Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 1605-1611
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5775

On primary isolated submodules

Ghaleb Ahmed[a,*], Tamara Alshareef[a]

[a]*Department of Mathematics, College of Education for Pure Science, Ibn-Al-Haitham, University of Baghdad, Baghdad, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

Let L be a left module over a ring S with identity. In this paper, the concept of primary isolated submodules is introduced. We look for relations between this class of submodules and related modules. A number of facts and characterizations that concern is gained. The aim of this work is to introduce and study the primary isolated submodules as a generalization of isolated submodules. A submodule A of L is primary isolated if for each proper B of A, there is a primary submodule C of L, $B \subseteq C$ but A $\nsubseteq C$. Some properties are gained and we look for any relationship between this type of modules and other related modules.

Keywords: Primary isolated submodules, Primary lifted submodules, Primary radical submodules, Primary submodules, Prime submodules.

1. Introduction

Throughout this work, S is denoted a ring has an identity and L denotes a left S-module. Notation $\subseteq$ indicates inclusion. A proper submodule A of an S-module L is prime if $IB \subseteq A$ for an ideal I of S and a submodule B of L, then $B \subseteq A$ or $IL \subseteq A$ [3]. The (prime) radical for any submodule A (in L), denoted by $\mathrm{rad}_L(A) = \bigcap_{A\subseteq B} B$ where B is prime of L. If A is not in any prime, then the radical of A is defined to be $\mathrm{rad}_L(A) = L$. Further, $L \neq A$ is said to be radical (in L) if $A = rad_L(A)$. Prime B of a submodule A of L can be lifted to L if there is prime C of L with $B = A \cap C$ [3]. $L \neq A$ is named primary if whenever sl $\in A$, for $s \in S$ and $l \in L$, then $l \in A$ or $s^n L \subseteq A$ for a positive integer n [4].One can easily see that A is primary of L if and only if sB $\subseteq A$, for some $s \in S$ and submodule B of L, then $B \subseteq A$ or $s^n L \subseteq A$ for a positive integer n. The primary radical for any submodule A in L is indicated by $P\text{-}\mathrm{rad}_L(A) = \bigcap_{A\subseteq B} B$ where B is primary of L. If A is not

*Corresponding author
Email addresses: ghaleb.a.h@ihcoedu.uobaghdad.edu.iq (Ghaleb Ahmed), tamaraalshareef@yahoo.com (Tamara Alshareef)

Received: November 2021 *Accepted:* November 2021

contained in any primary, then the primary radical of A is dfined to be $P\text{-rad}_L(A) = L$ [2]. Moreover, a proper submodule A is primary radical in L if $A = P\text{-rad}_L(A)$. L is cocyclic when it includes an essential simple submodule [5]. A submodule A is essential in L if for any nonzero submodule B of L, $A \cap B \neq 0$ [5]. A submodule A of L is called isolated when any proper submodule B of A, there is prime C of L with $B \subseteq C$ while A $\nsubseteq C$ [3].

This work contains three sections. In section two, we introduce the primary isolated submodules. Some properties of this concept are discussed (Proposition 2.4 and Proposition 2.7). Further, we define the primary lifted submodules as a generalization of lifted submodules . By using the concept primary lifted submodules we will give a description of the concept of primary isolated submodules (Theorem 2.12). In section three, we give another description of primary isolated submodules by using the concept primary radical of submodules (Proposition 3.1). Moreover, we look for any relationships between primary isolated submodules and the primary radical of submodules or some other related modules (Proposition 3.2, Proposition 3.6, Proposition 3.7 and Theorem 3.11). In what follows, $\mathbb{Z}$, Z_{p^∞}, and $\mathrm{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ denote respectively, integers, the p-Prüfer group and the residue ring of integers modulo n.

2. Primary Isolated Submodules

Definition 2.1. *A is primary isolated when each proper submodule B of A, there exists a primary submodule C of L such that $B \subseteq C$ but $A \nsubseteq C$.*

Remark 2.2.

(1) *Isolated submodules are primary isolated but the reverse is not hold in general. For instance, $n\mathbb{Z}$ is a primary isolated submodule but it is not isolated of $\mathbb{Z}$ as $\mathbb{Z}$-module for each positive integer n.*

(2) *The zero submodule is always an isolated submodule and hence it is primary isolated.*

(3) *Every simple module is an isolated submodule and hence it is primary isolated.*

(4) *The simple submodules need not be primary isolated (and hence not isolated). For example, $<\overline{2}>$ is a simple submodule in the $\mathbb{Z}$-module $\mathbb{Z}_4$. However, $<\overline{2}>$ is not primary isolated in $\mathbb{Z}_4$ because there is no primary submodule A of $\mathbb{Z}_4$ such that $<\overline{0}>\subseteq A$ and $<\overline{2}>\nsubseteq A$*

(5) *All non-zero submodules of $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ are not primary isolated and hence they are not isolated. In fact all proper submodules of $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ are not primary and hence they are not prime.*

(6) *If J is a maximal (or prime) of L, then J may not be primary isolated (and hence not isolated). For example, $<\overline{2}>$ is maximal in $\mathbb{Z}_4$ as $\mathbb{Z}$-module but $<\overline{2}>$ is not primary isolated*

(7) *Even though $<\overline{2}>$ is not primary isolated in $\mathbb{Z}_4$ as a $\mathbb{Z}$-module, but $\mathbb{Z}_4$ is isolated and hence it is primary isolated.*

(8) *The maximal submodule of a local module L is not primary isolated and hence it is not isolated as we have seen in example (6).*

(9) *Let $N \subseteq H \subseteq M$. If H is primary isolated then N may not be primary isolated. $\mathbb{Z}_8$ as a $\mathbb{Z}$-module is primary isolated in $\mathbb{Z}_8$ while $<\overline{2}>$ is not primary isolated in $\mathbb{Z}_8$ since there is no primary submodule A of $\mathbb{Z}_8$ such that $<\overline{4}>\subseteq A$ and $<\overline{2}>\nsubseteq A$.*

(10) *Let $N \subseteq H \subseteq M$. If N is primary isolated then H may not be primary isolated in M. For example, $<\overline{0}>$ is a primary isolated submodule in $\mathbb{Z}_4$ as $\mathbb{Z}$-module. However, $<\overline{2}>$ is not a primary isolated submodule in $\mathbb{Z}_4$.*

(11) *All submodules of $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$ as $\mathbb{Z}$-module are isolated and hence they are primary isolated.*

(12) *Consider* $\mathbb{Z}_{12}$ *as a* $\mathbb{Z}$*-module. Then* $\mathbb{Z}_{12}$, $<\overline{4}>$, $<\overline{6}>$, $<\overline{3}>$ *are primary isolated . However,* $<\overline{2}>$ *is not primary isolated since there is no primary submodule* A *of* $\mathbb{Z}_{12}$ *such that* $<\overline{2}>\nsubseteq A$ *and* $<\overline{4}>\subseteq A$.

Proposition 2.3. *Let* B *be a submodule of* L *with* $B \nsubseteq A$ *for some primary submodule* A*. Yield*

(1) $A \cap B$ *is primary of* B.
(2) $\sqrt{[A \cap B :_R B]} = \sqrt{[A :_R L]}$.

Proof .

(1) $r \in R$ *and* $x \in B$, $rx \in A \cap B$ *implies* $rx \in A$ *. So that either* $x \in A$ *and hence* $x \in A \cap B$ *or* $r^n L \subseteq A$ *for* n *and thus* $r^n B = r^n L \cap r^n B \subseteq A \cap B$ *as desired.*
(2) *Let* $r \in \sqrt{[A \cap B :_R B]}$ *then* $r^n \in [A \cap B :_R B]$ *and so* $r^n B \subseteq A \cap B$ *implies* $r^n B \subseteq A$ *then for each* $x \in B$, $r^n x \in A$*. This implies either* $x \in A$ *and it follows* $B \subseteq A$ *which is a contradiction or* $(r^n)^m L \subseteq A$ *for some positive integer* m*. Put* $t = nm$ *so we have* $r^t L \subseteq A$ *and hence* $r \in \sqrt{[A :_R L]}$*. Let* $r \in \sqrt{[A :_R L]}$ *then* $r^n L \subseteq A$ *implies* $r^n B \subseteq A$*. But* $r^n B \subseteq B$ *so that* $r^n B \subseteq A \cap B$ *means* $r \in \sqrt{[A \cap B :_R B]}$.

□

Proposition 2.4. *If* A *is a primary isolated of* L *then each proper submodule of* A *is cotained in primary of* A*.*
Proof .*Let* B *and* A *be submodules of* L *with* B *is proper in* A *and* A *is a primary isolated. So there is primary* C *of* L, $B \subseteq C$ *and* $A \nsubseteq C$ *implies* $B = B \cap A \subseteq C \cap A$*. By Proposition 2.3,* $C \cap A$ *is primary of* A *as desired.* □

Example 2.5. *Every proper submodule of* $<\overline{2}>$ *in* $\mathbb{Z}_4$ *as* $\mathbb{Z}$*-module is contained in a prime submodule and hence it is contained in primary. However,* $<\overline{2}>$ *is not primary isolated.*

Lemma 2.6. *Let* $L = A \oplus B$ *be a direct sum of two submodules* A *and* B *as* S*-module. If* C *is primary of* A *then* $C \oplus B$ *is primary of* M*.*
Proof .*Let* $L = A \oplus B$ *be direct sum.* $r \in R$, $x \in L$, $rx \in C \oplus B$ *for some primary* C *of* A*. On the other hand,* $x = a + b$ *for some* $a \in A$ *and* $b \in B$ *and hence* $rx = c + d$ *for some* $c \in C$ *and* $d \in B$ *and hence* $ra + rb = r(a + b) = rx = c + d$*. This implies* $-c + ra = d - rb \in A \cap B = 0$ *and so* $ra = c$*. It follows that* $ra \in A$*. Further,* C *is a primary submodule of* A *and* $a \in A$ *it follows that either* $a \in C$ *or* $r^n A \subseteq C$ *for positive* n*. If* $a \in C$ *then* $x = a + b \in C \oplus B$*. If* $r^n A \in C$ *then* $r^n M = r^n(A + B) = r^n A + r^n B \subseteq C + B$ *as desired.* □

Proposition 2.7. *Let* A *be a summand of* M*. If each proper of* A *is included in primary of* A *then* A *is primary isolated.*
Proof .*Let* $M = A \oplus B$ *and* D *be proper of* A*. By hypotheisis, there is primary* C *of* A *with* $D \subseteq C$*. By Lemma (2.6),* $C \oplus B$ *is primary of* M*. Further,* $D \subseteq C \oplus B$ *and* $A \nsubseteq C \oplus B$ *it follows that* A *is primary isolated.* □

Corollary 2.8. *If* A *is finitely generated summand of* L *then each proper of* A *is included in primary of* A*.*
Proof .*Let* A *be a finitely generated summand of* L *and* B *be proper in* A*. So there is maximal* C *of* L *such that* $B \subseteq C$*.* □

Corollary 2.9. *Each finitely generated summand of L is primary isolated.*
Proof . *Directly obtained by Proposition 2.7 and Corollary 2.8.* $\square$

Example 2.10. *Consider $M=\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_p$ as $\mathbb{Z}$-module and $A=\mathbb{Z}_{p^\infty}\oplus 0$ is summand of M. However, A is not finitely generated , it is not primary isolated and every submodule of A is not primary.*

Definition 2.11. *Let A be a submodule of L. We call a primary submodule B of A can be primary lifted to L if there is primary C of L with $B = A \cap C$.*

Theorem 2.12. *The next are the same*

1. *A is primary isolated of M.*
2. *For each each proper C of A, there is primary B of A, $C \subseteq B$ and B can be primary lifited to M.*

Proof .
(1)$\Longrightarrow$(2) Let A be primary isolated of M and C be proper of A. Via Definition (2.1), there is primary D of M such that $C \subseteq D$ and $A \nsubseteq D$. By Proposition (2.3), $A\cap D$ is primary of A. Put $B = A\cap D$. Thus $C \subseteq B$ and B can be primary lifited to M as desired.
(2)$\Longrightarrow$(1) It is clear. $\square$

Theorem 2.13. *Let I be primary of R and A of M. The next are equal*

1. *An P-primary submodule B of A can be primary lifted to M*
2. *$A \cap PM \subseteq B$.*

Proof .
(1)$\Longrightarrow$(2) Assume that (1) is hold. Then $B = A\cap C$ for some P-primary C of M, implies $A\cap PM \subseteq A\cap C = B$.
(2)$\Longrightarrow$(1) Let $A \cap PM \subseteq B$ where B is P-primary of A. By modular law, $A \cap (B + PM) = (A\cap B) + (A\cap PM) = B$. A submodule T of M maximal to the properties $B + PM \subseteq T$ and $T \cap A = B$. Also T is proper in M since if $T = M$ implies that $A = B$ which is a contradiction. $r \in R$ and S of M with T is proper in S and $rS \subseteq T$. By the choice of T, then B is proper in $S\cap A$. But $r^n (S\cap A) \subseteq r^nS \cap r^nA \subseteq T\cap A = B$ and so $r^n \in P$. Therefore $r^nM \subseteq PM \subseteq T$ and hence T is primary of M. This means that B can be primary lifted to M. $\square$

3. Primary Isolated Submodules and Primary Radical

Proposition 3.1. *Let M be an T-module and A of M. The next are the same*

1. *For each proper submodule B of A, $P\text{-}rad_M(A) \neq P\text{-}rad_M(B)$.*
2. *A is primary isolated.*

Proof .
(1)$\Longrightarrow$(2) Let B is a proper submodule of A, so by (1) $P\text{-}rad_M(A) \neq P\text{-}rad_M(B)$. This means that there is primary C of M, $B \subseteq C$ and $A \nsubseteq C$ as desired.
(2)$\Longrightarrow$(1) by the same argument of (1)$\Longrightarrow$(2). $\square$

Proposition 3.2. *Let A be a submodule of M and $r \in R$. Then $P\text{-}rad_M(r^nA) = P\text{-}rad_M(A\cap r^nM)$ for n.*
Proof . *It is clear that $r^nA \subseteq A\cap r^nM$. Then $P\text{-}rad_M(r^nA) \subseteq P\text{-}rad_M(A\cap r^nM)$. The other inclusion, for each primary B of M, $r^cA \subseteq B$ for positive c, $A \subseteq B$ or $(r^c)^tM \subseteq B$ for positive t. If we choose $n = tc$ then $A\cap r^nM \subseteq P\text{-}rad_M(r^nA)$ and hence $P\text{-}rad_M(A\cap r^nM) \subseteq P\text{-}rad_M(r^nA)$.* $\square$

Corollary 3.3. *Let A be a submodule of M over a principal ideal ring S and I of S. Then P-$rad_M(I^nA) = P$-$rad_M(A \cap I^nM)$ for n.*
Proof . *Via similar way of Proposition 3.2.* □

Corollary 3.4. *Let A be primary isolated and $r \in S$. Then $A \subseteq r^nM$ if and only if $A = r^nA$.*
Proof . *Assume $A \subseteq r^nL$. Then $A = A \cap r^nL \subseteq P$-$rad_M(A \cap r^nL) =$ P-$rad_M(r^nA)$ by Proposition (3.2). Because of A is primary isolated of M it follows that $A = r^nA$. The reverse side is clear.* □

Corollary 3.5. *Let A be a submodule of M over a principal ideal ring S and T of S. Yield $A \subseteq T^nM$ if and only if $A = T^nA$.*
Proof . *By similar way of Corollary 3.4.* □

Proposition 3.6. *Let A be a submodule of M with r^nA is primary radical for an element $r \in R$ and for a positive integer n. Then $A \cap r^nM = r^nA$.*
Proof . *Directly by Proposition 3.2.* □

Proposition 3.7. *Let A be a submodule of M over a principal ideal ring R and with I^nA is primary radical of M for an ideal I of R for a positive integer n. Then $A \cap I^nM = I^nA$.*
Proof . *By similar way of Proposition 3.6.* □

The following lemma is stated and proved in [3], we give it for completeness

Lemma 3.8. *Let M be a cocyclic F-module with S an essential simple submodule, $S \cap IM = IS$ for any left primitive ideal I of T. Implies the zero submodule is prime.*
Proof . *Let $I = \{a \in F : aS = 0\}$. Yield I is left primitive of F. Further, $S \cap IM = IS = 0$, but S is essential in M so $IM = 0$. Assume J be an ideal of F and $0 \neq A$ submodule of M, $JA = 0$. Because S is the intersection of each nonzero submodules of M, implies that $S \subseteq A$ and hence $JS = 0$. Thus $J \subseteq I$ it follows that $JM = 0$. This means that $< 0 >$ is prime.* □

Proposition 3.9. *The next are equal.*

(1) *Each submodule is isolated of M .*
(2) *Each proper submodule is radical.*
(3) *$A \cap IM = IA$ for each A of M and for each ideal I of T.*
(4) *$A \cap PM = PA$ for every A of M and for every left primitive I of T.*
(5) *For every $x \in M$ and $r \in R$, $rx = rtrx$ for some element $t \in T$.*
(6) *$\frac{T}{ann_R(x)}$ is regular for every element $x \in M$.*
(7) *$J \cap I = JI$ for any two ideals I and J of R.*
(8) *$r = rtr$ for every element $r \in R$ and for some element $t \in R$.*

Proof .
(1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$(3) $\Longleftrightarrow$ (4) by [3] and (3) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) by [6]. □

Compare the following lemma with lemma 3.8

Lemma 3.10. *Let M be a cocyclic over a principal ideal ring T with an essential simple submodule S such that $S \cap I^nM = I^nS$ for n. Yield the zero submodule is primary.*
Proof . *By similar way of Lemma 3.8.* □

We finish our work by the following result

Theorem 3.11. *The next are the same for M over a principal ideal ring S.*

(1) *Any submodule is primary.*
(2) *Every proper submodule is primary radical.*
(3) $A \cap I^n M = I^n A$ *for each A of M and each ideal I of S and for some positive integer n.*
(4) $A \cap I^n M = I^n A$ *for each A of M and left primitive I of S.*
(5) *For every $x \in M$ and $r \in R$, $r^n x = r^n t r^n x$ for element $t \in R$ and for positive integer n.*
(6) $\frac{R}{ann_R(x)}$ *is n-regular ring for every element $x \in M$*
(7) $J \cap I^n = J I^n$ *for every ideal I and J of R.*
(8) $r^n = r^n t r^n$ *for every element $r \in R$, for element $t \in R$.*

Proof .
(1)$\Longrightarrow$(2) Let H be a proper of M. We show that $P\text{-}rad_M(H) = H$. It is clear that $P\text{-}rad_M(H) \supseteq H$. Suppose that H is a proper submodule in $P\text{-}rad_M(H)$. By (1), $P\text{-}rad_M(H)$ is primary isolated implies that there is primary A of M such that $A \supseteq H$ and $A \not\supseteq P\text{-}rad_M(H)$ which is a contradiction. Thus $P\text{-}rad_M(H) = H$.
(2)$\Longrightarrow$(3) Assume that every submodule is contained properly in M is primary radical so that for each A of M, and I an ideal of R we have, $I^n A = P\text{-}rad_M(r^n A)$ and $A \cap I^n M = P\text{-}rad_M(A \cap r^n M)$. By Corollary 3.4, $P\text{-}rad_M(I^n A) = P\text{-}rad_M(A \cap I^n M)$ and hence $A \cap I^n M = I^n A$.
(3)$\Longrightarrow$(4) It is clear.
(4)$\Longrightarrow$(2) Let H be proper of M. For each $x \in M$ and $x \notin H$, let A_x be a submodule in M maximal with the properties $H \subseteq A_x$ and $x \notin A_x$. Obviously, $H = \bigcap_{x\in M} A_x$ and H is one of the elements of this family $\{A_x\}_{x\in M}$. Let $x \in M$ and $x \notin H$. Put $B_x =< x > +A_x$. We claim that $\frac{B_x}{A_x}$ is an essential simple submodule in $\frac{M}{A_x}$. To show this. Suppose that $\frac{B_x}{A_x}$ is not simple in $\frac{M}{A_x}$ so that there exists a nonzero a proper submodule $\frac{D_x}{A_x}$ in $\frac{B_x}{A_x}$ for some submodule D_x of M. Now, there are two cases, if $x \notin D_x$ and by assumption $H \subseteq A_x \subseteq D_x$ which is contradicts with maximality of A_x with respect to H. If $x \in D_x$, then $\frac{D_x}{A_x} = \frac{D_x}{A_x} \cap \frac{B_x}{A_x} = \frac{D_x}{A_x} \cap \frac{<x>+A_x}{A_x} = \frac{D_x \cap (<x>+A_x)}{A_x} = \frac{(D_x \cap <x>)+(D_x \cap A_x)}{A_x} = \frac{<x>+A_x}{A_x}$ which is a contradiction. This means that $\frac{B_x}{A_x}$ must be a simple submodule in $\frac{M}{A_x}$. Now we prove that $\frac{B_x}{A_x}$ is an essential submodule of $\frac{M}{A_x}$. Suppose that $\frac{B_x}{A_x}$ is not essential in $\frac{M}{A_x}$ so there exists nonzero a proper submodule $\frac{D_x}{A_x}$ of $\frac{M}{A_x}$ with $\frac{D_x}{A_x} \cap \frac{<x>+A_x}{A_x} = 0_{\frac{M}{A_x}}$. By similar way of above argument, if $x \notin D_x$ lead us to a contradiction. If $x \in D_x$, then $0_{\frac{M}{A_x}} = \frac{D_x}{A_x} \cap \frac{B_x}{A_x} = \frac{<x>+A_x}{A_x}$ which is a contradiction. Hence $\frac{M}{A_x}$ is a cocyclic R-module. By (4), $B_x \cap I^n M = I^n B_x$ for each left primitive I, implies $\frac{B_x}{A_x} \cap I^n \frac{M}{A_x} = I^n \frac{B_x}{A_x}$. By Theorem 3.11, A_x is primary of M for each $x \in M$ and $x \notin H$.
(2)$\Longrightarrow$(1) Let A be proper of M and B be proper of A. By (2), $B = P\text{-}rad_M(B)$ is primary of M and $B \subseteq B$ and $A \nsubseteq B$ as desired. If $A = M$, then A is a summand of M and because every proper submodule is primary radical (and hence primary) so by Proposition 3.6, M is primary isolated.
(3) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) by [1]. $\square$

4. Conclusion

In this work we present the primary isolated submodules as generalization of isolated submodules. We see that many properties of isolated submodules can be extended to primary isolated submodules.

References

[1] G. Ahmed, *Some Generalization of Regular Modules*, Ph. D. Thesis, College of science, University of Baghdad, 2015.

[2] G. Ahmed, *The primary radical of submodules*, Iraqi J. Sci. to appear.
[3] R.L. McCasland and P.F. Smith, *On isolated submodules*, Commut. Alge. 34(8) (2006) 2977–2988.
[4] P.F. Smith, *Primary modules over commutative rings*, Glasgow Math. J. 43(1) (2001) 103–111.
[5] R. Wisbauer, *Foundations of Modules and Rings Theory*, Philadelphia, Gordon and Breach, 1991.
[6] S.M. Yaseen, *F-Regular Modules*, M.Sc. Thesis, College of science, University of Baghdad, 1993.