliability function of KD using "maximum likelihood estimator" as the classical method, Bayes estimator by assuming joint informative priors represented by (gamma- exponential, gamma-chi-squared, chi-squared-exponential and gamma priors) under asymmetric loss functions (De-groot and NLENEX loss functions) and using expansion methods (Bernstein polynomials and Power function) to estimate the reliability function. Compare the efficiency of these estimators, using Mont-Carlo simulation method in terms of integrated mean squared error (IMSE).

The probability density function of KD random variable is given by [8].

$$f(t;\varnothing,v) = (\varnothing \;\; v \;\; t^{v-1}(1-t^{v})^{\varnothing-1}; \qquad 0<t \;<1\;, \quad \varnothing, v>0 \tag{1.1}$$

where $\varnothing$ and v are shape parameters respectively. Here we assume that v is the known shape parameter. The corresponding cumulative distribution function (cdf) is given by:

$$F(t;\varnothing,v) = 1-(1-t^{v})^{\varnothing}; \qquad 0<t \;<1; \quad \varnothing, \;\; v>0 \tag{1.2}$$

The reliability and failure rate functions of KD are given, respectively by:

$$R(t) = 1-F(t;\varnothing,v) = (1-t^{v})^{\varnothing}; \qquad 0<t \;<1 \;\; ; \;\; \varnothing, \;\; v>0 \tag{1.3}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\varnothing \;\; v \;\; t^{v-1}}{1-t^{v}}; \qquad 0<t \;<1 \;\; ; \;\; \varnothing, \;\; v>0 \tag{1.4}$$

2. Maximum Likelihood Estimator Method (MLEM)

The maximum likelihood method is attributed to Fisher. However, the method can be traced back to the works the 18-century scientists Lambert and Bernoulli. Fisher introduced the method of maximum likelihood in his first statistical publications in (1912) [6], and he developed it in [22]. Let $t = (t_1, t_2, \ldots, t_n)$ be the life time of random sample of size n drawn independently from KD defined by (1.1). Then the likelihood function for the given sample observations is:

$$L(\varnothing, v\,|t) = \prod_{i=1}^{n} f(t_i\,|\varnothing, v\;) = \varnothing^{v} v^{n} \prod_{i=1}^{n} [(t_i)^{v-1}\,(1-t_i^{v})^{\varnothing-1}]$$

Then

$$L(\varnothing, v\,|t) = \;\varnothing^{n} v^{n} e^{(v-1)\sum_{i=1}^{n}\ln(t_i)} e^{(\varnothing-1)\sum_{i=1}^{n}\ln(1-t_i^{v})} \tag{2.1}$$

We take the natural logarithm for the likelihood function so we get the function:

$$\ln L(\varnothing, v\,|t) = n\ln\varnothing + n\ln v(v-1)\sum_{i=1}^{n}\ln(t_i) + (\varnothing-1)\sum_{i=1}^{n}\ln(1-t_i^v) \tag{2.2}$$

The partial derivative for log-likelihood function with respect to unknown parameter $\varnothing$ is:

$$\frac{\partial \ln L}{\partial\varnothing} = \frac{n}{\varnothing} + \sum_{i=1}^{n}\ln(1-t_i^v) \tag{2.3}$$

Then we equate the partial derivate force to zero, and get the following formula:

$$\begin{aligned} &\frac{n}{\varnothing} + \sum_{i=1}^{n}\ln(1-t_i^v) = 0 \\ &\widehat{\varnothing}_{\mathrm{ML}} = \frac{-n}{\sum_{i=1}^{n}\ln(1-t_i^v)} = \frac{-n}{T}, \quad \textit{where } T = \sum_{i=1}^{n}\ln(1-t_i^v) \end{aligned} \tag{2.4}$$

Then

$$\widehat{R}(t)_{\mathrm{ML}} = (1-t^v)^{\widehat{\varnothing}_{\mathrm{ML}}} \tag{2.5}$$

3. Standard Bayes Estimator Method (SBEM)

The standard Bayesian estimator method (SBEM) assumes that the random sample $t_1, t_2, \ldots t_n$ taken from population with pdf $f(t;\varnothing)$. However the unknown parameter $\varnothing$ is considered to be as random variable in some real situation. There is often additional information available about $\varnothing$ (that means there is a prior knowledge exit about the parameter $\varnothing$. This method is based on the notation g($\varnothing$) that represent the prior distribution for parameter $\varnothing$ which come from prior knowledge, additional information and past experience. The Bayes estimator depend on the probability density function (posterior pdf) which includes information from previous knowledge and sample information [21]. The steps of standard Bayes estimator method are as the followings:

1. Finding conditional density function for parameter $\varnothing$ given the random variable $t_1, t_2, \ldots t_n$:

$$\pi(\varnothing\,|t\ ) = \frac{L(\varnothing\,|t\ )\,g(\varnothing)}{\int_{\varnothing} L(\varnothing\,|t\ )\,g(\varnothing)\,d\varnothing} \tag{3.1}$$

   is called posterior density function there are two variables terms in Eq. (3.1): one term is the likelihood function L($\varnothing\,|t$), and the second is the prior probability of the parameter, g($\varnothing$) [3].

2. Using loss functions $L(\widehat{\varnothing}, \varnothing)$ which is defined to be real function satisfying:
   (a) $L(\widehat{\varnothing}, \varnothing) \geq 0$ for all possible estimator $\widehat{\varnothing}$ and all parameter $\varnothing$.
   (b) $L(\widehat{\varnothing}, \varnothing) = 0$ for $\widehat{\varnothing} = \varnothing$ [20].

   There are two function which as the following:

   - The De-groot loss function (weighted balance loss function) [2, 5].
   - Non-linear exponential (NLINEX) loos function [10].

3. Finding the risk function for parameter $\widehat{\varnothing}$.

$$\textit{Risk}\ \left(\widehat{\varnothing}\right) = E_{\pi}\left[\ L(\widehat{\varnothing}, \varnothing)\right] = \int_{\varnothing} L\left(\widehat{\varnothing}, \varnothing\right)\pi(\varnothing|t)\,d\varnothing \tag{3.2}$$

   where $\widehat{\varnothing}$ is an estimate of $\varnothing$. If $\widehat{\varnothing}$=$\varnothing$ there is no loss if $\widehat{\varnothing} < \varnothing$ we call it under estimation, on the other hand if $\widehat{\varnothing} > \varnothing$ then we call it overestimation [24]. The value of $\widehat{\varnothing}$ which minimize the risk function is called the standard Bayes estimator.

3.1. Joint Posterior Density Function Using Gamma and Exponential Priors

The most widely used prior distribution of the parameter ∅ is the gamma distribution with hyper-parameters α and β with probability density function given by [18].

$$g_1(\varnothing) = \frac{\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing\beta}; \qquad \varnothing > 0 \ , \alpha, \beta > 0 \tag{3.3}$$

The first posterior density function of the unknown parameter ∅ of KD have been obtained by combining the likelihood function (2.1) with the density function of gamma prior (3.3) and using (3.1) as:

$$\begin{aligned} \pi_1(\varnothing | t) &= \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing\beta} \varnothing^n \nu^n * e^{(\nu-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln(1-t_i)}}{\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \varnothing^{n-1} e^{-\varnothing\beta} \varnothing^{n-1} e^{-\varnothing\beta} * \varnothing^n \nu^n e^{(\nu-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln(1-t_i)} d\varnothing} \\ &= \frac{\varnothing^{n+\alpha-1} e^{-\varnothing(\beta-T)} (\beta - T)^{n+\alpha}}{\Gamma(n+\alpha)} \end{aligned} \tag{3.4}$$

where T as (2.4). The second prior distributing is exponential distribution with hyper parameter 'c' with probability density function given by [18]

$$g_2(\varnothing) = ce^{-\varnothing c}; \qquad \varnothing > 0 \ , \quad c > 0 \tag{3.5}$$

Can combining the likelihood (2.1) with the density function of exponential prior (3.5) via (3.1), results the second posterior density function of ∅.

$$\begin{aligned} \pi_2(\varnothing | t) &= \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \ * \ ce^{-\varnothing c}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln(1-t_i^v)} d\varnothing} \\ &= \frac{\varnothing^n e^{-\varnothing(c-T)} \ (c-T)^{n+1}}{\Gamma(n+\alpha)} \end{aligned} \tag{3.6}$$

By combining (3.3) and (3.5), obtain the double prior distribution (gamma-exponential) for ∅ as [19]:

$$g_3(\varnothing) = g_1(\varnothing)\, g_2(\varnothing) \ = \frac{c\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing(\beta+c)}; \quad \varnothing > 0, \quad \alpha, \beta, c > 0 \tag{3.7}$$

Hence, the posterior distribution based on this double prior distribution of ∅ for given data t can be obtained, using (3.5), (3.6) and via (3.1) as :

$$\begin{aligned} \pi_3(\varnothing | t) &= \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \ * \ \frac{c\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing(\beta+c)}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \ \frac{c\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing(\beta+c)} d\varnothing} \\ &= \frac{\varnothing^{n+\alpha-1} e^{-\varnothing(\beta+c-T)} (\beta + c - T)^{n+\alpha}}{\Gamma(n+\alpha)} \end{aligned} \tag{3.8}$$

The probability density function (3.8) is similar to gamma distribution G (α_1, β_1) where $\alpha_1 = (n + \alpha)$ and $\beta_1 = \ (\beta + c - T)$

3.2. Joint Posterior Density Function Using Gamma and Chi-Squared Priors

Chi-squared priors of the parameter ∅ with hyper- parameter C_2 defined by the following density [4];

$$g_4(\varnothing) = \frac{e^{-\frac{\varnothing}{2}} \ \varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}; \qquad \varnothing > 0, \quad c_2 > 0 \tag{3.9}$$

Combining the likelihood function (2.1) with the density function of chi-squared prior (3.9) and using (3.1), results the fourth posterior density function of ∅ .

$$\pi_4(\varnothing|t) = \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{e^{\frac{-\varnothing}{2}} \ \varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{e^{\frac{-\varnothing}{2}} \ \varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)} \, d\varnothing}$$
$$= \frac{\varnothing^{n+\frac{c_2}{2}-1} e^{-\varnothing\left(\frac{1}{2}-T\right)} \left(\frac{1}{2}-T\right)^{n+\frac{c_2}{2}}}{\Gamma\left(n+\frac{c_2}{2}\right)} \tag{3.10}$$

By combining (3.3) and (3.9), obtain the double prior distribution (gamma-chi- squared) for ∅ as [17]:

$$g_5(\varnothing) = g_1(\varnothing) \ . \ g_4(\varnothing) = \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing\left(\beta+\frac{1}{2}\right)}}{\Gamma\left(\alpha+\frac{c_2}{2}\right) 2^{\frac{c_2}{2}}}; \qquad \varnothing > 0, \quad \alpha, \beta, c_2 > 0 \tag{3.11}$$

Hence, the posterior distribution of ∅ based on this double prior distribution of ∅ for given data t can be obtained, using (2.1) and (3.11), as:

$$\pi_5(\varnothing|t) = \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing\left(\beta-\frac{1}{2}\right)}}{\Gamma\left(\alpha+\frac{c_2}{2}\right) 2^{\frac{c_2}{2}}}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing\left(\beta-\frac{1}{2}\right)}}{\Gamma\left(\alpha+\frac{c_2}{2}\right) 2^{\frac{c_2}{2}}} d\varnothing}$$
$$= \frac{\varnothing^{n+\alpha+\frac{c_2}{2}-2} e^{-\varnothing\left(\beta+\frac{1}{2}-T\right)} \left(\beta+\frac{1}{2}-T\right)^{n+\alpha+\frac{c_2}{2}-1}}{\Gamma\left(n+\alpha+\frac{c_2}{2}-1\right)} \tag{3.12}$$

The probability density in (3.12) is similar to gamma distribution G (α_2, β_2) where $\alpha_2 = \left(n+\alpha+\frac{c_2}{2}-1\right)$ and $\beta_2 = \ \left(\beta+\frac{1}{2}-T\right)$.

3.3. Joint Posterior Density Function Using Chi-Squared and Exponential Priors

In a similar manner, assume both the prior distributions have pdfs given by (3.5) and (3.9). Hence, the double prior distribution ∅ becomes [17]

$$g_6(\varnothing) = g_2(\varnothing) \ . \ g_4(\varnothing) = \frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing\left(c+\frac{1}{2}\right)}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}; \qquad \varnothing > 0, \quad c, c_2 > 0 \tag{3.13}$$

and the poster for distribution of ∅ given the data t, based on this double prior distribution, comes out to be (2.1) and (3.13), as:

$$\pi_6(\varnothing|t) = \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing\left(c+\frac{1}{2}\right)}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing\left(c+\frac{1}{2}\right)}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)} \, d\varnothing}$$
$$= \frac{\varnothing^{n+\frac{c_2}{2}-1} e^{-\varnothing\left(c+\frac{1}{2}-T\right)} \left(c+\frac{1}{2}-T\right)^{n+\frac{c_2}{2}}}{\Gamma\left(n+\frac{c_2}{2}\right)} \tag{3.14}$$

The probability density function in (3.14) is similar to gamma distribution G (α_3, β_3), where $\alpha_3 = \left(n + \frac{c_2}{2}\right)$ and $\beta_3 = \ (c + \frac{1}{2} - T)$.

3.4. Posterior Density Function Using Gamma Distribution

Here, we consider only a single gamma prior distribution for $\varnothing$, given by (3.3) [17, 9] and corresponding posterior distribution for $\varnothing$ as (3.4). Which is also a gamma distribution $\mathrm{G}(\alpha_4, \beta_4)$ with parameter $\alpha_4 = (n + \alpha)$ and $\beta_4 = \ (\beta - T)$, thus in all the cases of the different types of double Prior distribution and in the case of a single prior distribution, the posterior distribution of $\varnothing$ given the data t becomes a gamma distribution.

4. Loss Functions under Study

Some Bayesian estimators are obtained based on two loss function which are: De-groot loss function (weighted balance loss function) and Non-Linear exponential (NLINEX) loss function as asymmetric loss functions.

4.1. De-groot Loss Function (Weighted Balance Loss Function)

In Bayesian estimation, we consider a type of loss function which is classified as asymmetric function, was introduced by [5], which is widely used in most estimation problems. It can be defined as [2, 14]:

$$L(\widehat{R}(t), R(t)) = \frac{\left(R(t) - \ \widehat{R}(t)\right)^2}{\widehat{R}^2(t)} \tag{4.1}$$

According to (3.2) by taking the derivative of loss function (4.1) with respect to $\widehat{R}(t)$ and setting it equal to zero, the Bayes estimator of R(t) based on De-groot loss function, denoted by $\widehat{R}(t)_{\mathrm{Bd}}$, can obtained as:

$$\widehat{R}(t)_{\mathrm{Bd}} = \frac{E_{\pi(R^2(t)|t \ \ )}}{E_{\pi(R(t)|t \ \ )}} \tag{4.2}$$

4.2. Non- Liner Exponential (NLINEX) Loss Function

In this paper we have proposed a new loss function that is a symmetric in nature and non-linear function of the error called non-linear exponential (NLINEX) loss function was proposed by Islam et al. (2004) is linear combination of LINEX loss function and squared error loss function [10].
For NLINEX loss function, the Bayes estimator of R(t) is:

$$\widehat{R}(t)_{\mathrm{BNL}} = \frac{-\left[\ln E_\pi \left(e^{- \ \ c_1 R(t)}\right) - 2E_\pi \left(R(t)\right)\right]}{(c_1 + 2)} \tag{4.3}$$

where E_π stands for posterior expectation.

4.3. Bayes Estimators under the De-groot Loss Function (Weighted Balance Loss Function)

In this subsections, we obtain Bayes estimators of R(t) for KD corresponding to different posterior distributions.
*Corresponding to $\pi_{\mathbf{3}} (\varnothing \,|t \ \ )$

Under the gamma- exponential prior distribution, using (3.2) and (3.8), the Bayes estimator for R(t) corresponding to $\pi_3(\varnothing\,|t\,)$ can be found as:

$$\begin{aligned} E_{\pi_3}\left(\left. R(t)\right| t\right) &= \int_0^\infty (1-t^v)^{\varnothing} \frac{\varnothing^{n+\alpha-1} \quad e^{-\varnothing(\beta+c-T)}(\beta+c-T)^{n+\alpha}}{\Gamma(n+\alpha)} d\varnothing \\ &= \frac{(\beta+c-T)^{n+\alpha}}{(\beta+c-T-\ln(1-t^v))^{n+\alpha}} \end{aligned} \tag{4.4}$$

Again

$$\begin{aligned} E_{\pi_3}\left(\left. R^2(t)\right| t\right) &= \int_0^\infty \left((1-t^v)^{\varnothing}\right)^2 \frac{\varnothing^{n+\alpha-1} \quad e^{-\varnothing(\beta+c-T)}(\beta+c-T)^{n+\alpha}}{\Gamma(n+\alpha)} d\varnothing \\ &= \frac{(\beta+c-T)^{n+\alpha}}{(\beta+c-T-2\ln(1-t^v))^{n+\alpha}} \end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.2), the Bayes estimator of reliability function based on De-groot loss function under the assumption of gamma-exponential prior information is given by:

$$\widehat{R}(t)_{\text{Bdge}} = \left[\frac{\beta+c-T-\ln(1-t^v)}{\beta+c-T-2\ln(1-t^v)}\right]^{n+\alpha} \tag{4.6}$$

*Corresponding to $\pi_{\mathbf{5}}\left(\left.\varnothing\right| t\right)$

Under the gamma-chi-squared prior distribution, using (3.2) and (3.12), the Bayes estimator for R(t) corresponding to $\pi_5\left(\left.\varnothing\right| t\right)$ can be found as:

$$E_{\pi_5}\left(\left. R(t)\right| t\right) = \frac{(\beta+\frac{1}{2}-T)^{n+\alpha+\frac{c_2}{2}-1}}{(\beta+\frac{1}{2}-T-\ln(1-t^v))^{n+\alpha+\frac{c_2}{2}-1}} \tag{4.7}$$

$$E_{\pi_5}\left(R^2(t)\,|t\,\right) = \frac{(\beta+\frac{1}{2}-T)^{n+\alpha+\frac{c_2}{2}-1}}{(\beta+\frac{1}{2}-T-2\ln(1-t^v))^{n+\alpha+\frac{c_2}{2}-1}} \tag{4.8}$$

Substituting (4.7) and (4.8) in (4.2), the Bayes estimator of reliability function based on De-groot loss function under the assumption of gamma-chi-squared prior information is given by:

$$\widehat{R}(t)_{\text{Bdgch}} = \left[\frac{\beta+\frac{1}{2}-T-\ln(1-t^v)}{\beta+\frac{1}{2}-T-2\ln(1-t^v)}\right]^{n+\alpha+\frac{c_2}{2}-1} \tag{4.9}$$

*Corresponding to $\pi_{\mathbf{6}}\left(\left.\varnothing\right| t\right)$

Under the chi-squared-exponential prior distribution, using (3.2) and (3.14), the Bayes estimator for R(t) corresponding to $\pi_6(\varnothing\,|t\,)$ can be found as:

$$E_{\pi_6}(R(t)\,|t\,) = \frac{(c+\frac{1}{2}-T)^{n+\frac{c_2}{2}}}{(c+\frac{1}{2}-T-\ln(1-t^v))^{n+\frac{c_2}{2}}} \tag{4.10}$$

With

$$E_{\pi_6}\left(R^2(t)\,|t\,\right) = \frac{(c+\frac{1}{2}-T)^{n+\frac{c_2}{2}}}{(c+\frac{1}{2}-T-2\ln(1-t^v))^{n+\frac{c_2}{2}}} \tag{4.11}$$

Substituting (4.10) and (4.11) in (4.2) the Bayes estimator of reliability function based on De-groot loss function under the assumption of chi-squared-exponential prior information is given by:

$$\widehat{R}(t)_{\text{Bdche}} = \left[\frac{c+\frac{1}{2}-T-\ln\left(1-t^{v}\right)}{c+\frac{1}{2}-T-2\ln\left(1-t^{v}\right)}\right]^{n+\frac{c_2}{2}} \tag{4.12}$$

*Corresponding to $\pi_{\mathbf{1}}\left(\varnothing\,|t\ \right)$

From (3.2) and (3.4), the Bayes estimator of R(t) based on De-groot loss function corresponding to $\pi_1\left(\varnothing\,|t\ \right)$ can be found as:

$$E_{\pi_1}\left(R(t)\,|t\ \right) = \frac{\left(\beta-T\right)^{n+\alpha}}{\left(\beta-T-\ln\left(1-t^{v}\right)\right)^{n+\alpha}} \tag{4.13}$$

And

$$E_{\pi_1}\left(R^2(t)\,|t\ \right) = \frac{\left(\beta-T\right)^{n+\alpha}}{\left(\beta-T-2\ln\left(1-t^{v}\right)\right)^{n+\alpha}} \tag{4.14}$$

Substituting (4.13) and (4.14) in (4.2), the Bayes estimator of reliability function based on De-groot loss function under the assumption of gamma prior information is given by:

$$\widehat{R}(t)_{\text{Bdg}} = \left[\frac{\beta-T-\ln\left(1-t^{v}\right)}{\beta-T-2\ln\left(1-t^{v}\right)}\right]^{n+\alpha} \tag{4.15}$$

4.4. Bayes Estimators under the Non – Linear Exponential (NLINEX) Loss Function

In this subsection, based on NLINEX loss function, we obtain Bayes estimators of R(t) for KD corresponding to different posterior distributions.

*Corresponding to $\pi_{\mathbf{3}}\left(\varnothing\,|t\ \right)$

From (3.2) and (3.8), the Bayes estimator of R(t) based on NLINEX loss function corresponding to $\pi_3\left(\varnothing\,|t\ \right)$ can be found as:

$$E_{\pi_3}\left(e^{-c_1R(t)}\,|t\ \right) = \int_0^{\infty} e^{-c_1\left(1-t^{v}\right)^{\varnothing}} \frac{\varnothing^{n+\alpha-1}e^{-\varnothing(\beta+c-T)}\left(\beta+c-T\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)} d\varnothing$$

Let $e^{t} = \sum_{D=0}^{\infty}\frac{t^{D}}{D!}$

$$\begin{aligned} E_{\pi_3}\left(e^{-c_1R(t)}\,|t\ \right) &= \frac{\left(\beta+c-T\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)} \int_0^{\infty} \varnothing^{n+\alpha-1}\sum_{D=0}^{\infty}\frac{\left(-c_1\left(1-t^{v}\right)^{\varnothing}\right)^{D}}{D!}e^{-\varnothing(\beta+c-T)}d\varnothing \\ &= \sum_{D=0}^{\infty}\frac{\left(-c_1\right)^{D}}{D!}\left(\frac{\beta+c-T}{\beta+c-T-D\ln\left(1-t^{v}\right)}\right)^{n+\alpha} \end{aligned}$$

$$\ln E_{\pi_3}\left(e^{-c_1R(t)}\,|t\ \right) = \ln\left[\sum_{D=0}^{\infty}\frac{\left(-c_1\right)^{D}}{D!}\left(\frac{\beta+c-T}{\beta+c-T+D\ln\left(1-t^{v}\right)}\right)^{n+\alpha}\right] \tag{4.16}$$

Substituting (4.4) and (4.16) in (4.3), the Bayes estimator of reliability function based on NLINEX loss function under the assumption of gamma-exponential prior information is given by:

$$\widehat{R}(t)_{\text{BNlge}} = \frac{-\left[\ln\left\{\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta+c-T}{\beta+c-T-\text{D}\ln(1-t^v)}\right)^{n+\alpha}\right\}-2\left(\frac{\beta+c-T}{\beta+c-T-\ln(1-t^v)}\right)^{n+\alpha}\right]}{c_1+2} \tag{4.17}$$

*Corresponding to $\pi_{\mathbf{5}}\,(\varnothing\,|t\;)$

From (3.2) and (3.12), the Bayes estimator of R(t) based on NLINEX loss function corresponding to $\pi_5\,(\varnothing\,|t\;)$ can be found as:

$$\begin{aligned} E_{\pi_5}\left(e^{-c_1R(t)}\,|t\;\right) &= \sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T-D\ln(1-t^v)}\right)^{n+\alpha+\frac{c_2}{2}-1} \\ \ln E_{\pi_5}\left(e^{-c_1R(t)}\,|t\;\right) &= \ln\left[\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T-D\ln(1-t^v)}\right)^{n+\alpha+\frac{c_2}{2}-1}\right] \end{aligned} \tag{4.18}$$

Substituting (4.7) and (4.18) in (4.3), the Bayes estimator of reliability function based on NLINEX loss function under the assumption of gamma-chi-squared prior information is given by:

$$\widehat{R}(t)_{\text{BNLgch}} = \frac{-\left[\ln\left\{\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T-D\ln(1-t^v)}\right)^{n+\alpha+\frac{c_2}{2}-1}\right\}-2\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T-\ln(1-t^v)}\right)^{n+\alpha+\frac{c_2}{2}-1}\right]}{c_1+2} \tag{4.19}$$

*Corresponding to $\pi_{\mathbf{6}}\,(\varnothing\,|t\;)$

From (3.2) and (3.14), the Bayes estimator of R(t) based on NLINEX loss function corresponding to $\pi_6\,(\varnothing\,|t\;)$ can be found as:

$$\begin{aligned} E_{\pi_6}\left(e^{-c_1R(t)}\,|t\;\right) &= \sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T-D\ln(1-t^v)}\right)^{n+\frac{c_2}{2}} \\ \ln E_{\pi_6}\left(e^{-c_1R(t)}\,|t\;\right) &= \ln\left[\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T-D\ln(1-t^v)}\right)^{n+\frac{c_2}{2}}\right] \end{aligned} \tag{4.20}$$

Substituting (4.10) and (4.20) in (4.3), the Bayes estimator of reliability function based on NLINEX loss function under the assumption of chi-squared-exponential prior information is given by:

$$\widehat{R}(t)_{\text{BNLche}} = \frac{-\left[\ln\left\{\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{c+\frac{1}{2}-T}{(c+\frac{1}{2}-T-D\ln(1-t^v))}\right)^{n+\frac{c_2}{2}}\right\}-2\left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T-\ln(1-t^v)}\right)^{n+\frac{c_2}{2}}\right]}{c_1+2} \tag{4.21}$$

*Corresponding to $\pi_{\mathbf{1}}\,(\varnothing\,|t\;)$

From (3.2) and (3.4), the Bayes estimator of R(t) based on NLINEX loss function corresponding to $\pi_1\,(\varnothing\,|t\;)$ can be found as:

$$\begin{aligned} E_{\pi_1}\left(e^{-c_1R(t)}\,|t\;\right) &= \sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left[\frac{\beta-T}{\beta-T-D\ln(1-t^v)}\right]^{n+\alpha} \\ \ln E_{\pi_1}\left(e^{-c_1R(t)}\,|t\;\right) &= \ln\left[\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta-T}{\beta-T-D\ln(1-t^v)}\right)^{n+\alpha}\right] \end{aligned} \tag{4.22}$$

Substituting (4.13) and (4.22) in (4.3), the Bayes estimator of reliability function based on NLINEX loss function under the assumption of gamma prior information is given by:

$$\widehat{R}(t)_{\mathrm{BNLg}} = \frac{-\left[\ln\left\{\sum_{D=0}^{\infty}\frac{(-c_1)^D}{D!}\left(\frac{\beta-T}{\beta-T-D\ln(1-t^v)}\right)^{n+\alpha}\right\} - 2\left(\frac{\beta-T}{\beta-T-\ln(1-t^v)}\right)^{n+\alpha}\right]}{c_1+2} \tag{4.23}$$

5. Numerical Estimator of Reliability Function

Here, expansion methods by two ways (Bernstein polynomial and Power function) where applied to find the reliability function R(t) numerically.
Bernstein polynomials are given by [16, 15] as the formula:

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \qquad i = 0,\ 1,\ \ldots,\ n \quad ; \quad 0 < t < 1 \tag{5.1}$$

And power functions are given as [16]:

$$P_i(t) = t^i i = 0, 1, \ldots, n \tag{5.2}$$

where n is degree of polynomial.
Now, to find numerical estimator of reliability function R(t) start as follows:

$$R_n(t) = \sum_{i=0}^{n} k_i\ P_i(t), \qquad 0 < t < 1 \tag{5.3}$$

wherek_i are unknown coefficients and $P_i(t)$ are known functions. In the first way, we take $R_n(t) = R(t)_{\mathrm{BP}}$ and $P_i(t) = B_i^n(t)$ as in (5.1), but in the second way, we put $R_n(t) = R(t)_{\mathrm{PF}}$ and $P_i(t)$ as in (5.2) where $i = 0, 1, \ldots, n$. After that, let $\{t_0, t_1, \ldots, t_n\}$ be a set of points in the interval (0, 1), then (5.3) becomes:

$$R_n(t_j) = \sum_{i=0}^{n} k_i\ P_i(t_j) \qquad j = 0,\ \ldots,\ n \tag{5.4}$$

Now, from (1.3) find $R_n(t_j)$ and substitute into (5.4), we obtain

$$\sum_{i=0}^{n} k_i\ P_i(t_j) = \left(1 - t_j^v\right)^{\varnothing}, \qquad j = 0,\ \ldots,\ n$$

That is, we have in the two ways, linear systems of (n) equations and (n) unknown $k_i, i = 0, 1, \ldots, n$. Finally, solve these systems for (k_i) using Gauss-elimination to find the numerical estimate for reliability function R(t) by these ways called $\widehat{R}(t)_{\mathrm{BP}}$ when using Bernstein polynomial and $\widehat{R}(t)_{\mathrm{PF}}$ when using power function.

6. Simulation Study

The simulation study state that used to estimate R(t) of KD can be summarized by the following steps:

Step 1. • In this step, it has been set default values of parameters and constants for simulation experiments summarized in the following table.

Table 1: Default Values of Parameters and Constants that have been used in Simulation Experiments

Simple size	n	***10, 15, 25, 50, 100***
Shape parameter	$\varnothing$	***1.5, 2, 2.5***
Hyper-Parameter-Gamma-Exponential	α	***3***
	β	***2***
	C	***1.5***
Hyper-Parameter-Gamma-Chi-Squared	α	***3***
	β	***2***
	C_2	***2***
Hyper-Parameter-Chi-Squared- Exponential	C	***1.5***
	C_2	***2***
Hyper-Parameter-Gamma	α	***3***
	β	***2***
Number of Sample Replicate	L	***1000***

- The shape parameters of KD which are varied into nine cases to observe their effect on the estimates when $v > \varnothing$, $v = \varnothing$ and $v < \varnothing$.
- The values of NLINEX loss function constant (C_1) used are different values which are indicated in the tables.
- Select different values of time (t = 0.2, 0.4, 0.6 and 0.8) to calculate the estimating reliability function in order to compute IMSE.
- The simulation study process is replicate 1000 times to get independent samples from different sizes.

Step 2 At this step, we are generating random samples as follows:
Suppose that U is a random variable with uniform distribution in (0, 1), then the data of this distribution can be created using the inverse transformation method of the cdf where:

$$U = F(t) \tag{6.1}$$
$$t = F^{-1}(U) \tag{6.2}$$

Now, substituting equation (1.2) in equation (6.1), we get

$$U_i = F(t_i) = \ 1 - \ (1 - t_i^v)^{\varnothing} \ ; \qquad t > 0 \ ; \ \ \varnothing, \ v > 0$$

Simplify this equation, we have

$$t_i = \left[\, 1 - \ (1 - U_i)^{\frac{1}{\varnothing}} \right]^{\frac{1}{v}}; \qquad i = 1, \ \ldots, \ n \tag{6.3}$$

Step 3 Calculate the non- Bayes, Bayes and numerical estimators of the reliability function R(t) of KD according to the formulas that we have obtained in the previous chapter.

Step 4 Compare the different estimation methods for the reliability function according to integrated mean squared error (IMSE). The best estimator is the estimator that gives the smallest value of IMSE. Where IMSE is given as:

$$\text{IMSE}\ \left(\widehat{R}\,(t)\right) = \frac{1}{L}\sum_{j=1}^{L}\left[\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\widehat{R}_j(t_i) - \ R(t_i)\right)^2\right] \tag{6.4}$$

where

$$L : \text{is the number of sample replicated,}$$
$$n_t : \text{is the number of times.}$$
$$\widehat{R}_j(t_i) : \text{is the estimate of R(t) at the} j^{th} \text{replicate and } i^{th} \ time.$$

7. Simulation Results for Estimating the Reliability Function

Table (2) include nine different cases, which contains the IMSE values for non- Bayes, Bayes and numerical estimators of the reliability function R(t) of KD.

- From case (I) – case (VIIII) the best prior of Bayes estimator based on De-groot loss function is gamma except case (III) (n=15, 25, 50) is gamma – chi – squared. As well as case (VIIII) with all sample sizes based on NLINEX loss function also is gamma prior function.
- From case (I) to case (III) for all sample sizes and (n=100) for all cases the best prior of Bayes estimator based on NLINEX loss function is gamma – exponential.
- Case (VIIII) for all sample sizes and (n=100) with all different cases the best loss function is DeGroot.
- The IMSE values associated with numerical estimator is the best than non- Bayes and Bayes estimates with different cases and all sample sizes.
- From expansion method, the power function is the best estimator than Bernstein polynomial with different cases and all sample sizes.
- From case (I) to case (V) with n=10, Bayes methods based on NLINEX for all priors are the best than non-Bayes method.

Table 2: IMSE Values for Non-Bayes, Numerical and Bayes Estimators of the Reliability Function R(t) of Kumaraswamy Distribution with Different cases

Case (I): $\varnothing = 1.5, \upsilon = 1$

n	*Non-Bayes Method ML* $\widehat{R}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.7*	
10	*0.0088*	*1.1697e-08*	*2.1137e-07*	*Gamma-Exponential*	*0.0093*	*0.0037*	*NLINEX*
				Gamma-Chi-squared	*0.0064*	*0.0048*	*NLINEX*
				Chi-squared- Exponential	*0.0115*	*0.0051*	*NLINEX*
				Gamma	*0.0057*	*0.0063*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
15	*0.0052*	*1.1372e-08*	*4.2221e-07*	*Gamma-Exponential*	*0.0061*	*0.0029*	*NLINEX*
				Gamma-Chi-squared	*0.0044*	*0.0037*	*NLINEX*
				Chi-squared- Exponential	*0.0070*	*0.0036*	*NLINEX*
				Gamma	*0.0040*	*0.0046*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
25	*0.0024*	*1.4922e-10*	*1.5764e-10*	*Gamma-Exponential*	*0.0025*	*0.0018*	*NLINEX*
				Gamma-Chi-squared	*0.0020*	*0.0026*	*Degroot*
				Chi-squared- Exponential	*0.0028*	*0.0021*	*NLINEX*
				Gamma	*0.0019*	*0.0032*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
50	*0.0016*	*3.1416e-07*	*2.6580e-05*	*Gamma-Exponential*	*0.0018*	*0.0017*	*NLINEX*
				Gamma-Chi-squared	*0.0016*	*0.0020*	*Degroot*
				Chi-squared- Exponential	*0.00184*	*0.00179*	*NLINEX*
				Gamma	*0.0015*	*0.0022*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*8.1941e-04*	*2.6632e-08*	*1.7109e-06*	*Gamma-Exponential*	*8.1482e-04*	*0.0013*	*Degroot*
				Gamma-Chi-squared	*7.7602e-04*	*0.0015*	*Degroot*
				Chi-squared- Exponential	*8.3722e-04*	*0.0014*	*Degroot*
				Gamma	*7.7047e-04*	*0.0016*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (II): $\varnothing = 1.5, \upsilon = 1.5$

n	*Non-Bayes Method ML* $\widehat{R}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.5*	
10	*0.0070*	*1.8446e-06*	*9.3265e-05*	*Gamma-Exponential*	*0.0070*	*0.0033*	*NLINEX*
				Gamma-Chi-squared	*0.0047*	*0.0039*	*NLINEX*
				Chi-squared- Exponential	*0.0086*	*0.0045*	*NLINEX*
				Gamma	*0.0042*	*0.0051*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
15	*0.0045*	*7.9495e-05*	*7.9494e-05*	*Gamma-Exponential*	*0.0040*	*0.0024*	*NLINEX*
				Gamma-Chi-squared	*0.0030*	*0.0033*	*Degroot*
				Chi-squared- Exponential	*0.0046*	*0.0031*	*NLINEX*
				Gamma	*0.0028*	*0.0042*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
25	*0.0031*	*1.0273e-04*	*5.1829e-04*	*Gamma-Exponential*	*0.0031*	*0.0023*	*NLINEX*
				Gamma-Chi-squared	*0.0026*	*0.0028*	*Degroot*
				Chi-squared- Exponential	*0.0034*	*0.0027*	*NLINEX*
				Gamma	*0.0025*	*0.0032*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
50	*0.0015*	*1.0789e-04*	*9.7091e-04*	*Gamma-Exponential*	*0.00151*	*0.00150*	*NLINEX*
				Gamma-Chi-squared	*0.0014*	*0.0018*	*Degroot*
				Chi-squared- Exponential	*0.00159*	*0.00163*	*Degroot*
				Gamma	*0.0013*	*0.0020*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*8.0394e-04*	*2.1465e-04*	*2.2332e-04*	*Gamma-Exponential*	*7.8715e-04*		*Degroot*
				Gamma-Chi-squared	*7.5675e-04*	*0.0011* *0.0013*	*Degroot*
				Chi-squared- Exponential	*8.0993e-04*	*0.0012*	*Degroot*
				Gamma	*7.5379e-04*	*0.0014*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (III): $\varnothing = 1.5, \upsilon = 2$

n	*Non-Bayes Method ML* $\hat{R}(t)_{ML}$	*Numerical Methods* $\hat{R}(t)_{PF}$	$\hat{R}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\hat{R}(t)_{Bd}$	*NLINEX* $\hat{R}(t)_{BNL}$ *C1=6.3*	*Best Loss*
10	*0.0069*	*7.0296e-06*	*7.3608e-06*	*Gamma-Exponential*	*0.0043*	*0.0035*	*NLINEX*
				Gamma-Chi-squared	*0.00304*	*0.00444*	*Degroot*
				Chi-squared- Exponential	*0.0053*	*0.00443*	*NLINEX*
				Gamma	*0.00301*	*0.0058*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
15	*0.0044*	*1.9674e-04*	*1.9677e-04*	*Gamma-Exponential*	*0.0024*	*0.0029*	*Degroot*
				Gamma-Chi-squared	*0.0020*	*0.0039*	*Degroot*
				Chi-squared- Exponential	*0.0029*	*0.0035*	*Degroot*
				Gamma	*0.0022*	*0.0049*	*Degroot*
Best Prior					*Gamma-Chi-squared*	*Gamma-Exponential*	
25	*0.0025*	*6.2288e-06*	*6.8328e-06*	*Gamma-Exponential*	*0.0020*	*0.0027*	*NLINEX*
				Gamma-Chi-squared	*0.00178*	*0.0031*	*Degroot*
				Chi-squared- Exponential	*0.0023*	*0.0030*	*NLINEX*
				Gamma	*0.00180*	*0.0035*	*Degroot*
Best Prior					*Gamma-Chi-squared*	*Gamma-Exponential*	
50	*0.0013*	*1.1068e-05*	*2.1882e-05*	*Gamma-Exponential*	*0.00117*	*0.0021*	*Degroot*
				Gamma-Chi-squared	*0.00111*	*0.0023 Degroot*	
				Chi-squared- Exponential	*0.00124*	*0.0022*	*Degroot*
				Gamma	*0.00112*	*0.0025*	*Degroot*
Best Prior					*Gamma-Chi-squared*	*Gamma-Exponential*	
100	*6.5547e-04*	*6.3067e-06*	*1.1253e-05*	*Gamma-Exponential*	*6.2488e-04*	*0.0016*	*Degroot*
				Gamma-Chi-squared	*6.0519e-04*	*0.00173*	*Degroot*
				Chi-squared- Exponential	*6.4388e-04*	*0.00167*	*Degroot*
				Gamma	*6.0607e-04*	*0.0018*	*Degroot*
Best Prior				*Gamma*		*Gamma-Exponential*	

Case (IV): $\varnothing = 2, \upsilon = 1$

n	*Non-Bayes Method ML* $\hat{R}(t)_{ML}$	*Numerical Methods* $\hat{R}(t)_{PF}$	$\hat{R}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\hat{R}(t)_{Bd}$	*NLINEX* $\hat{R}(t)_{BNL}$ *C1=6.7*	*Best Loss*
10	*0.0076*	*7.9550e-12*	*3.3980e-10*	*Gamma-Exponential*	*0.0148*	*0.0050*	*NLINEX*
				Gamma-Chi-squared	*0.0090*	*0.0037*	*NLINEX*
				Chi-squared- Exponential	*0.0154*	*0.0053*	*NLINEX*
				Gamma	*0.0069*	*0.0041*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
15	*0.0044*	*1.8012e-13*	*4.5343e-07*	*Gamma-Exponential*	*0.0091*	*0.0034*	*NLINEX*
				Gamma-Chi-squared	*0.0058*	*0.0028*	*NLINEX*
				Chi-squared- Exponential	*0.0090*	*0.0035*	*NLINEX*
				Gamma	*0.0046*	*0.0032*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
25	*0.0028*	*1.1183e-10*	*1.1543e-07*	*Gamma-Exponential*	*0.0046*	*0.0023*	*NLINEX*
				Gamma-Chi-squared	*0.0032*	*0.0024*	*NLINEX*
				Chi-squared- Exponential	*0.0044*	*0.0025*	*NLINEX*
				Gamma	*0.0028*	*0.0027*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Exponential*	
50	*0.0015*	*5.3879e-13*	*4.7104e-09*	*Gamma-Exponential*	*0.0020*	*0.0016*	*NLINEX*
				Gamma-Chi-squared	*0.0016*	*0.0019*	*Degroot*
				Chi-squared- Exponential	*0.0019*	*0.0018*	*NLINEX*
				Gamma	*0.0015*	*0.0021*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*6.8867e-04*	*6.8118e-14*	*4.3495e-09*	*Gamma-Exponential*	*8.4651e-04*	*0.0012*	*Degroot*
				Gamma-Chi-squared	*7.3056e-04*	*0.0014*	*Degroot*
				Chi-squared- Exponential	*8.1164e-04*	*0.0013*	*Degroot*
				Gamma	*6.9396e-04*	*0.0015*	*Degroot*
Best Prior				*Gamma*		*Gamma-Exponential*	

Case (V): $\varnothing = 2, \upsilon = 2$

n	*Non-Bayes Method ML* $\widehat{R}(t)_{ML}$	*Numerical Methods* $\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.3*	*Best Loss*
10	*0.0066*	*2.1068e-16*	*6.7644e-08*	*Gamma-Exponential*	*0.0108*	*0.0051*	*NLINEX*
				Gamma-Chi-squared	*0.0065*	*0.0035*	*NLINEX*
				Chi-squared- Exponential	*0.0110*	*0.0053*	*NLINEX*
				Gamma	*0.0051*	*0.0036*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
15	*0.0040*	*8.2005e-17*	*1.8320e-06*	*Gamma-Exponential*	*0.0066*	*0.0033*	*NLINEX*
				Gamma-Chi-squared	*0.0042*	*0.0027*	*NLINEX*
				Chi-squared- Exponential	*0.0064*	*0.0034*	*NLINEX*
				Gamma	*0.0034*	*0.0029*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
25	*0.0026*	*9.4340e-18*	*1.3795e-07*	*Gamma-Exponential*	*0.0035*	*0.00214*	*NLINEX*
				Gamma-Chi-squared	*0.0025*	*0.00212*	*NLINEX*
				Chi-squared- Exponential	*0.0033*	*0.0023*	*NLINEX*
				Gamma	*0.0022*	*0.0024*	*Degroot*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
50	*0.0012*	*1.7223e-17*	*7.5901e-07*	*Gamma-Exponential*	*0.0016*	*0.0013*	*NLINEX*
				Gamma-Chi-squared	*0.0013*	*0.0014*	*Degroot*
				Chi-squared- Exponential	*0.0015*	*0.0014*	*NLINEX*
				Gamma	*0.0012*	*0.0016*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*6.4105e-04*	*3.6564e-16*	*3.8306e-05*	*Gamma-Exponential*	*7.2598e-04*	*9.4643e-04*	*Degroot*
				Gamma-Chi-squared	*6.4401e-04*	*0.00104*	*Degroot*
				Chi-squared- Exponential	*7.0504e-04*	*0.00101*	*Degroot*
				Gamma	*6.2109e-04*	*0.0011*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (VI): $\varnothing = 2, \upsilon = 2.5$

n	*Non-Bayes Method ML* $\widehat{R}(t)_{ML}$	*Numerical Methods* $\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.2*	*Best Loss*
10	*0.0061*	*3.6220e-11*	*6.7801e-05*	*Gamma-Exponential*	*0.0093*	*0.0062*	*NLINEX*
				Gamma-Chi-squared	*0.0057*	*0.00454*	*NLINEX*
				Chi-squared- Exponential	*0.0095*	*0.0063*	*NLINEX*
				Gamma	*0.00447*	*0.00453*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
15	*0.0036*	*1.6579e-11*	*5.5866e-05*	*Gamma-Exponential*	*0.0056*	*0.0044*	*NLINEX*
				Gamma-Chi-squared	*0.00358*	*0.00365*	*Degroot*
				Chi-squared- Exponential	*0.0055*	*0.0045*	*NLINEX*
				Gamma	*0.0029*	*0.00374*	*Degroot*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
25	*0.0024*	*1.5355e-11*	*1.7781e-04*	*Gamma-Exponential*	*0.0033*	*0.0034*	*Degroot*
				Gamma-Chi-squared	*0.0024*	*0.0032*	*Degroot*
				Chi-squared- Exponential	*0.0031*	*0.0035*	*Degroot*
				Gamma	*0.0021*	*0.0033*	*Degroot*
Best Prior					*Gamma*	*Chi-squared- Exponential*	
50	*0.0013*	*2.1401e-11*	*5.4844e-05*	*Gamma-Exponential*	*0.00149*	*0.00254*	*Degroot*
				Gamma-Chi-squared	*0.00124*	*0.00257*	*Degroot*
				Chi-squared- Exponential	*0.00145*	*0.00262*	*Degroot*
				Gamma	*0.00118*	*0.0027*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*5.6892e-04*	*2.8685e-11*	*1.4389e-05*	*Gamma-Exponential*	*6.3248e-04*	*0.00197*	*Degroot*
				Gamma-Chi-squared	*5.6297e-04*	*0.00202*	*Degroot*
				Chi-squared- Exponential	*6.1509e-04*	*0.00201*	*Degroot*
				Gamma	*5.4445e-04*	*0.0021*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (VII): $\varnothing = 2.5, v = 1$

n	*Non-Bayes Method ML* $\hat{R}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.7*	
10	*0.0067*	*5.7729e-11*	*4.4406e-10*	*Gamma-Exponential*	*0.0213*	*0.0083*	*NLINEX*
				Gamma-Chi-squared	*0.0126*	*0.0045*	*NLINEX*
				Chi-squared- Exponential	*0.0199*	*0.0070*	*NLINEX*
				Gamma	*0.0092*	*0.0038*	*NLINEX*
Best Prior					*Gamma*	*Gamma*	
15	*0.0043*	*6.2985e-11*	*6.6039e-10*	*Gamma-Exponential*	*0.0123*	*0.0050*	*NLINEX*
				Gamma-Chi-squared	*0.0074*	*0.0033*	*NLINEX*
				Chi-squared- Exponential	*0.0107*	*0.0043*	*NLINEX*
				Gamma	*0.0056*	*0.0032*	*NLINEX*
Best Prior					*Gamma*	*Gamma*	
25	*0.0027*	*1.0059e-10*	*1.0392e-09*	*Gamma-Exponential*	*0.0061*	*0.0030*	*NLINEX*
				Gamma-Chi-squared	*0.0039*	*0.0026*	*NLINEX*
				Chi-squared- Exponential	*0.0052*	*0.0029*	*NLINEX*
				Gamma	*0.0032*	*0.0028*	*NLINEX*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
50	*0.0013*	*8.1316e-11*	*7.7175e-09*	*Gamma-Exponential*	*0.0023*	*0.00187*	*NLINEX*
				Gamma-Chi-squared	*0.0017*	*0.00197*	*Degroot*
				Chi-squared- Exponential	*0.00197*	*0.00198*	*Degroot*
				Gamma	*0.0014*	*0.00213*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	
100	*6.7134e-04*	*6.3975e-11*	*3.5846e-08*	*Gamma-Exponential*	*9.5667e-04*	*0.0016*	*Degroot*
				Gamma-Chi-squared	*7.7514e-04*	*0.0017*	*Degroot*
				Chi-squared- Exponential	*8.5302e-04*	*0.0017*	*Degroot*
				Gamma	*7.1372e-04*	*0.0018*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (VIII): $\varnothing = 2.5, v = 2.5$

n	*Non-Bayes Method ML* $\hat{R}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.2*	
10	*0.0055*	*1.1321e-05*	*1.1363e-05*	*Gamma-Exponential*	*0.0137*	*0.0077*	*NLINEX*
				Gamma-Chi-squared	*0.0079*	*0.0043*	*NLINEX*
				Chi-squared- Exponential	*0.0123*	*0.0065*	*NLINEX*
				Gamma	*0.0056*	*0.0035*	*NLINEX*
Best Prior					*Gamma*	*Gamma*	
15	*0.0038*	*1.6873e-04*	*1.6869e-04*	*Gamma-Exponential*	*0.0087*	*0.0053*	*NLINEX*
				Gamma-Chi-squared	*0.0052*	*0.0035*	*NLINEX*
				Chi-squared- Exponential	*0.0075*	*0.0046*	*NLINEX*
				Gamma	*0.0040*	*0.0031*	*NLINEX*
Best Prior					*Gamma*	*Gamma*	
25	*0.0025*	*1.6410e-05*	*1.6460e-05*	*Gamma-Exponential*	*0.0047*	*0.0034*	*NLINEX*
				Gamma-Chi-squared	*0.0031*	*0.0027*	*NLINEX*
				Chi-squared- Exponential	*0.0040*	*0.0031*	*NLINEX*
				Gamma	*0.00256*	*0.00262*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
50	*0.0011*	*4.7859e-05*	*4.7859e-05*	*Gamma-Exponential*	*0.0017*	*0.00186*	*Degroot*
				Gamma-Chi-squared	*0.0013*	*0.00177*	*Degroot*
				Chi-squared- Exponential	*0.0015*	*0.00184*	*Degroot*
				Gamma	*0.0011*	*0.00182*	*Degroot*
Best Prior					*Gamma*	*Gamma-Chi-squared*	
100	*4.9603e-04*	*1.3847e-04*	*1.3846e-04*	*Gamma-Exponential*	*6.4333e-04*	*0.00128*	*Degroot*
				Gamma-Chi-squared	*5.2344e-04*	*0.001317*	*Degroot*
				Chi-squared- Exponential	*5.7383e-04*	*0.001315*	*Degroot*
				Gamma	*4.8787e-04*	*0.0014*	*Degroot*
Best Prior					*Gamma*	*Gamma-Exponential*	

Case (VIIII): $\varnothing = 2.5, v = 3$

n	*Non-Bayes Method ML* $\widehat{R}(t)_{ML}$	*Numerical Methods* $\widehat{R}(t)_{PF}$	$\widehat{R}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{R}(t)_{Bd}$	*NLINEX* $\widehat{R}(t)_{BNL}$ *C1=6.2*	*Best Loss*
				Gamma-Exponential	*0.0122*	*0.0139*	*Degroot*
10	*0.0051*	*7.4200e-04*	*7.4337e-04*	*Gamma-Chi-squared*	*0.0071*	*0.0094*	*Degroot*
				Chi-squared- Exponential	*0.0110*	*0.0123*	*Degroot*
				Gamma	*0.0051*	*0.0080*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
				Gamma-Exponential	*0.0071*	*0.0101*	*Degroot*
15	*0.0032*	*1.1390e-05*	*1.2363e-05*	*Gamma-Chi-squared*	*0.0041*	*0.0075*	*Degroot*
				Chi-squared- Exponential	*0.0059*	*0.0089*	*Degroot*
				Gamma	*0.0030*	*0.0068*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
				Gamma-Exponential	*0.0039*	*0.0079*	*Degroot*
25	*0.0022*	*8.5459e-04*	*8.5784e-04*	*Gamma-Chi-squared*	*0.0026*	*0.0067*	*Degroot*
				Chi-squared- Exponential	*0.0033*	*0.0073*	*Degroot*
				Gamma	*0.0021*	*0.0063*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
				Gamma-Exponential	*0.0017*	*0.0061*	*Degroot*
50	*0.0010*	*7.4647e-04*	*7.4731e-04*	*Gamma-Chi-squared*	*0.0012*	*0.0056*	*Degroot*
				Chi-squared- Exponential	*0.0014*	*0.0058*	*Degroot*
				Gamma	*0.0011*	*0.0055*	*Degroot*
Best Prior					*Gamma*	*Gamma*	
				Gamma-Exponential	*6.6585e-04*	*0.0051*	*Degroot*
100	*5.1865e-04*	*6.2696e-04*	*6.2749e-04*	*Gamma-Chi-squared*	*5.5110e-04*	*0.0050*	*Degroot*
				Chi-squared- Exponential	*6.0138e-04*	*0.0051*	*Degroot*
				Gamma	*5.1582e-04*	*0.0049*	*Degroot*
Best Prior					*Gamma*	*Gamma*	

8. Conclusions

Depending on simulation results based on complete data, the most essential conclusions are summarized by:

1. More results the best prior of Bayes estimator based on DeGroot loss function is gamma prior function. As well as case (VIIII) with all sample sizes based on NLINEX loss function also is gamma prior function.
2. Case (VIIII) for all sample sizes and (n=100) with all different cases the best loss function is De-groot.
3. For all sample sizes and for all cases the IMSE values associated with numerical estimators are better than non-Bayes and Bayes estimates.
4. Expansion method by using power function is the best from Bernstein polynomial to estimate IMSE values for all cases and for all sample sizes.

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