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Relation theoretic results via simulation function with applications

Swati Antal[a,*], Anita Tomar[b], U. C. Gairola[a]

[a]H.N.B. Garhwal University, BGR Campus Pauri, Pauri Garhwal-246001, Uttarakhand, India
[b]Government Degree College Thatyur (Tehri Garhwal), Uttarakhand, India

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Abstract

We introduce Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction to investigate the existence of a single fixed point under a binary relation. In the sequel, we demonstrate that a variety of contractions are obtained as consequences of our contraction. Also, we provide illustrative examples to demonstrate the significance of Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction in the existence of a fixed point for a discontinuous map via binary relation. The paper is concluded by applications to solve an integral equation and a nonlinear matrix equation.

Keywords: (Binary relation, transitive relation, Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction, $\mathcal{R}$-continuity)
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1. Introduction and preliminaries

The Banach contraction principle is the original and eminent outcome of the metric fixed point theory formulated by S. Banach [6]. It confirms the existence and uniqueness of a contraction on a complete metric space thereby providing an effective method to find a fixed point. Fixed point theory plays a significant role in analysis to solve differential equation, integral equation, nonlinear matrix equation and so on (for instance, [4], [5], [12], [16], [18], [21], [25], [27], [28] and so on). Motivated by the fact that on utilizing a simulation function, different contractive conditions are expressible in a simple and unified way, Khojasteha et al. [13] familiarized a simulation function to study the fixed point for $\mathcal{Z}$-contraction type operators. On the other hand, Turinici [29] instigated the perception of

*Swati Antal
Email addresses: antalswati11@gmail.com (Swati Antal), anitatmr@yahoo.com (Anita Tomar), ucgairola@rediffmail.com (U. C. Gairola)

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order-theoretic fixed point and Ran and Reurings [21] rediscovered an order-theoretic variant of the Banach contraction principle. Recently Tomar et al. [27] established a fixed of a set-valued map in a partial Pompeiu-Hausdorff metric space for a relation theoretic contractions and provided a novel answer to the open question posed by Rhoades [22] on continuity at a fixed point.

In this paper, we familiarize Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction under an arbitrary binary relation to examine the existence and uniqueness of a fixed point of a discontinuous single valued mapping in a noncomplete metric space. In the sequel, with the help of examples and remarks, we demonstrate that Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction includes, extends, unifies, and improves a large number of non-linear contractions existing in the literature and marks supremacy over all those contractions wherein the continuity of mapping and completeness (or closedness) of the whole space/subspaces are assumed for the existence of a fixed point. Lastly, we utilize our contraction to solve an integral equation and a nonlinear matrix equation to demonstrate the effectiveness of our result. The solution of a matrix equation is also validated by a numerical example in which the matrix is assumed to be Hermitian. However in most of the examples existing in the literature authors have considered a symmetric matrix (For instance: Long et al. [16], Sawangsup and Sintunavarat [26]). It is worth mentioning here that the symmetric matrix is a Hermitian matrix but the Hermitian matrix is not a symmetric matrix thereby revealing the prominence of our results over others existing in the literature.

Definition 1.1. *[13] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a simulation function if:*

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(\mathfrak{t},\mathfrak{s}) < \mathfrak{s} - \mathfrak{t}$, $\mathfrak{s},\mathfrak{t} > 0$;
(ζ_3) $\{\mathfrak{t}_n\}$ *and* $\{\mathfrak{s}_n\}$ *are sequences in* $(0,\infty)$ *satisfying* $\lim_{n\to\infty} \mathfrak{t}_n = \lim_{n\to\infty} \mathfrak{s}_n > 0$, *then* $\limsup_{n\to\infty} \zeta(\mathfrak{t}_n,\mathfrak{s}_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Example 1.2. *[13] If $\phi, \psi : [0,\infty) \to [0,\infty)$ are two continuous functions and $\mathfrak{t},\mathfrak{s} \in [0,\infty)$ such that $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, satisfy*

(i) $\zeta(\mathfrak{t},\mathfrak{s}) = \psi(\mathfrak{s}) - \phi(\mathfrak{t})$, *where* $\psi(\mathfrak{t}) = \phi(\mathfrak{t}) = 0$ *iff* $\mathfrak{t} = 0$ *and* $\psi(\mathfrak{t}) < \mathfrak{t} \leq \phi(\mathfrak{t})$.

(ii) $\zeta(\mathfrak{t},\mathfrak{s}) = \mathfrak{s} - \dfrac{\phi(\mathfrak{t},\mathfrak{s})}{\psi(\mathfrak{t},\mathfrak{s})}\mathfrak{t}$, *where* $\phi(\mathfrak{t},\mathfrak{s}) > \psi(\mathfrak{t},\mathfrak{s})$.

(iii) $\zeta(\mathfrak{t},\mathfrak{s}) = \mathfrak{s} - \phi(\mathfrak{s}) - \mathfrak{t}$, *where* $\phi(\mathfrak{t}) = 0$ *iff* $\mathfrak{t} = 0$.

Then in (i)-(iii), ζ is a simulation function.

In what follows $(\mathcal{X}, d)$, $\mathcal{R}$, $\mathbb{N}$, and $\mathbb{N}_0$ respectively, stand for a metric space, a non-empty binary relation defined on a non-empty set $\mathcal{X}$, the set of natural numbers, and the set of whole numbers.

Definition 1.3. *[15] A binary relation $\mathcal{R}$ on a non-empty set $\mathcal{X}$ is defined as a subset of $\mathcal{X} \times \mathcal{X}$. We say that "$\upsilon$ is $\mathcal{R}$-related to ω" iff $(\upsilon,\omega) \in \mathcal{R}$.*

Definition 1.4. *[17] $\mathcal{R}$ is complete if either $(\upsilon,\omega) \in \mathcal{R}$ or $(\omega,\upsilon) \in \mathcal{R}$ (i.e. $[\upsilon,\omega] \in \mathcal{R}$), $\forall\, \upsilon,\omega \in \mathcal{X}$.*

Definition 1.5. *[1] Let $\mathcal{T}$ be a self-mapping defined on a non-empty set $\mathcal{X}$. Then $\mathcal{R}$ is $\mathcal{T}$-closed if*

$$(\upsilon,\omega) \in \mathcal{R} \Rightarrow (\mathcal{T}\upsilon, \mathcal{T}\omega) \in \mathcal{R}, \;\; \upsilon,\omega \in \mathcal{X}.$$

Definition 1.6. *[15] The symmetric closure $\mathcal{R}^s$ is the smallest symmetric relation containing $\mathcal{R}$, i.e., $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.*

Proposition 1.7. *[1] If $\mathcal{R}$ is $\mathcal{T}$-closed, then $\mathcal{R}^s$ is also $\mathcal{T}$-closed.*

Definition 1.8. *[1] A sequence $\{\upsilon_n\}$ in $\mathcal{X}$ is $\mathcal{R}$-preserving if*

$$(\upsilon_n, \upsilon_{n+1}) \in \mathcal{R}, \;\; n \in \mathbb{N}_0.$$

Definition 1.9. *[14] $\mathcal{R}$ is transitive if $(\upsilon, \omega) \in \mathcal{R}$ and $(\omega, \rho) \in \mathcal{R}$ implies that $(\upsilon, \rho) \in \mathcal{R}$*

Definition 1.10. *[1] $\mathcal{R}$ is d-self-closed if $\{\upsilon_n\}$ is an $\mathcal{R}$-preserving sequence and if*

$$\upsilon_n \xrightarrow{d} \upsilon \;\; as \;\; n \to \infty,$$

then there exists a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ with $[\upsilon_{n_k}, \upsilon] \in \mathcal{R}$, $k \in \mathbb{N}$.

Definition 1.11. *[2] $(\mathcal{X}, d)$ is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{X}$ converges to a point in $\mathcal{X}$.*

Remark 1.12. *[2] A complete metric space is $\mathcal{R}$-complete. However, the reverse is not essentially true. Further, if $\mathcal{R}$ is universal relation, then completeness coincides with $\mathcal{R}$-completeness.*

Definition 1.13. *[2] A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is $\mathcal{R}$-continuous at υ if for any $\mathcal{R}$-preserving sequence $\{\upsilon_n\}$ with $\upsilon_n \xrightarrow{d} \upsilon$, $\mathcal{T}(\upsilon_n) \xrightarrow{d} \mathcal{T}(\upsilon)$. Further, $\mathcal{T}$ is $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at every point of $\mathcal{X}$.*

Remark 1.14. *[2] Every continuous mapping is $\mathcal{R}$-continuous. However, the reverse is not essentially true. Further, if $\mathcal{R}$ is universal relation, then continuity coincides with $\mathcal{R}$-continuity.*

Definition 1.15. *[25] A subset D of $\mathcal{X}$ is $\mathcal{R}$-directed if for each pair of points $\upsilon, \omega \in D$, there exists $\rho \in \mathcal{X}$ satisfying $(\upsilon, \rho) \in \mathcal{R}$ and $(\omega, \rho) \in \mathcal{R}$.*

Definition 1.16. *[14] Let $\mathcal{X}$ be a nonempty set, $\mathcal{Y} \subseteq \mathcal{X}$. Then, $\mathcal{R}|_{\mathcal{Y}}$, the restriction of $\mathcal{R}$ to $\mathcal{Y}$, is the set $\mathcal{R} \cap \mathcal{Y}^2$ (i.e. $\mathcal{R}|_{\mathcal{Y}} := \mathcal{R} \cap \mathcal{Y}^2$). In fact, $\mathcal{R}|_{\mathcal{Y}}$ is a relation on $\mathcal{Y}$ induced by $\mathcal{R}$.*

Definition 1.17. *[3] $\mathcal{R}$ is locally transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{\upsilon_n\} \subset \mathcal{X}$ (range $\mathcal{Y} := \upsilon_n : n \in \mathbb{N}_0$), the binary relation $\mathcal{R}|_{\mathcal{Y}}$ is transitive ($\mathcal{R}|_{\mathcal{Y}}$: the restriction of $\mathcal{R}$ to $\mathcal{Y}$).*

Definition 1.18. *[14] For $\upsilon, \omega \in \mathcal{X}$, a path of length k in $\mathcal{R}$ from υ to ω is a finite sequence $\{\rho_0, \rho_1, \rho_2, ..., \rho_k\} \subset \mathcal{X}$ satisfying:*
(i) $\rho_0 = \upsilon$ and $\rho_k = \omega$,
(ii) $(\rho_i, \rho_{i+1}) \in \mathcal{R}$ for each i $(0 \le i \le k-1)$ (k is a natural number).
Clearly, a path of length k necessitates $k+1$ elements of $\mathcal{X}$, which are not essentially distinct.

In the following

$\mathcal{X}(\mathcal{T}; \mathcal{R}) := \{\upsilon \in \mathcal{X} : (\upsilon, \mathcal{T}\upsilon) \in \mathcal{R}\}$, where $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\gamma(\upsilon, \omega, \mathcal{R})$ is the class of all paths in $\mathcal{R}$ from υ to ω.

2. Main result

We introduce Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction mapping under an arbitrary binary relation $\mathcal{R}$.

Definition 2.1. *Let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a mapping of a metric space $(\mathcal{X},d)$ equipped with a binary relation $\mathcal{R}$ and $\zeta\in\mathcal{Z}$. If*

$$\zeta(d(\mathcal{T}\upsilon,\mathcal{T}\omega),M_{\mathcal{T}}(\upsilon,\omega))\geq 0,\ \ \upsilon,\omega\in\mathcal{X},\ \ (\upsilon,\omega)\in\mathcal{R}, \tag{2.1}$$

where

$$M_{\mathcal{T}}(\upsilon,\omega)=max\Big\{d(\upsilon,\omega),d(\upsilon,\mathcal{T}\upsilon),d(\omega,\mathcal{T}\omega),\frac{d(\upsilon,\mathcal{T}\omega)+d(\omega,\mathcal{T}\upsilon)}{2}\Big\}$$

then $\mathcal{T}$ is called a Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction.

Proposition 2.2. *If $(\mathcal{X},d)$ is a metric space equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ a Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction with respect to $\zeta\in\mathcal{Z}$, then the following complement each other*
(i) $\zeta(d(\mathcal{T}\upsilon,\mathcal{T}\omega),M_{\mathcal{T}}(\upsilon,\omega))\geq 0$, $\forall\ \upsilon,\omega\in\mathcal{X}$ with $(\upsilon,\omega)\in\mathcal{R}$,
(ii) $\zeta(d(\mathcal{T}\upsilon,\mathcal{T}\omega),M_{\mathcal{T}}(\upsilon,\omega))\geq 0$, $\forall\ \upsilon,\omega\in\mathcal{X}$ with $[\upsilon,\omega]\in\mathcal{R}$.

Proof . Clearly, $(ii)\Rightarrow(i)$ is trivial.
Conversely, let (i) be true. Taking $\upsilon,\omega\in\mathcal{X}$ and $[\upsilon,\omega]\in\mathcal{R}$. In case $(\upsilon,\omega)\in\mathcal{R}$, then (ii) follows from (i).

Now let $(\omega,\upsilon)\in\mathcal{R}$, then using (i) and the symmetry of the metric d,

$$\zeta(d(\mathcal{T}\upsilon,\mathcal{T}\omega),M_{\mathcal{T}}(\upsilon,\omega))=\zeta(d(\mathcal{T}\omega,\mathcal{T}\upsilon),M_{\mathcal{T}}(\omega,\upsilon))\geq 0$$

where

$$\begin{aligned} M_{\mathcal{T}}(\upsilon,\omega) &= max\Big\{d(\upsilon,\omega),d(\upsilon,\mathcal{T}\upsilon),d(\omega,\mathcal{T}\omega),\frac{d(\upsilon,\mathcal{T}\omega)+d(\omega,\mathcal{T}\upsilon)}{2}\Big\}\\ &= max\Big\{d(\omega,\upsilon),d(\omega,\mathcal{T}\omega),d(\upsilon,\mathcal{T}\upsilon),\frac{d(\omega,\mathcal{T}\upsilon)+d(\upsilon,\mathcal{T}\omega)}{2}\Big\}\\ &= M_{\mathcal{T}}(\omega,\upsilon)\end{aligned}$$

this shows that $(i)\Rightarrow(ii)$. □

Now we utilize Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction to establish the fixed-point.

Theorem 2.3. *Let $(\mathcal{X},d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ be a self-mapping on $\mathcal{X}$. Let the following hypotheses hold:*

(a) there exist $\mathcal{Y}\subseteq\mathcal{X}$, $\mathcal{T}\mathcal{X}\subseteq\mathcal{Y}\subseteq\mathcal{X}$ so that $(\mathcal{Y},d)$ is $\mathcal{R}$- complete,
(b) $\mathcal{X}(\mathcal{T};\mathcal{R})\neq\phi$,
(c) $\mathcal{R}$ is $\mathcal{T}$-closed and $\mathcal{R}$ is transitive.
(d) $\mathcal{T}$ is a Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction,
(e) either $\mathcal{R}|_{\mathcal{Y}}$ is d-self-closed or $\mathcal{T}$ is $\mathcal{R}$-continuous.

Then $\mathcal{T}$ has a fixed point.

Proof . Since $\mathcal{X}(\mathcal{T};\mathcal{R}) \neq \phi$. Let $\upsilon_0 \in \mathcal{X}(\mathcal{T};\mathcal{R})$. Construct a Picard sequence $\{\upsilon_n\}$ as $\upsilon_{n+1} = \mathcal{T}\upsilon_n$, $\forall\ n \in \mathbb{N}_0$. Since $(\upsilon_0, T\upsilon_0) \in \mathcal{R}$ and $\mathcal{R}$ is $\mathcal{T}$-closed,

$$(\mathcal{T}\upsilon_0, \mathcal{T}^2\upsilon_0), (\mathcal{T}^2\upsilon_0, \mathcal{T}^3\upsilon_0), ..., (\mathcal{T}^n\upsilon_0, \mathcal{T}^{n+1}\upsilon_0), ... \in \mathcal{R}.$$

Thus

$$(\upsilon_n, \upsilon_{n+1}) \in \mathcal{R}, \tag{2.2}$$

and the sequence $\{\upsilon_n\}$ is $\mathcal{R}$-preserving. Since $\mathcal{T}$ is Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction, we have

$$\zeta(d(\mathcal{T}\upsilon_{n-1}, \mathcal{T}\upsilon_n), M_{\mathcal{T}}(\upsilon_{n-1}, \upsilon_n)) \geq 0, \tag{2.3}$$

where

$$\begin{aligned} M_{\mathcal{T}}(\upsilon_{n-1}, \upsilon_n) &= max\Big\{d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_{n-1}, \mathcal{T}\upsilon_{n-1}), d(\upsilon_n, \mathcal{T}\upsilon_n), \\ &\quad \frac{d(\upsilon_n, \mathcal{T}\upsilon_{n-1}) + d(\upsilon_{n-1}, \mathcal{T}\upsilon_n)}{2}\Big\} \\ &= max\Big\{d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_n, \upsilon_{n+1}), \\ &\quad \frac{d(\upsilon_n, \upsilon_n) + d(\upsilon_{n-1}, \upsilon_{n+1})}{2}\Big\} \\ &= max\Big\{d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_n, \upsilon_{n+1}), \frac{d(\upsilon_{n-1}, \upsilon_{n+1})}{2}\Big\} \\ &= max\Big\{d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_n, \upsilon_{n+1})\Big\}. \end{aligned}$$

From (2.3) we get

$$0 \leq \zeta\big(d(\upsilon_n, \upsilon_{n+1}), M_{\mathcal{T}}(\upsilon_{n-1}, \upsilon_n)\big) = \zeta\big(d(\upsilon_n, \upsilon_{n+1}), max(d(\upsilon_{n-1}, \upsilon_n), d(\upsilon_n, \upsilon_{n+1})\big).$$

Suppose that $d(\upsilon_n, \upsilon_{n+1}) > d(\upsilon_{n-1}, \upsilon_n)$ for some $n \in \mathbb{N}_0$, then from (2.1)

$$0 \leq \zeta\big(d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_n, \upsilon_{n+1})\big),$$

a contradiction. Thus $d(\upsilon_n, \upsilon_{n+1}) < d(\upsilon_{n-1}, \upsilon_n)$, $n \in \mathbb{N}_0$ and

$$0 \leq \zeta\big(d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\big).$$

So $\{d(\upsilon_n, \upsilon_{n+1})\}$ is a decreasing sequence of non-negative real numbers. Therefore, it is convergent. Suppose

$$\lim_{n\to\infty} d(\upsilon_n, \upsilon_{n+1}) = r \geq 0.$$

If $r > 0$ then from (2.2) and (ζ_3)

$$0 \leq \limsup_{n\to\infty} \zeta\big(d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n\big) < 0,$$

a contradiction. So $r = 0$, i.e.,

$$\lim_{n\to\infty} d(\upsilon_n, \upsilon_{n+1}) = 0. \tag{2.4}$$

Now we shall show that $\{\upsilon_n\}$ is a Cauchy sequence. If possible, let $\{\upsilon_n\}$ be not a Cauchy sequence. Then by Lemma 4 [20], there exist $\epsilon > 0$ and subsequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that the sequences

$$\{d(\upsilon_{m_k}, \upsilon_{n_k}), d(\upsilon_{m_k}, \upsilon_{n_k+1}), d(\upsilon_{m_k-1}, \upsilon_{n_k}), d(\upsilon_{m_k-1}, \upsilon_{n_k+1}), d(\upsilon_{m_k+1}, \upsilon_{n_k+1})\} \\ \to \epsilon \; as \; k \to +\infty.$$

Thus

$$\lim_{k\to\infty} M_{\mathcal{T}}(\upsilon_{m_k}, \upsilon_{n_k}) = \lim_{k\to\infty} max\Big\{d(\upsilon_{m_k}, \upsilon_{n_k}), d(\upsilon_{m_k}, \mathcal{T}\upsilon_{m_k}), d(\upsilon_{n_k}, \mathcal{T}\upsilon_{n_k}), \\ \frac{d(\upsilon_{m_k}, \mathcal{T}\upsilon_{n_k}) + d(\upsilon_{n_k}, \mathcal{T}\upsilon_{m_k})}{2}\Big\} = \epsilon.$$

Thus applying (d) with $\upsilon = \upsilon_{m_k}$ and $\omega = \upsilon_{n_k}$, we get

$$0 \le \limsup_{n\to\infty}(d(\upsilon_{m_k+1}, \upsilon_{n_k+1}), M_{\mathcal{T}}(\upsilon_{m_k}, \upsilon_{n_k})) < 0,$$

a contradiction. Thus $\{\upsilon_n\}$ is a Cauchy sequence in $\mathcal{X}$.

Since $\{\upsilon_n\} \subseteq \mathcal{T}\mathcal{X} \subseteq \mathcal{Y}$, therefore $\{\upsilon_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{Y}$. Since $(\mathcal{Y}, d)$ is $\mathcal{R}$-complete, there exists $p \in \mathcal{Y}$ satisfying $\upsilon_n \xrightarrow{d} p$. If $\mathcal{T}$ is $\mathcal{R}$-continuous,

$$p = \lim_{n\to\infty} \upsilon_{n+1} = \lim_{n\to\infty} \mathcal{T}\upsilon_n = \mathcal{T} \lim_{n\to\infty} \upsilon_n = \mathcal{T}p.$$

Hence p is a fixed point of $\mathcal{T}$.

On the other hand, if $\mathcal{R}$ is d-self-closed. Since $\{\upsilon_n\}$ is an $\mathcal{R}$-preserving sequence and

$$\upsilon_n \xrightarrow{d} \upsilon,$$

there exists a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ with

$$[\upsilon_{n_k}, \upsilon] \in \mathcal{R}|_{\mathcal{Y}}, \quad \forall\ k \in \mathbb{N}_0.$$

Using (d) and Proposition 2.2, $[\upsilon_{n_k}, \upsilon] \in \mathcal{R}$ and $\upsilon_{n_k} \xrightarrow{d} \upsilon$, we have

$$\begin{aligned} 0 \le \zeta(d(\upsilon_{n_k+1}, \mathcal{T}\upsilon), M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon)) &= \zeta(d(\mathcal{T}\upsilon_{n_k}, \mathcal{T}\upsilon), M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon)) \\ &< M_{\mathcal{T}}(\upsilon_{n_k}, x) - d(\mathcal{T}\upsilon_{n_k}, \mathcal{T}\upsilon) \end{aligned}$$

$$\Longrightarrow \qquad d(\mathcal{T}\upsilon_{n_k}, \mathcal{T}\upsilon) \le M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon),$$

where

$$M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon) = max\Big\{d(\upsilon_{n_k}, \upsilon), d(\upsilon_{n_k}, \upsilon_{n_k+1}), d(\upsilon, \mathcal{T}\upsilon), \frac{d(\upsilon_{n_k}, \mathcal{T}\upsilon) + d(\upsilon, \upsilon_{n_k+1})}{2}\Big\}.$$

By (2.4) and taking limit $k \to \infty$

$$\lim_{k\to\infty} d(\upsilon_{n_k+1}, \mathcal{T}\upsilon) = \lim_{k\to\infty} M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon) = d(\upsilon, \mathcal{T}\upsilon) > 0.$$

From the condition (ζ_3)

$$0 \le \limsup_{k\to\infty} \zeta(d(\upsilon_{n_k+1}, \mathcal{T}\upsilon), M_{\mathcal{T}}(\upsilon_{n_k}, \upsilon)) < 0,$$

a contradiction and hence $d(\upsilon, \mathcal{T}\upsilon) = 0$, i.e., $\mathcal{T}\upsilon = \upsilon$, i.e., υ is a fixed point of $\mathcal{T}$. □

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, if*

(f) $\gamma(\upsilon,\omega,\mathcal{R})\neq\phi$.

Then $\mathcal{T}$ *has a unique fixed point.*

Proof . Let υ^*,ω^* be two fixed points of $\mathcal{T}$ so that $\upsilon^*\neq\omega^*$. Since $\gamma(\upsilon^*,\omega^*,\mathcal{R})\neq\phi$, there exists a path $\{\rho_0,\rho_1,\rho_2,...,\rho_k\}$ of some finite length k in $\mathcal{R}$ from υ to ω so that

$$\rho_0=\upsilon^*,\rho_k=\omega^*,(\rho_i,\rho_{i+1})\in\mathcal{R},\ \ i=0,1,2,...,k-1.$$

Since $\mathcal{R}$ is transitive,

$$(\rho_0,\rho_k)\in\mathcal{R}.$$

Therefore,

$$\begin{aligned}0\le\zeta(d(\mathcal{T}\rho_0,\mathcal{T}\rho_k),M_{\mathcal{T}}(\rho_0,\rho_k)) &< M_{\mathcal{T}}(\rho_0,\rho_k)-d(\mathcal{T}\rho_0,\mathcal{T}\rho_k)\\ &= M_{\mathcal{T}}(\upsilon^*,\omega^*)-d(\upsilon^*,\omega^*)=0,\end{aligned}$$

a contradiction. Thus $\mathcal{T}$ has a unique fixed point. □

The following examples are given to appreciate the effectiveness of Theorem 2.4 and to validate the result proved herein.

Example 2.5. *Let* $\mathcal{X}=[0,4)$ *equipped with a usual metric and a binary relation* $\mathcal{R}=\big\{(0,0),(0,1),(0,2),(0,3),(1,1),(1,3),(2,2)\big\}$. *Let a self-mapping* $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ *be defined by*

$$\mathcal{T}(\upsilon)=\begin{cases}0, & \upsilon\in[0,1],\\ 1, & \upsilon\in(1,4).\end{cases}$$

Let $\mathcal{Y}=[0,1]$, *then clearly* $\mathcal{T}\upsilon=\{0,1\}\subset\mathcal{Y}$ *and* $\mathcal{Y}$ *is* $\mathcal{R}$*-complete. Evidently,* $\mathcal{T}$ *is not continuous but* $\mathcal{R}$ *is* $\mathcal{T}$*-closed and transitive. For* $\upsilon=0$, $\mathcal{T}\upsilon=0$, $(\upsilon,\mathcal{T}\upsilon)\in\mathcal{R}$, *i.e.,* $\mathcal{X}(\mathcal{T},\mathcal{R})\neq\phi$. *If we take any* $\mathcal{R}$*-preserving sequence* $\{\upsilon_n\}$ *with*

$$\upsilon_n\xrightarrow{d}\upsilon \text{ and } (\upsilon_n,\upsilon_{n+1})\in\mathcal{R},\forall\, n\in\mathbb{N}_0,$$

$(\upsilon_n,\upsilon_{n+1})\in\mathcal{R}|_{\mathcal{Y}}$, $\forall\ n\in\mathbb{N}_0$ *and there exists* $N\in\mathbb{N}_0$ *such that* $\upsilon_n=\upsilon\in\{0,1\}$, $\forall\ n\ge N$. *A subsequence* $\{\upsilon_{n_k}\}=\{\upsilon\}$, $\forall\ k\in\mathbb{N}_0$, *in such a way that* $(\upsilon_{n_k},\upsilon)\in\mathcal{R}|_{\mathcal{Y}}$, $\forall\ k\in\mathbb{N}_0$. *Therefore,* $\mathcal{R}|_{\mathcal{Y}}$ *is d-self-closed.*

Let $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ *be defined by* $\zeta(\mathfrak{t},\mathfrak{s})=\frac{4}{5}\mathfrak{s}-\mathfrak{t}$, $\mathfrak{s},\mathfrak{t}\in[0,\infty)$. *Now, with a view to check that* $\mathcal{T}$ *is Ćirić type* $\mathcal{Z}_{\mathcal{R}}$*-contraction, let* $(\upsilon,\omega)=\{(0,2),(0,3),(1,3)\}$ *(as in rest of cases* $d(\mathcal{T}\upsilon,\mathcal{T}\omega)=0$*), we have*

$$\zeta\big(d(\mathcal{T}\upsilon,\mathcal{T}\omega),M_{\mathcal{T}}(\upsilon,\omega)\big)=\frac{4}{5}M_{\mathcal{T}}(\upsilon,\omega)-d(\mathcal{T}\upsilon,\mathcal{T}\omega)\ge0$$

$\Longrightarrow$

$$d(\mathcal{T}\upsilon,\mathcal{T}\omega)\le\frac{4}{5}M_{\mathcal{T}}(\upsilon,\omega). \tag{2.5}$$

If $(\upsilon,\omega)=\{(0,2),(0,3),(1,3)\}$, *then from (2.5)*

$$d(\mathcal{T}\upsilon,\mathcal{T}\omega)=d(0,1)=1\le\frac{4}{5}M_{\mathcal{T}}(\upsilon,\omega).$$

Thus all the hypotheses of Theorem 2.4 (Theorem 2.3) are verified and 0 *is the only fixed point of* $\mathcal{T}$. *Noticeably, neither* $(\mathcal{X},d)$ *is complete nor* $\mathcal{R}$*-complete.*

Example 2.6. *Let $\mathcal{X} = [0,4]$ be equipped with a usual metric and a binary relation $\mathcal{R} = \{(1,1),(3,3),(4,4),(1,2),(1,4),(3,4)\}$. Let a self-mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by*

$$\mathcal{T}(\upsilon) = \begin{cases} 1, & \upsilon \in [0,1), \\ 3, & \upsilon \in [1,2), \\ 4, & \upsilon \in [2,4] \end{cases}$$

Let $\mathcal{Y} = [1,4]$. Clearly, $\mathcal{T}\upsilon = \{1,3,4\} \subset \mathcal{Y} \subset \mathcal{X}$ where $\mathcal{Y}$ is $\mathcal{R}$-complete, $\mathcal{T}$ is not continuous but $\mathcal{R}$ is $\mathcal{T}$-closed and transitive. For $\upsilon = 3$, $\mathcal{T}\upsilon = 4$, $(\upsilon, \mathcal{T}\upsilon) \in \mathcal{R}$, i.e., $\mathcal{X}(\mathcal{T}, \mathcal{R}) \neq \phi$. If we take any $\mathcal{R}$-preserving sequence $\{\upsilon_n\}$ with

$$\upsilon_n \xrightarrow{d} \upsilon \text{ and } (\upsilon_n, \upsilon_{n+1}) \in \mathcal{R}, \forall\, n \in \mathbb{N}_0.$$

Here, one may notice that $(\upsilon_n, \upsilon_{n+1}) \in \mathcal{R}|_{\mathcal{Y}}$, $\forall\, n \in \mathbb{N}_0$ and there exists $N \in \mathbb{N}_0$ such that $\upsilon_n = \upsilon \in \{1,3,4\}$, $\forall\, n \geq N$. Choose a subsequence $\{\upsilon_{n_k}\} = \{\upsilon\}$, $\forall\, k \in \mathbb{N}_0$, in such a way that $(\upsilon_{n_k}, \upsilon) \in \mathcal{R}|_{\mathcal{Y}}$, $\forall\, k \in \mathbb{N}_0$. Therefore, $\mathcal{R}|_{\mathcal{Y}}$ is d-self closed.

Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\zeta(\mathfrak{t}, \mathfrak{s}) = \frac{6}{7}\mathfrak{s} - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0,\infty)$. Now, with a view to check that $\mathcal{T}$ is Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction, let $(\upsilon, \omega) = \{(1,2),(1,4)\}$ (as in rest of cases $d(\mathcal{T}\upsilon, \mathcal{T}\omega) = 0$), we have

$$\zeta\big(d(\mathcal{T}\upsilon, \mathcal{T}\omega), M_{\mathcal{T}}(\upsilon,\omega)\big) = \frac{6}{7} M_{\mathcal{T}}(\upsilon,\omega) - d(\mathcal{T}\upsilon, \mathcal{T}\omega) \geq 0$$

$\Longrightarrow$

$$d(\mathcal{T}\upsilon, \mathcal{T}\omega) \leq \frac{6}{7} M_{\mathcal{T}}(\upsilon,\omega). \tag{2.6}$$

Case (i): *If $(\upsilon, \omega) = (1,2)$, then from (2.6)*

$$d(\mathcal{T}1, \mathcal{T}2) = d(3,4) = 1 \leq \frac{6}{7} M_{\mathcal{T}}(1,2) = 1.71.$$

Case (ii): *If $(\upsilon, \omega) = (1,4)$, then from (2.6)*

$$d(\mathcal{T}1, \mathcal{T}4) = d(3,4) = 1 \leq \frac{6}{7} M_{\mathcal{T}}(1,4) = 2.57.$$

shows that condition (d) is verified. Thus, all the hypotheses of Theorem 2.4 (Theorem 2.3) are true and $\upsilon = 4$ is the only fixed point of $\mathcal{T}$.

Remark 2.7. *In Example 2.6, observe that at $(\upsilon, \omega) = (1,2)$,*

$$d(\mathcal{T}1, \mathcal{T}2) = d(3,4) = 1 > k = kd(1,2),$$

for any $k \in [0,1)$. Thus neither the result of Alam and Imdad [1] nor the result of Sawangsup et al. [26] is applicable in the present example.

Remark 2.8. *It is worth mentioning that Theorem 2.4 is a genuine extension and improvement of Alam and Imdad [1] in a metric space, in view of the fact that we have neither used the completeness of whole space nor its subspace. Rather, we used a relatively weaker notion like: $\mathcal{R}$ completeness of any subspace of the whole space. Further, we replaced continuity of an involved map by its $\mathcal{R}$-continuity or d-self closedness of restriction of $\mathcal{R}$ to $\mathcal{Y}$ (One may check by a simple calculation that in Examples 2.5 and 2.6 involved map is not even $\mathcal{R}$-continuous. However, restriction of $\mathcal{R}$ to $\mathcal{Y}$ is d-self closed). Moreover, Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction is a significant generalization of Banach contraction used by Alam and Imdad [1]. In fact, binary relation $\mathcal{R}$ is a near-order and it is nonreflexive and nonsymmetric. Consequently, it is none of tolerance, preorder, strict order, or partial order. Further, it is never a symmetric closure of a binary relation $\mathcal{R}$.*

Remark 2.9. *Theorem 2.3 is true if the transitivity of $\mathcal{R}$ is substituted by local transitivity [3].*

In particular, on taking $\mathcal{Y} = \mathcal{X}$ in Theorem 2.4, we have:

Corollary 2.10. *Theorem 2.3 is true if (a) is replaced by*
(a') $(\mathcal{X}, d)$ is $\mathcal{R}$-complete.

In view of Remark 1.12 and Remark 1.14, the following natural result is obtained:

Corollary 2.11. *Theorem 2.3 remains true if $\mathcal{R}$-completeness of $(\mathcal{Y}, d)$ is replaced by completeness in (a) and $\mathcal{R}$-continuity is replaced by continuity in (e).*

Remark 2.12. *Noticeably, Theorem 2.4 , Corollary 2.10, and 2.11 are relation theoretic variants of Olgun et al. [19]. It interesting to see(Examples 2.5 and 2.6) that $\mathcal{T}$ is neither a $\mathcal{Z}$-contraction [13] nor a generalized $\mathcal{Z}$-contraction [19] and consequently does not satisfy the hypotheses of [13], [15] and related results ([1], [6], [8] - [10], [23] and so on).*

Corollary 2.13. *Conclusions of the Theorem 2.3, Corollaries 2.10 and 2.11 remain true if the condition (d) is replaced by*

$$\zeta(d(\mathcal{T}\upsilon, \mathcal{T}\omega), d(\upsilon, \omega)) \geq 0, \;\; \upsilon, \omega \in \mathcal{X} \;\; with \;\; (\upsilon, \omega) \in \mathcal{R}.$$

It is interesting to see here that Corollary 2.13 is an improved relation theoretic version of Khojasteh et al. [13].

Remark 2.14. *A variety of extensions and improvements of known contractions are obtained on varying the elements of $\mathcal{Z}_{\mathcal{R}}$. For instance :*

(i) *Relation Theoretic version of Banach Contraction [6] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_B(\mathfrak{t}, \mathfrak{s}) = \lambda\mathfrak{s} - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$.

(ii) *Relation Theoretic version of Rhoades type contraction [23] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_R(\mathfrak{t}, \mathfrak{s}) = \mathfrak{s} - \phi(\mathfrak{s}) - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$, *where* $\phi : [0, \infty) \to [0, \infty)$ *is lower semi continuous function and* $\phi^{-1}(0) = \{0\}$.

(iii) *Relation Theoretic version of Geraghty contraction [10] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_G(\mathfrak{t}, \mathfrak{s}) = \mathfrak{s}\phi(\mathfrak{s}) - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$, *where* $\phi : [0, +\infty) \to [0, 1)$ *be a mapping such that* $\limsup_{t \to r^+} \phi(t) < 1$, $r > 0$.

(iv) *Relation Theoretic version of Boyd and Wong contraction [7] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_{BW}(\mathfrak{t}, \mathfrak{s}) = \eta(\mathfrak{s}) - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$, *where* $\eta : [0, +\infty) \to [0, +\infty)$ *is an upper semi continuous mapping,* $\eta(\mathfrak{t}) < \mathfrak{t}$, $\mathfrak{t} > 0$ *and* $\eta(0) = 0$.

(v) *Relation Theoretic version of Branciari contraction [8] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_B(\mathfrak{t}, \mathfrak{s}) = \mathfrak{s} - \int_0^{\mathfrak{t}} \phi(u)du$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$, *where* $\phi : [0, \infty) \to [0, \infty)$ *is such that* $\int_0^{\epsilon} \phi(\mathfrak{t})d\mathfrak{t}$ *exists and* $\int_0^{\epsilon} \phi(\mathfrak{t})d\mathfrak{t} > \epsilon$, $\epsilon > 0$.

(vi) *Relation Theoretic version of Hierro et al. [24] is obtained on taking* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *as* $\zeta_U(\mathfrak{t}, \mathfrak{s}) = \mathfrak{s}h(\mathfrak{t}, \mathfrak{s}) - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$, *where* $h : [0, +\infty) \to [0, +\infty)$ *is a mapping, such that* $h(\mathfrak{t}, \mathfrak{s}) < 1$ *and* $\limsup_{n \to \infty} h(\mathfrak{t}_n, \mathfrak{s}_n) < 1$ *provided that* $\{\mathfrak{t}_n\}$ *and* $\{\mathfrak{s}_n\} \subset (0, +\infty)$ *are sequences satisfying* $\lim_{n \to \infty} \mathfrak{t}_n = \lim_{n \to \infty} \mathfrak{s}_n$.

Remark 2.15. *(i) It is interesting to mention that relation-theoretic contractions are comparatively weaker than standard contractions since these hold only for the elements in the relation under consideration (see, in Examples 2.5, 2.6) and in the established results Ćirić type $\mathcal{Z}_{\mathcal{R}}$-contraction is assumed to hold only for transitive relation. Consequently, we are able to particularize the existing results to a variety of situations.*

(ii) Noticeably, in Rhoades [23], ϕ is taken to be continuous, nondecreasing, and $\lim_{t\to\infty}\phi(t)=\infty$. However, in view of Remark 2.14 (ii) we may replace it with a lower semicontinuity of ϕ. Consequently, our version is improved and extended than Rhoades [23].

3. Applications

3.1. Application to an integral equation

We solve an integral equation under some binary relation using Theorem 2.3. Let

$$\upsilon(\mathfrak{t}) = q(\mathfrak{t}) + \alpha\int_a^b k(\mathfrak{s},\mathfrak{t})f(\mathfrak{s},\upsilon(\mathfrak{s}))d\mathfrak{t},\ \ \mathfrak{t}\in\mathcal{I}=[a,b] \tag{3.1}$$

where υ is an unknown function on $\mathcal{I}=[a,b]$, $q:\mathcal{I}\to\mathbb{R}$, $k:\mathcal{I}\times\mathcal{I}\to\mathbb{R}$ and $f:\mathcal{I}\times\mathbb{R}\to\mathbb{R}$, $\alpha>0$ a parameter and $\mathcal{X}:=C(\mathcal{I},\mathbb{R})$.

Theorem 3.1. *Consider (3.1), such that*

(i) $\sup_{t\in\mathcal{I}}\int_a^b k(\mathfrak{s},\mathfrak{t})d\mathfrak{s} \le \dfrac{1}{\alpha(b-a)}$, $\mathfrak{t},\mathfrak{s}\in\mathcal{I}$

(ii) $|f(\mathfrak{s},\upsilon(\mathfrak{s}))-f(\mathfrak{s},\omega(\mathfrak{s}))| \le |\upsilon(\mathfrak{s})-\omega(\mathfrak{s})|$, $\forall\ \upsilon,\omega\in\mathbb{R}$.
Then equation (3.1) has a solution in $\mathcal{X}$.

Proof . Define $\mathcal{T}: C(\mathcal{I},\mathbb{R})\to C(\mathcal{I},\mathbb{R})$ by

$$\mathcal{T}\upsilon(\mathfrak{t}) = q(\mathfrak{t}) + \alpha\int_a^b k(\mathfrak{s},\mathfrak{t})f(\mathfrak{s},\upsilon(\mathfrak{s}))d\mathfrak{s},\ \ \mathfrak{t}\in\mathcal{I}=[a,b]$$

and a binary relation

$$\mathcal{R}=\{(\upsilon,\omega)\in C(\mathcal{I},\mathbb{R})\times C(\mathcal{I},\mathbb{R})|\upsilon(t)\le\omega(\mathfrak{t}),\forall\ \mathfrak{t}\in\mathcal{I}\}.$$

Here $C(\mathcal{I},\mathbb{R})$ is equipped with

$$d(\upsilon,\omega)=\sup_{\mathfrak{t}\in\mathcal{I}}|\upsilon(\mathfrak{t})-\omega(\mathfrak{t})|,\ \upsilon,\omega\in C(\mathcal{I},\mathbb{R})$$

and $(C(\mathcal{I},\mathbb{R}),d)$ is $\mathcal{R}$-complete.
Take $\mathcal{R}$-preserving sequence $\{\upsilon_n\}$ as $\upsilon_n\xrightarrow{d}\upsilon$. Then,

$$\upsilon_0(\mathfrak{t})\le\upsilon_1(\mathfrak{t})\le\upsilon_2(\mathfrak{t})\le ...\le\upsilon_n(\mathfrak{t})\le\upsilon_{n+1}(\mathfrak{t})\le ...,\mathfrak{t}\in\mathcal{I}.$$

and convergence to $\upsilon(\mathfrak{t})$ implies that $\upsilon_n(\mathfrak{t}) \leq \upsilon(\mathfrak{t})$, $n \in \mathbb{N}_0$, i.e., we can choose a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ such that $[\upsilon_{n_k}, \upsilon] \in \mathcal{R}$, $n \in \mathbb{N}_0$. Hence, $\mathcal{R}$ is d-self-closed.
If $(\upsilon, \omega) \in \mathcal{R}$, i.e., $\upsilon(\mathfrak{t}) \leq \omega(\mathfrak{t})$, $\alpha > 0$ and $k(\mathfrak{t}, \mathfrak{s}) \geq 0$, we get

$$\begin{aligned} (\mathcal{T}\upsilon)(\mathfrak{t}) &= q(\mathfrak{t}) + \int_a^b k(\mathfrak{s}, \mathfrak{t}) f(\mathfrak{s}, \upsilon(\mathfrak{s})) d\mathfrak{s} \\ &\leq q(\mathfrak{t}) + \int_a^b k(\mathfrak{s}, \mathfrak{t}) f(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s} \\ &= (\mathcal{T}\omega)(\mathfrak{t}), \end{aligned}$$

which shows that $(\mathcal{T}\upsilon, \mathcal{T}\omega) \in \mathcal{R}$, i.e., $\mathcal{R}$ is $\mathcal{T}$-closed.
Now for $(\upsilon, \omega) \in \mathcal{R}$,

$$\begin{aligned} d(\mathcal{T}\upsilon, \mathcal{T}\omega) &= sup_{\mathfrak{t}\in\mathcal{I}} |\mathcal{T}\upsilon(\mathfrak{t}) - \mathcal{T}\omega(\mathfrak{t})| \\ &= sup_{\mathfrak{t}\in\mathcal{I}} |\alpha| \int_a^b |k(\mathfrak{s}, \mathfrak{t})||f(\mathfrak{s}, \upsilon(\mathfrak{s})) - f(\mathfrak{s}, \omega(\mathfrak{t}))| d\mathfrak{s} \\ &= \alpha sup_{\mathfrak{t}\in\mathcal{I}} \Big\{ \int_a^b |k(\mathfrak{s}, \mathfrak{t})| d\mathfrak{s} \Big\} \Big\{ \int_a^b |f(\mathfrak{s}, \upsilon(\mathfrak{s})) - f(\mathfrak{s}, \omega(\mathfrak{s}))| d\mathfrak{s} \Big\} \\ &\leq \frac{1}{b-a} \Big\{ \int_a^b |f(\mathfrak{s}, \upsilon(\mathfrak{s})) - f(\mathfrak{s}, \omega(\mathfrak{s}))| d\mathfrak{s} \Big\} \\ &\leq \frac{1}{b-a} \int_a^b |\upsilon(\mathfrak{s}) - \omega(\mathfrak{s})| d\mathfrak{s} \\ &\leq \frac{1}{b-a} \int_a^b d(\upsilon, \omega) d\mathfrak{s} \\ &= d(\upsilon, \omega) \leq M_{\mathcal{T}}(\upsilon, \omega). \end{aligned}$$

This proves that $\mathcal{T}$ satisfies hypothesis (d) of Theorem 2.3. Now let $p \in C(\mathcal{I}, \mathbb{R})$ be a solution of (3.1), i.e.,

$$p(\mathfrak{t}) \leq q(\mathfrak{t}) + \alpha \int_a^b k(\mathfrak{s}, \mathfrak{t}) f(\mathfrak{s}, p(\mathfrak{t})) ds = (\mathcal{T}p)(\mathfrak{t})$$

$\Longrightarrow (p, \mathcal{T}p) \in \mathcal{R}$, i.e., $X(\mathcal{T}, \mathcal{R}) \neq \phi$.

Now, let $\rho = max\{\upsilon, \omega\}$. Then $\upsilon(\mathfrak{t}) \leq \rho(\mathfrak{t})$ and $\omega(\mathfrak{t}) \leq \rho(\mathfrak{t})$, i.e., $(\upsilon, \rho) \in \mathcal{R}$ and $(\omega, \rho) \in \mathcal{R}$. So, the sequence $\{\upsilon, \rho, \omega\}$ describes a path joining υ to ω in $\mathcal{R}$. Consequently, all the hypotheses of Theorem 2.4 are verified and we conclude that $\mathcal{T}$ has a unique fixed point, i.e., the integral equation has a solution. □

3.2. Application to a nonlinear matrix equation

Now we utilize Corollary 2.13 to solve the nonlinear matrix equation. Let $\mathcal{C}(\mathfrak{n})$ be the set of all complex matrices of order $\mathfrak{n}$. In the following, the symbol $\|.\|$ denotes the spectral norm of a matrix $\mathcal{A}$, i.e., $\|\mathcal{A}\| = \sqrt{\lambda^+(\mathcal{A}^*\mathcal{A})}$, $\lambda^+(\mathcal{A}^*\mathcal{A})$ is the largest eigenvalue of $(\mathcal{A}^*\mathcal{A})$, where $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}$. Further, $\|.\|_{tr}$ denotes the trace norm of $\mathcal{A}$ and $\|\mathcal{A}\|_{tr} = \sum_{j=1}^{\mathfrak{n}} s_j(\mathcal{A})$, $s_j(\mathcal{A}), j = 1, 2, 3, ..., \mathfrak{n}$, are the singular values of $\mathcal{A} \in \mathcal{C}(\mathfrak{n})$. The set of all Hermitian matrices of order $\mathfrak{n}$, $\mathcal{H}(\mathfrak{n}) \subset \mathcal{C}(\mathfrak{n})$, induced by this trace norm, is a Banach space. Clearly, $(\mathcal{H}(\mathfrak{n}), \preceq)$ is a partially ordered set.

Theorem 3.2. *Consider the nonlinear matrix equation*

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i, \tag{3.2}$$

where $\mathcal{A}_i$ is an arbitrary matrix of order $\mathfrak{n}$, $i = 1, 2, ..., \mathfrak{m}$ and $\mathcal{G} : \mathcal{H}(\mathfrak{n}) \to \mathcal{P}(\mathfrak{n})$ is a continuous order-preserving mapping satisfying $\mathcal{G}(0) = 0$. Here $\mathcal{P}(\mathfrak{n})$ is the positive definite matrix of order $\mathfrak{n}$. Let there exists a positive real number $\mathcal{M}$ satisfying

(i) $|tr(\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X})| \leq \frac{1}{\mathcal{M}}[\psi|tr(\mathcal{Y} - \mathcal{X})|]$, $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(\mathfrak{n})$, $(\mathcal{X}, \mathcal{Y}) \in \preceq$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous function so that $\psi(t) < t$, $\forall\ t > 0$.

(ii) $\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i \mathcal{A}_i^* \prec \mathcal{M}\mathcal{I}_{\mathfrak{n}}$, $\mathcal{I}_{\mathfrak{n}}$ is the identity matrix of order $\mathfrak{n}$.

(iii) $\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i \succ 0$.

Then the matrix equation (3.2) has a unique Hermitian solution.
Further, the iteration

$$\mathcal{X}_{\mathfrak{n}} = \mathcal{Q} + \sum_{i=1}^{\mathfrak{n}} \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_{\mathfrak{n}-1}) \mathcal{A}_i, \tag{3.3}$$

$\mathcal{X}_0 \in \mathcal{H}(\mathfrak{n})$ such that $\mathcal{X}_0 \preceq \mathcal{Q} + \sum_{i=1}^{\mathfrak{n}} \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_0) \mathcal{A}_i$, converges in the sense of trace norm $\|.\|_{tr}$, to the solution of the nonlinear matrix equation (3.2).

Proof . Define $\mathcal{T} : \mathcal{H}(\mathfrak{n}) \to \mathcal{H}(\mathfrak{n})$ as

$$\mathcal{T}(\mathcal{X}) = \mathcal{Q} + \sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i, \ \forall \mathcal{X} \in \mathcal{H}(\mathfrak{n}). \tag{3.4}$$

Clearly, $\mathcal{T}$ is well defined and $\preceq$ on $\mathcal{H}(\mathfrak{n})$ is $\mathcal{T}$-closed. The fixed point of $\mathcal{T}$ is the solution of (3.2). Here, we assert that $\mathcal{T}$ is a $\mathcal{Z}_{\mathcal{R}_{\preceq}}$-contraction mapping with respect to ζ, $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $\zeta(\mathfrak{t}, \mathfrak{s}) = \psi(\mathfrak{s}) - \mathfrak{t}$, $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$. By (i) $(\mathcal{X}, \mathcal{Y}) \preceq$, i.e., $\mathcal{X} \preceq \mathcal{Y}$ implies $\mathcal{G}(\mathcal{X}) \preceq \mathcal{G}(\mathcal{Y})$. Therefore

$$\begin{aligned}
\|\mathcal{T}(\mathcal{Y}) - \mathcal{T}(\mathcal{X})\|_{tr} &= tr(\mathcal{T}(\mathcal{Y}) - \mathcal{T}(\mathcal{X})) \\
&= tr\Big(\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i^* (\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X})) \mathcal{A}_i\Big) \\
&= \sum_{i=1}^{\mathfrak{m}} tr(\mathcal{A}_i^* (\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X})) \mathcal{A}_i) \\
&= \sum_{i=1}^{\mathfrak{m}} tr(\mathcal{A}_i \mathcal{A}_i^* (\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X}))) \\
&= tr\Big(\big(\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i \mathcal{A}_i^*\big)(\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X}))\Big) \\
&\leq \Big(\|\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i \mathcal{A}_i^*\|\Big) \|\mathcal{G}(\mathcal{Y}) - \mathcal{G}(\mathcal{X}))\|_{tr}
\end{aligned}$$

(since $0 \le tr(\mathcal{A}\mathcal{B}) \le \|\mathcal{A}\| tr(\mathcal{B})$, $\mathcal{A} \succeq 0$ and $\mathcal{B} \succeq 0$, by Lemma 3.1 [21]).

$$\begin{aligned} &\le \frac{\|\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i\mathcal{A}_i^*\|}{\mathcal{M}}[\psi(\|\mathcal{Y}-\mathcal{X}\|_{tr})] \\ &< \psi(\|\mathcal{Y}-\mathcal{X}\|_{tr}) \end{aligned}$$

(since $\mathcal{A} \prec \mathcal{I}_{\mathfrak{n}}$, implies $\|A\| < 1$, $\mathcal{A} \in \mathcal{H}(\mathfrak{n})$, by Lemma 2.2 [16]),
i.e.,

$$0 < \psi(\|\mathcal{Y}-\mathcal{X}\|_{tr}) - \|\mathcal{T}(\mathcal{Y}) - \mathcal{T}(\mathcal{X})\|_{tr}$$

Hence,

$$0 \le \zeta(\|\mathcal{T}(\mathcal{Y}) - \mathcal{T}(\mathcal{X})\|_{tr}, \|\mathcal{Y}-\mathcal{X}\|_{tr}). \tag{3.5}$$

This proves that $\mathcal{T}$ is a $\mathcal{Z}_{\mathcal{R}_{\preceq}}$-contraction. Since $\sum_{i=1}^{\mathfrak{m}} \mathcal{A}_i^*\mathcal{G}(\mathcal{Q})\mathcal{A}_i \succ 0$, $\mathcal{Q} \preceq \mathcal{T}(\mathcal{Q})$ and hence $\mathcal{H}(\mathfrak{n})(\mathcal{T};\preceq) \ne \phi$. So, there exists $\mathcal{Q} \in \mathcal{H}(\mathfrak{n})(\mathcal{T};\preceq)$. Thus, all the hypotheses of Corollary 2.13 are verified and there exists $\chi \in \mathcal{H}(\mathfrak{n})$ so that $\mathcal{T}(\chi) = \chi$, i.e., the matrix equation (3.2) has a solution.

Since $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(\mathfrak{n})$, there exists a greatest lower bound and a least upper bound, $\gamma(\upsilon, \omega, \mathcal{R}) \ne \phi$, $\upsilon, \omega \in \mathcal{H}(\mathfrak{n})$. Using Corollary 2.13, $\mathcal{T}$ has a unique fixed point in $\mathcal{H}(\mathfrak{n})$ and consequently, Equation (3.2) has a solution in $\mathcal{H}(\mathfrak{n})$. □

Next, we give a numerical example with an appropriate graph to validate the authenticity and visualize the related concepts of Theorem 3.2.

Example 3.3. *Let, in Equation (3.2) , for $i = 3$,*

$\mathcal{Q} = \begin{pmatrix} 3 & 1.5i & 0 & 0 \\ -1.5i & 3 & 1.5i & 0 \\ 0 & -1.5i & 3 & 1.5i \\ 0 & 0 & -1.5i & 3 \end{pmatrix}$ *is a Hermitian positive definite matrix of order 4,*

$\mathcal{A}_1 = \begin{pmatrix} 0.0010 & 0.2100 & 0.0231 & 0 \\ 0.1200 & 0.2110 & 0 & 0.3120 \\ 0.0110 & 0.5010 & 0.0020 & 0.1210 \\ 0.0090 & 0.0100 & 0.1212 & 0.1800 \end{pmatrix}$, $\mathcal{A}_2 = \begin{pmatrix} 0 & 0.3100 & 0.0140 & 0.2150 \\ 0.0020 & 0.0100 & 0.3120 & 0 \\ 0.1410 & 0.0500 & 0.0690 & 0.0120 \\ 0 & 0.1210 & 0.0020 & 0.0910 \end{pmatrix}$ *and*

$\mathcal{A}_3 = \begin{pmatrix} 0.1210 & 0.1201 & 0.0141 & 0.0090 \\ 0 & 0.1290 & 0.0250 & 0.0295 \\ 0.0030 & 0 & 0.2190 & 0 \\ 0.2190 & 0.1540 & 0.0030 & 0.0010 \end{pmatrix}$ *are matrices of order 4.*

Define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$, $t \in [0,\infty)$. Define a continuous order preserving mapping $\mathcal{G} : \mathcal{H}(\mathfrak{n}) \to \mathcal{P}(\mathfrak{n})$ as $\mathcal{G}(\mathcal{X}) = \mathcal{X}$. Clearly , $\mathcal{G}(0) = 0$. So, using Equation (3.2)

$$\mathcal{X} = \mathcal{Q} + \mathcal{A}_1^*(\mathcal{X})\mathcal{A}_1 + \mathcal{A}_2^*(\mathcal{X})\mathcal{A}_2 + \mathcal{A}_3^*(\mathcal{X})\mathcal{A}_3. \tag{3.6}$$

One may verify that all the hypotheses of Theorem 3.2 are true for $\mathcal{M} = \frac{1}{2}$.
Now, consider the iteration (3.3) for $\mathcal{G}(\mathcal{X}) = \mathcal{X}_{\mathfrak{n}-1}$ and $i = 3$

$$\mathcal{X}_{\mathfrak{n}} = \mathcal{Q} + \mathcal{A}_1^*(\mathcal{X}_{\mathfrak{n}-1})\mathcal{A}_1 + \mathcal{A}_2^*(\mathcal{X}_{\mathfrak{n}-1})\mathcal{A}_2 + \mathcal{A}_3^*(\mathcal{X}_{\mathfrak{n}-1})\mathcal{A}_3, \tag{3.7}$$

$\mathcal{X}_0 = \mathcal{Q}$ and the error $E_0 = \|\mathcal{X}_{\mathfrak{n}} - \mathcal{X}_{\mathfrak{n}-1}\|_{tr}$. After fourteen iterations, we approximate a solution of equation (3.6) as:

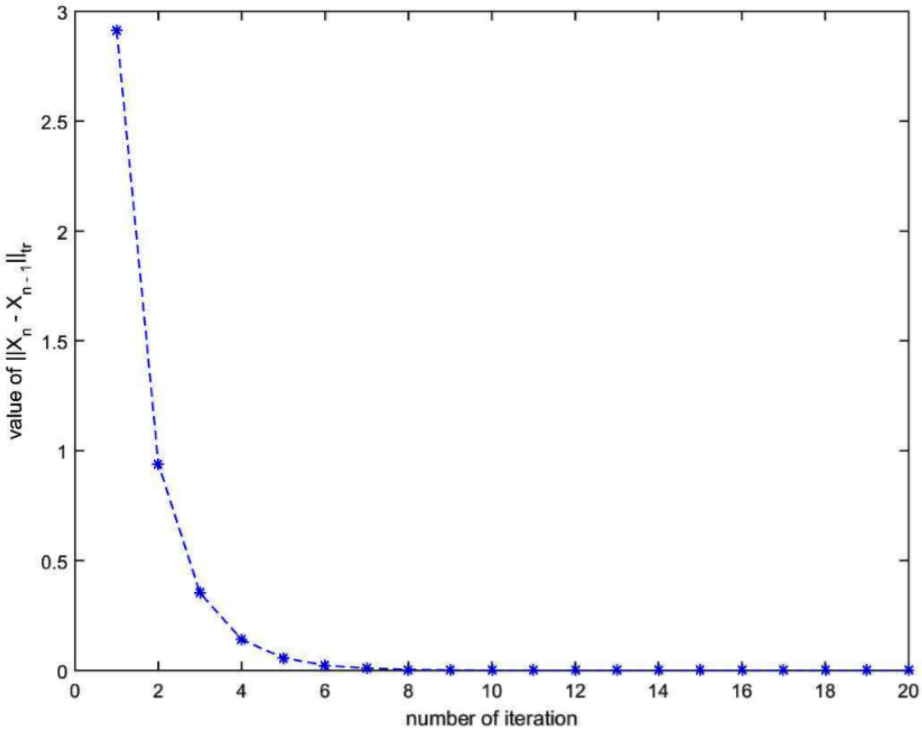

Figure 1:

$$\tilde{\mathcal{X}} \approx \mathcal{X}_{14} = \begin{pmatrix} 3.4128 & 0.4581 + 1.6030i & 0.1065 - 0.1415i & 0.2639 + 0.0476i \\ 0.4581 - 1.6030i & 5.1871 & 0.2644 + 1.6940i & 1.0585 + 0.0116i \\ 0.1065 + 0.1415i & 0.2644 - 1.6940i & 3.7938 & 0.2235 + 1.3899i \\ 0.2639 - 0.0476i & 1.0585 - 0.0116i & 0.2235 - 1.3899i & 4.0485 \end{pmatrix}$$

with $E_{14} = 1.8488e - 05.$

The error of the iteration process (3.7) for Equation (3.6) is shown in Figure 1.

Conclusion

We have proved the existence of a single fixed point for relation theoretic variants of Ćirić type contraction [9] via simulation function. Our theorems and corollaries are sharpened versions of the well-known results, wherein completeness and continuity are replaced by their $\mathcal{R}$ analogues, which are comparatively weaker notions. Examples and applications to find the solution of an integral equation and a nonlinear matrix equation substantiate the utility of these extensions. It is interesting to see that a matrix equation solved is similar to discrete-time algebraic Riccati equation [11] arising in the infinite-horizon optimal control problems.

References

[1] A. Alam and M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl. 17(4) (2015) 693–702.
[2] A. Alam, and M. Imdad, *Relation-theoretic metrical coincidence theorems*, Filomat 31(14) (2017) 4421–4439.
[3] A. Alam and M. Imdad, *Nonlinear contractions in metric spaces under locally T-transitive binary relations*, Fixed Point Theory 19 (2018) 13–24.

[4] L.A. Alnasera, D. Lateefa, H. A. Fouada and J. Ahmadc, *Relation theoretic contraction results in F-metric spaces.* J. Nonlinear Sci. Appl. 12 (2019) 337–344.
[5] S. Antal and U.C. Gairola, *Generalized Suzuki type α-$\mathcal{Z}$-contraction in b-metric space*, J. Nonlinear Sci. Appl. 13 (2020) 212–222.
[6] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3(1) (1922) 133–181.
[7] D.W. Boyd and J.S.W. Wong, *Nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969), 458–464.
[8] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Sci. 29 (9)(2002) 531–536.
[9] Lj B. Ćirić, *A generalization of Banach's contraction principle spaces*, Proc. Amer. Math. Soc. 45(2) (1974) 267–273.
[10] M.A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. 40 (1973) 604–608.
[11] W.L. De Koning, *Infinite horizon optimal control of linear discrete time systems with stochastic parameters*, Automatica 18(4) (1982) 443–453.
[12] D. Khantwal and U.C. Gairola, *An extension of Matkowski's and Wardowski's fixed point theorems with applications to functional equations*, Aequationes Math. 93(2) (2019) 433–443.
[13] F. Khojasteh, S. Shukla and S. Radenovic, *A new approach to the study of fixed point theorems via simulation functions*, Filomat 29(6) (2015) 1189–1194.
[14] B. Kolman, R.C. Busby and S. Ross, *Discrete Mathematical Structures*, 3rd ed., PHI Pvt. Ltd., New Delhi, 2000.
[15] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, 1964.
[16] J.H. Long, X.Y. Hu and L. Zhang, *On the Hermitian positive definite solution of the nonlinear matrix equation* $X + A^*x^{-1}A + B^*X^{-1}B = I$, Bull. Braz. Math. Soc. 39(3) (2015) 317–386.
[17] R.D. Maddux, *Relation algebras: Studies in Logic and the Foundations of Mathematics*, Elsevier B. V., Amsterdam, 2006.
[18] J. Matkowski, *Integrable solutions of functional equations*, Dissertations Math. 127(68) (1975).
[19] M. Olgun, Ö. Biçer and T. Alyildiz, *A new aspect to Picard operators with simulation functions*, Turkish J. Math. 40(4) (2016) 832–837.
[20] S. Radenović, F. Vetro and J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math. **50** (1) (2017), 223–230.
[21] A. Ran and M. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. 132(5) (2003) 1435–1443.
[22] B.E. Rhoades, *Contractive definitions and continuity*, Contemp. Math. 72 (1988) 233–245.
[23] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. (TMA) 47 (2001) 2683–2693.
[24] A.F. Roldán-López-de-Hierro, E. Karapinar, C. Roldán-López-de-Hierro and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. 275 (2015) 345–355.
[25] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. 13 (2012) 82–97.
[26] K. Sawangsup and W. Sintunavarat, *On modified $\mathcal{Z}$-contractions and an iterative scheme for solving nonlinear matrix equations*, J. Fixed Point Theory Appl. 20(2) (2018).
[27] A. Tomar, M. Joshi, S.K. Padaliya, B. Joshi and A. Diwedi, *Fixed point under set-valued relation-theoretic nonlinear contractions and application*, Filomat 33(14) (2019) 4655–4664.
[28] A. Tomar and M. Joshi, *Relation-theoretic nonlinear contractions in an F-metric space and applications*, Rend. Circ. Mat. Palermo 2 (2020) 1–18.
[29] M. Turinici, *Abstract comparison principles and multivariable gronwall-bellman inequalities*, J. Math. Anal. Appl. 117(1) (1986) 100–127.