Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 1827-1853
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23332.2521

A split common fixed point and null point problem for Lipschitzian $J-$quasi pseudocontractive mappings in Banach spaces

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(Communicated by Madjid Eshaghi Gordji)

Abstract

A split common fixed point and null point problem (SCFPNPP) which includes the split common fixed point problem, the split common null point problem and other problems related to the fixed point problem and the null point problem is studied. We introduce a Halpern–Ishikawa type algorithm for studying the split common fixed point and null point problem for Lipschitzian $J-$quasi pseudocontractive operators and maximal monotone operators in real Banach spaces. Moreover, we establish a strong convergence results under some suitable conditions and reduce our main result to the above-mentioned problems. Finally, we applied the study to split feasibility problem (SFP), split equilibrium problem (SEP), split variational inequality problem (SVIP) and split optimization problem (SOP).

Keywords: Split common fixed point problem, split common null point problem, $J-$quasi pseudocontractive operators, maximal monotone operators, Halpern–Ishikawa type algorithm.
2010 MSC: 47H09, 47H10, 47J05, 65J15.

1. Introduction

The split feasibility problem (SFP) and the split common null point problem (SNCPP); see, for instance, [5, 11, 14, 38, 15, 63] have been studied by many researchers. However, we have not found many results outside Hilbert spaces. The first extension of SFP to Banach spaces appears in [46]. This algorithm was later extended to multiple-sets split feasibility problem (MSSFP) in [59]. A very

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Received: March 2021 *Accepted:* May 2021

recent contribution for the SFP is [48]. [53] also solves the split common null point problem in Banach spaces.

Let E be a strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\|x - z\| \leq \|x - y\|, \forall y \in C. \tag{1.1}$$

Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C.

Let B be a mapping of E into 2^E.The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multivalued mapping B is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0 \ \forall \ x, y \in dom(B), \ u \in Bx, \ v \in By \tag{1.2}$$

There are two iterative methods for approximating fixed points of a nonexpansive mapping. One is introduced by [36] and the other by [28]. The iteration procedure of Mann's type for approximating fixed points of a nonexpansive mapping S is the following: $x_1 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. The iteration procedure of Halpern's type is the following: $u \in K, x_1 \in K$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Iterative method for approximating fixed points of Lipschitz pseudocontractive maps which map nonempty convex compact subsets K of H into itself was introduced by [30] as follows: The sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$\begin{array}{rcl} y_n & = & (1 - \beta_n) x_n + \beta_n T x_n \\ x_{n+1} & = & (1 - \alpha_n) x_n + \alpha_n T y_n \end{array} \quad n \geq 1. \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0, 1] satisfying the conditions (i) $0 \leq \alpha_n \leq \beta_n \leq 1$, (ii) $\lim_{n\to\infty} \beta_n = 0$, (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

A monotone operator B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator B on E and $r > 0$, the operator

$$J_r = (J + rB)^{-1} : H \to \ \mathrm{dom}(B) \tag{1.6}$$

is called the resolvent of B for r. It is known that J_r is $J-$firmly nonexpansive. Given a positive constant α, a mapping $A : C \to H$ is said to be $\alpha-$inverse strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \ \forall \ x, y \in C. \tag{1.7}$$

An operator h is called averaged if there exists a $J-$nonexpansive operator $N : D \to H$ and a number $\alpha \in (0, 1)$ such that

$$h = (1 - \alpha) J + \alpha N \tag{1.8}$$

where J is the duality map.

Let E be a real Banach space and E^* the dual of E. A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**nonexpansive** if

$$\|Tx - Ty\| \leq \|Jx - Jy\| \tag{1.9}$$

$\forall x,\, y \in K$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|Jx - p\| \tag{1.10}$$

$\forall x \in K,\, p \in F(T)$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be **strictly** $J-$**quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| < \|x - p\| \tag{1.11}$$

$\forall x \notin F(T), p \in F(T)$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be **strongly** $J-$**quasi-nonexpansive** if T is $J-$quasi-nonexpansive and

$$Jx_n - Tx_n \longrightarrow 0 \tag{1.12}$$

whenever $\{x_n\}$ is a bounded sequence in H and $\|jx_n - p\| - \|Tx_n - p\| \longrightarrow 0$ for some $p \in F(T)$
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**firmly nonexpansive if**

$$\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle \tag{1.13}$$

$\forall x,\, y \in K$ and $n \geq 1$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**firmly quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\|^2 \leq \|Jx - p\|^2 - \|Jx - Tx\|^2, \tag{1.14}$$

$\forall x \in K, p \in F(T)$ and $n \geq 1$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $k-$**strictly pseudocontractive** if there exists a $k \in [0, 1)$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.15}$$

$\forall x\,, y \in E$.
If $k = 1$ in (1.15), then T is called a **pseudocontractive mapping**.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $k-$**strictly** $J-$**pseudocontractive** if there exists a $k \in [0, 1)$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(J - T)x - (J - T)y\|^2, \tag{1.16}$$

$\forall x\, y \in E$ and $n \geq 1$.
If $k = 1$ in (1.16), then T is called a $J-$**pseudocontractive mapping** [20, 21]. Equivalently,

$$\langle x - y, Tx - Ty\rangle \leq \langle x - y, Jx - Jy\rangle. \tag{1.17}$$

$\forall x, y \in E$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**demicontractive** if $F(T) \neq \emptyset$ and there exists a $k \in [0, 1)$ such that

$$\langle Jx - Tx, Jx - p\rangle \geq \lambda\|Jx - Tx\|^2, \tag{1.18}$$

$\forall x \in K, p \in F(T)$.
A mapping $T : E \longrightarrow 2^{E^*}$ is said to be $J-$**quasi pseudocontractive**) if

$$\langle Jx - Tx, Jx - p\rangle \geq 0, \tag{1.19}$$

$\forall x \in K, p \in F(T)$.

Remark 1. *We easily observe that the class of $J-$quasi pseudocontractive operators includes the class of operators defined in equations* (1.9) - (1.18)*.*

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r \tag{1.20}$$

where J is the duality mapping on E. This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such J_r, $r > 0$ are called the metric resolvents of B. [53, 52] extended the result of SCFPP to Banach spaces. Furthermore, by using the methods of [40, 41, 49] and metric projections, [53] proved a strong convergence theorem for metric resolvents of maximal monotone operators in two Banach spaces. Also [55] considered the split common null point problem with generalized resolvents of maximal monotone operators in two Banach spaces.

The following questions were riased:

Open problem 1 [16]: It is of interest to define a space E_1 that is a real Banach space more general than real Hilbert spaces, E_2 as defined in the paper of [57] and an iterative algorithm for solving split common fixed point problem involving a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping such that the sequence generated by the algorithm converges strongly to a solution of the problem.

Open problem 2 [58]: Can we construct a new inertial algorithm for solving the SCNPP for two set-valued mappings in Banach spaces without prior knowledge of the operator norm $\|A\|$?

Open problem 3 The question of how to solve the split common null point problem for generalized resolvents in two Banach spaces was posed by [29].

Unfortunately, developing algorithms for approximating solutions of inclusions of type $0 \in Bu$ when $B : E \to 2^{E^*}$ is of monotone-type has not been very fruitful. Part of the difficulty seems to be that all efforts made to apply directly the geometric properties of Banach spaces developed from the mid 1980s to the early 1990s proved abortive. Furthermore, the technique of converting the inclusion $0 \in Bu$ into a fixed point problem for $T := I - B : E \to E$ is not applicable since, in this case when B is monotone, B maps E into E^* and the identity map does not make sense.

Fortunately, [4] (see also, [3]) introduced a Lyapunov functional $\Phi : E \times E \to \mathbb{R}$ which signalled the beginning of the development of new geometric properties of Banach spaces which are appropriate for studying iterative methods for approximating solutions of $0 \in Bu$ when $B : E \to 2^{E^*}$ is of monotone-type. Geometric properties so far obtained have rekindled enormous research interest on iterative methods for approximating solutions of equation $0 \in Bu$ where B is of the monotone-type, and other related problems [1, 4, 17, 19, 37, 39, 43, 51, 64]). A new class of maps $T := (J - B)$ is $J-$quasi pseudocontractive if and only if B is monotone and using the notion of $J-$fixed points (which has also been defined as semi-fixed point, duality fixed point, see e.g., [64, 35]) to prove that if E is a uniformly convex and uniformly smooth real Banach space with dual E^*.

Motivated by the works of [32, 54, 57, 58], we study a split common fixed point and null point problem which is more general than the problem [54]. Our problem can be reduced to the split common fixed point problem, the split common null problem and other problems which are connected with the fixed point problem and the null point problem. We also introduce a Halpern–Ishikawa type algorithm for studying the split common fixed point and null point problem for Lipschtzian $J-$quasi pseudocontractive operators and maximal monotone operators, and prove a strong convergence theorem of the proposed algorithm under some suitable conditions in real Banach spaces.

Finally, we applied the study to Split Feasibility Problem (FEP), Split Equilibrium Problem (SEP), Split Variational Inequality Problem (SVIP) and Split Optimization Problem (SOP).

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\} \tag{2.1}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every$\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2 \tag{2.2}$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [18, 22].

The duality mapping J from E into $2^{E^*}$is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.3}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.4}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.4) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.4) is attained uniformly for $x, y \in U$. The classical L_p spaces for $1 < p < \infty$ are uniformly convex and uniformly smooth. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J^* on E^*. For more details, see [18, 22, 33, 50]. We know the following result:

Lemma 2.1. *[50] Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\Phi : E \times E \to \mathbb{R}$ by

$$\Phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E \tag{2.5}$$

Define a map $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,\ \forall x \in E, x^* \in E^* \tag{2.6}$$

Then, it is easy to see that

$$V(x, x^*) = \Phi(x, J^{-1}(x^*)),\ \forall x \in E, x^* \in E^* \tag{2.7}$$

Observe that, in a Hilbert space H, $\Phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$\begin{aligned} (\|x\| - \|y\|)^2 &\leq \Phi(x, y) \leq (\|x\| + \|y\|)^2 && (2.8)\\ \Phi(x, y) &= \Phi(x, z) + \Phi(z, y) + \|x\|^2 + 2\langle x - z, Jz - Jy\rangle, && (2.9)\\ 2\langle x - y, Jz - Jw\rangle &= \Phi(x, w) + \Phi(y, z) - \Phi(x, z) - \Phi(y, w). && (2.10) \end{aligned}$$

If E is additionally assumed to be strictly convex, then

$$\Phi(x, y) = 0 \text{ if and only if } x = y. \tag{2.11}$$

The following lemma was proved by [33].

Lemma 2.2. *[33] Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\Phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a smooth, strictly convex, and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\Phi(z, x) = \min_{y \in C} \Phi(y, x) \tag{2.12}$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. For example, see [2, 4, 33].

Lemma 2.3. *[2, 4, 33] Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty, closed, and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(i) $z = \Pi_C x_1$,

(ii) $\langle z - y, Jx_1 - Jz\rangle \geq 0,\ \forall y \in C$.

The following theorem is due to [8, 44]; see also [[50], Theorem 3.5.4].

Theorem 2.1. *[8, 44] Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^* \tag{2.13}$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$Jx \in Jx_r + rBx_r \tag{2.14}$$

This equation has a unique solution x_r. In fact, it is obvious from Theorem 2.1 that there exists a solution x_r of $Jx \in Jx_r + rBx_r$. Assume that $Jx \in Ju + rBu$ and $Jx \in Jv + rBv$. Then there exist $w_1 \in Au$ and $w_2 \in Av$ such that $Jx = Ju + rw_1$ and $Jx = Jv + rw_2$. So, we have that

$$\begin{aligned} 0 &= \langle u - v, Jx - Jx\rangle \\ &= \langle u - v, Ju + rw_1 - (Jv + rw_2)\rangle \\ &= \langle u - v, Ju - Jv + rw_1 - rw_2\rangle \\ &= \langle u - v, Ju - Jv\rangle + \langle u - v, rw_1 - rw_2\rangle \\ &= \Phi(u, v) + \Phi(v, u) + r\langle u - v, w_1 - w_2\rangle \\ &\geq \Phi(u, v) + \Phi(v, u) \end{aligned} \tag{2.15}$$

and hence $0 = \Phi(u, v) = \Phi(v, u)$. Since E is strictly convex, we have $u = v$. We define J_r by $x_r = J_r x$. Such $J_r,\ r > 0$ are called the generalized resolvents of B.

Definition 2.1. *A Banach space E is said to be an opial space (see for example [42]) if for each sequence $\{x_n\}_{n=1}^{\infty}$ in E which converges weakly to a point $x \in E$*

$$\liminf \|x_n - x\| < \liminf \|x_n - y\|, \tag{2.16}$$

for all $y \in E,\ \ y \neq x$.

Lemma 2.4. *[12] Let $T : H \to H$ be a strictly quasi-nonexpansive operator and $S : H \to H$ a quasi-nonexpansive operator. Suppose that $F(T) \cap F(S) \neq \emptyset$. Then $F(TS) = F(ST) = F(T) \cap F(S)$.*

Lemma 2.5. *Let E be a real Banach space, E^* the dual of E. Let $T : E \to 2^{E^*}$ be a continuous pseudocontractive mapping. Then*
(i) Fix(T) is a closed convex subset of C,
(ii) (J - T) is demiclosed at zero.

Lemma 2.6. *Let E_1 and E_2 be Banach spaces. Let $A : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to 2^{E_2^*}$ be a $J-$quasi-nonexpansive operator such that the equation $(J - T)Ax = 0$ has a solution. Let $V := J + (\frac{1}{\|A\|^2})A^*(T - J)A$, then the following hold:*
(i) $JAx \in F(T)$ if and only if $Jx \in F(V)$,
(ii) If $J-T$ is demiclosed at zero, then $J-V$ is also demiclosed at zero, (iii) V is quasi-nonexpansive.

Lemma 2.7. *(see [6, 62]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n,\ n \geq 0 \tag{2.17}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1],\ \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\limsup\limits_{n\to\infty} \beta_n \leq 0,$

(iii) $\gamma_n \geq 0,\ \sum\limits_{n=1}^{\infty} \gamma_n < \infty.$

Then $\lim\limits_{n\to\infty} s_n = 0.$

Lemma 2.8. *[3] Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$V(x,x^*)+2\langle J^{-1}x^*-x,y^*\rangle\leq V(x,x^*+y^*),\ \forall x\in E,x^*,y^*\in X^* \tag{2.18}$$

Lemma 2.9. *[3] Let E be a smooth real Banach space with dual E^*. Let $\Phi:E\times E\to\mathbb{R}$ be the Lyapounov functional. Then,*

$$\Phi(y,x)=\Phi(x,y)-2\langle x+y,Jx-Jy\rangle+2(\|x\|^2-\|y\|^2),\ \forall x,y\in E \tag{2.19}$$

Definition 2.2. *Let $T:E\to 2^{E^*}$, $J-T$ is called demiclosed at zero, if for any sequence $\{x_n\}\subset E$ and $x\in E$, we have $x_n\rightharpoonup x$ and $(J-T)x_n\to 0$, then $Jx\in Fix(T)$.*

Definition 2.3. *[20] ($J-$fixed point). Let E be an arbitrary normed space and E^* be its dual. Let $T:E\to 2^{E^*}$ be any mapping. A point $x\in E$ will be called a $J-$fixed point of T if and only if there exists $\eta\in Tx$ such that $\eta\in Jx$.*

Definition 2.4. *(Lower Semi-continuity). Let $f:X\to\mathbb{R}\cup\{+\infty\}$ be a map. Let $x_0\in D(f)$, then f is lower semicontinuous at x_0 if for each $\epsilon>0$ there exists $\delta>0$ such that $f(x_0)-\epsilon<f(x)$ for all $x\in B(x_0,\delta)$.*

Proposition 2.1. *Let H be a real Hilbert space and identify H^* with H, then $Jx=\{x\}$ for all $x\in H$, i.e The duality map J is the identity map.*

3. Main Results

Lemma 3.1. *Let E be an arbitrary real normed space and E^* be its dual space. Let $A:E\to 2^{E^*}$ be any mapping. Then A is monotone if and only if $T:=(J-A):E\to 2^{E^*}$ is $J-$quasi pseudocontractive.*

Proof: Let $x,y\in E$ be arbitrary. Suppose $A:=(J-T)$ is monotone, we prove that $T:=(J-A)$ is $J-$quasi pseudocontractive. Then, for every $\mu_x\in Ax$, $\tau_x\in Tx$ such that $\tau_x=Jx-\mu_x$, we have

$$\begin{aligned}\langle x-\tau_x,x-y\rangle &= \langle x-(Jx-\mu_x),x-y\rangle=\langle x-Jx+Ax,x-y\rangle\\ &= \langle x-Jx,x-y\rangle+\langle Ax,x-y\rangle\\ &\leq \langle x-Jx,x-y\rangle\end{aligned} \tag{3.1}$$

Hence, T is $J-$quasi pseudocontractive.

Conversely, suppose $T:=(J-A)$ is $J-$quasi pseudocontractive, we prove that $A:=J-T$ is monotone. For $x,y\in E$. Let $\mu_x\in Ax$, $\mu_y\in Ay$, $\tau_x\in Tx$, $\tau_y\in Ty$ such that $\mu_x=Jx-\tau_x$, $\mu_y=Jy-\tau_y$, we have

$$\begin{aligned}\langle \mu_x-\mu_x,x-y\rangle &= \langle Jx-Tx-(Jy-Ty),x-y\rangle\\ &= \langle Jx-Tx,x-y\rangle-\langle Jy-Ty,x-y\rangle\\ &\geq 0.\end{aligned} \tag{3.2}$$

Hence, A is monotone.

Lemma 3.2. *Let E_1 and E_2 be real Banach spaces. Let B_1 and B_2 be maximal monotone operators of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be generalized resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J - S)$ and $(J - T)$ are demiclosed at zero, any sequence $\{x_n\}_{n=0}^\infty$ generated by*

$$x_{n+1} = SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \tag{3.3}$$

converges weakly to a point $Jx^* \in \Gamma = \{Jx \in F(S) \cap B_1^{-1}0, JAx \in F(T) \cap B_2^{-1}0\}$, provided that $\Gamma \neq \emptyset$, $\gamma \in (0, \frac{1}{L})$ where $L = \|A * A\|$, $J_\lambda^{B_1} := (J + \lambda B_1)^{-1}$, $J_\lambda^{B_1 2} := (J + \lambda B_2)^{-1}$ and J is normalized duality mapping.

Proof: First we prove that the operator $\gamma A^*(J - TJ_\lambda^{B_2})A$ is $v-$inverse strongly monotone for some $v > \frac{1}{2}$ and therefore its complement $J - \gamma A^*(J - TJ_\lambda^{B_2})A$ is averaged. TJ_λ^B is $J-$firmly nonexpansive and therefore $\frac{1}{2}-$averaged, so

$$TJ_\lambda^B = \frac{J + N}{2} \tag{3.4}$$

for some nonexpansive operator $N : H_2 \to H_2$. Since

$$J - TJ_\lambda^B = \frac{J - N}{2} \tag{3.5}$$

it follows that $J - TJ_\lambda^B$ is $1-$inverse strongly monotone. Hence

$$\langle (J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay, Ax - Ay\rangle \geq \|(J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay\|^2. \tag{3.6}$$

Now

$$\begin{aligned}\|A^*(J \;&-\; TJ_\lambda^{B_2})Ax - A^*(J - TJ_\lambda^{B_2})Ay\|^2 \\ &= \langle A^*(J - TJ_\lambda^{B_2})Ax - A^*(J - TJ_\lambda^{B_2})Ay, A^*(J - TJ_\lambda^{B_2})Ax - A^*(J - TJ_\lambda^{B_2})Ay\rangle \\ &= \langle (J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay, AA^*(J - TJ_\lambda^{B_2})Ax - AA^*(J - TJ_\lambda^{B_2})Ay\rangle \\ &\leq L\|(J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay\|^2. \end{aligned} \tag{3.7}$$

$$\|(J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay\|^2 \geq \frac{1}{L}\|A^*(J - TJ_\lambda^{B_2})Ax - A^*(J - TJ_\lambda^{B_2})Ay\|^2 \tag{3.8}$$

Substitute equation (3.8) into (3.6) to obtain

$$\langle (J - TJ_\lambda^{B_2})Ax - (J - TJ_\lambda^{B_2})Ay, Ax - Ay\rangle \geq \frac{1}{L}\|A^*(J - TJ_\lambda^{B_2})Ax - A^*(J - TJ_\lambda^{B_2})Ay\|^2. \tag{3.9}$$

This implies that $A^*(J - TJ_\lambda^{B_2})A$ is $\frac{1}{L}-$inverse strongly monotone and $\gamma A^*(J - TJ_\lambda^{B_2})A$ is $\frac{1}{\gamma L}-$inverse strongly monotone. Since $\gamma \in (0, \frac{2}{L})$, then $\frac{1}{\gamma L} > \frac{1}{2}$. Thus $J - \gamma A^*(J - TJ_\lambda^{B_2})A$ is averaged. Since both $SJ^{B_1}|\lambda$ and $J - \gamma A^*(J - TJ_\lambda^{B_2})A$ are averaged, so is their composition

$$SJ_\lambda^{B_1}(J - \gamma A^*(J - TJ_\lambda^{B_2})A)$$

Therefore, by the Krasnosel'ski-Mann-Opial Theorem ([34, 36, 42]), the sequence $\{x_n\}_{n=0}^\infty$ generated by Algorithm (3.3) converges weakly to a fixed point x^* of the operator

$$SJ_\lambda^{B_1}(J - \gamma A^*(J - TJ_\lambda^{B_2})A).$$

It remains to show that $Jx^* \in \Gamma$. Let $Jz \in \Gamma$., that is,

$$Jz \in F(S) \cap B_1^{-1}0 \text{ and } Jz \in A^{-1}(F(T) \cap B_2^{-1}0) \tag{3.10}$$

Since

$$Jz \in F(S) \cap B_1^{-1}0 \Leftrightarrow z \in Fix(SJ_\lambda^{B_1}) \text{ and}$$
$$Jz \in A^{-1}(F(T) \cap B_2^{-1}0) \Leftrightarrow JAz \in F(T) \cap B_2^{-1}0 \Leftrightarrow JAz \in Fix(TJ_\lambda^{B_2}).$$

Note that if

$$\begin{aligned} JAz \in Fix(TJ_\lambda^{B_2}) &\Rightarrow JAz = TTJ_\lambda^{B_2}Az \\ JAz - TTJ_\lambda^{B_2}Az = 0 &\Rightarrow (J - TTJ_\lambda^{B_2})Az = 0 \\ \Rightarrow \gamma A^*(J - TTJ_\lambda^{B_2})Az = 0 &\Rightarrow Jz - \gamma A^*(J - TTJ_\lambda^{B_2})Az = Jz \end{aligned} \tag{3.11}$$

Also note that if

$$\begin{aligned} &Jz \in F(S) \cap B_1^{-1}0 \Leftrightarrow Jz \in Fix(SJ_\lambda^{B_1}) \text{ from (3.11)}, \\ &\quad \Rightarrow SJ_\lambda^{B_1}(Jz - \gamma A^*(J - TTJ_\lambda^{B_2})Az) = Jz. \end{aligned} \tag{3.12}$$

In addition,

$$\begin{aligned} (J - \gamma A^*(J - TJ_\lambda^{B_2})A)z &= Jz - \gamma A^*(J - TJ_\lambda^{B_2})Az \\ &= Jz - \gamma A^*JAz + A * TJ_\lambda^{B_2}Az \\ &= Jz - \gamma A^*JAz + \gamma A^*JAz \\ &= Jz \end{aligned}$$

we get $Jz \in Fix(J - \gamma A^*(J - TJ_\lambda^{B_2})A)$. Observe that any $Jz \in \Gamma$ is a fixed point of the averaged operator $SJ_\lambda^{B_1}(I - \gamma A^*(I - TJ_\lambda^{B_2})A)$. Indeed, by the above equalities we get

$$\begin{aligned} SJ_\lambda^{B_1}(J - \gamma A^*(J - TJ_\lambda^{B_2})A)z &= SJ_\lambda^{B_1}(Jz - \gamma A^*(J - TJ_\lambda^{B_2})Az) \\ &= SJ_\lambda^{B_1}z = Jz. \end{aligned} \tag{3.13}$$

Since $\Gamma \neq \emptyset$, we get from [[**?**], Proposition 2.2] (see also [[**?**], Lemma 2.1]), with the averaged operators $J - \gamma A^*(J - TJ_\lambda^{B_2})A$ and $SJ_\lambda^{B_1}$, that

$$\begin{aligned} Fix(SJ_\lambda^{B_1}) \cap Fix(J &- \gamma A^*(J - TJ_\lambda^{B_2})A) = Fix(SJ_\lambda^{B_1}(J - \gamma A^*(J - TJ_\lambda^{B_2})A) \\ &= Fix((J - \gamma A^*(J - TJ_\lambda^{B_2})A)SJ_\lambda^{B_1}) \end{aligned} \tag{3.14}$$

Since Jx^* is a fixed point of $SJ_\lambda^{B_1}(J - \gamma A^*(I - TJ_\lambda^{B_2})A$, we have $Jx^* \in Fix(SJ_\lambda^{B_1})$ and $Jx^* \in Fix(J - \gamma A^*(J - TJ_\lambda^{B_2})A)$. Now we need to show that $JAx^* \in Fix(TJ_\lambda^{B_2})$. Indeed, from $Jx^* \in Fix(J - \gamma A^*(J - TJ_\lambda^{B_2})A)$, we get

$$\begin{aligned} Jx^* - \gamma A^*(J - TJ_\lambda^{B_2})Ax^* &= Jx^* \\ A^*(J - TJ_\lambda^{B_2})Ax^* &= 0 \\ A^*JAx^* - A^*TJ_\lambda^{B_2}Ax^* &= Jx^* - A^*TJ_\lambda^{B_2}Ax^* = 0 \\ A^*TJ_\lambda^{B_2}Ax^* &= Jx^* \\ TJ_\lambda^{B_2}Ax^* &= JAx^*. \end{aligned}$$

This completes the proof.

Theorem 3.1. *Let E_1 and E_2 be real Banach spaces. Let B_1 and B_2 be maximal monotone operators of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be generalized resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J - S)$ and $(J - T)$ are demiclosed at zero. If the solution set of SCFPNPP is nonempty (that is, $\Gamma = \{Jx \in F(S) \cap B_1^{-1}0, JAx \in F(T) \cap B_2^{-1}0\} \neq \emptyset$). Suppose that x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by*

$$\begin{array}{rcl} x_{n+1} &=& J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ y_n &=& J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) \end{array} \quad n \geq 1. \tag{3.15}$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the conditions: (i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, (ii) $\sum_{n=1}^\infty \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^\infty \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| = \lim_{n\to\infty} \|JAx_n - TJ_\lambda^{B_2}Ax_n\| = 0$, *then* $\{x_n\}_{n=1}^\infty$ *converges strongly to* $p \in \Gamma$.

Proof : Applying equations (2.5), (3.15) and using Lemma 2.8, we compute as follows:

$$\begin{array}{rcl} \Phi(p, x_{n+1}) &=& \Phi(p, J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n)) \\ &=& \|p\|^2 - 2\langle p, \beta_n Jx_0 + (1-\beta_n)Jy_n\rangle \\ && +\|\beta_n Jx_0 + (1-\beta_n)Jy_n\|^2 \\ &=& \|p\|^2 - 2\langle p, \beta_n Jx_0\rangle - 2(1-\beta_n)\langle p, Jy_n\rangle \\ && +\|\beta_n Jx_0 + (1-\beta_n)Jy_n\|^2 \\ &\leq& \|p\|^2 - 2\langle p, \beta_n Jx_0\rangle - 2(1-\beta_n)\langle p, Jy_n\rangle + \beta_n\|x_0\|^2 \\ && +(1-\beta_n)\|y_n\|^2 - \beta_n(1-\beta_n)\|y_n - x_0\|^2 + \|p\|^2 - \|p\|^2 \\ &\leq& \beta_n\Phi(p, x_0) + (1-\beta_n)\Phi(p, y_n). \end{array} \tag{3.16}$$

Setting $x^* = \alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)$ and $y^* = (1-\alpha_n)Jx_n - (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)$ in Lemma 2.9, we have

$$\begin{array}{rcl} \Phi(p, y_n) &=& \Phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))) \\ &\leq& V(p, \alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) \\ &\leq& V(p, Jx_n - (1-\alpha_n)Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\ && +(1-\alpha_n)Jx_n - (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) \\ && -2\langle J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - p, \\ && (1-\alpha_n)Jx_n - (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \end{array}$$

$$
\begin{aligned}
&= V(p, Jx_n) - 2\langle J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - p, \\
&\quad (1-\alpha_n)Jx_n - (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)\langle J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - p, \\
&\quad Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)\langle J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - J^{-1}(Jx_n) + J^{-1}(Jx_n) - p, \\
&\quad Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)\langle J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - J^{-1}(Jx_n), \\
&\quad Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle J^{-1}(Jx_n) - p, Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)\langle \alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)) - Jx_n, \\
&\quad Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(p, x_n) - 2(1-\alpha_n)\langle -(1-\alpha_n)Jx_n + (1-\alpha_n)SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)), Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)^2\langle -Jx_n + SJ_\lambda^{B_1}(Jx_n \\
&\quad -\gamma A^*(J - TJ_\lambda^{B_2})Ax_n)), Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - p, JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) + Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - p, JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p, x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad +JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - p, JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - p, JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n), JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle
\end{aligned} \tag{3.17}
$$

Since S is $J-$quasi pseudocontractive, that is,

$$
\begin{aligned}
\langle JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n),& \\
J_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - p\rangle \geq 0&
\end{aligned} \tag{3.18}
$$

we have

$$
\begin{aligned}
&\Phi(p,y_n) = \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n), JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) \\
&\quad -SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - Jp + Jp - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - Jp\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jp - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\langle x_n - p, Jx_n - Jp\rangle + 2(1-\alpha_n)\langle x_n - p, Jx_n - Jp\rangle \\
&\quad -2(1-\alpha_n)\langle x_n - p, \gamma A^*(J - TJ_\lambda^{B_2})Ax_n\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\langle x_n - p, \gamma A^*(J - TJ_\lambda^{B_2})Ax_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\gamma\langle Ax_n - Ap, (J-TJ_\lambda^{B_2})Ax_n\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\gamma\langle Ax_n - TJ_\lambda^{B_2}Ax_n + TJ_\lambda^{B_2}Ax_n - Ap, (J-TJ_\lambda^{B_2})Ax_n\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\gamma\langle Ax_n - TJ_\lambda^{B_2}Ax_n, (J-TJ_\lambda^{B_2})Ax_n\rangle \\
&\quad -2(1-\alpha_n)\gamma\langle TJ_\lambda^{B_2}Ax_n - Ap, (J-TJ_\lambda^{B_2})Ax_n\rangle \\
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\gamma\langle Ax_n - Ap + Ap - TJ_\lambda^{B_2}Ax_n, JAx_n - TJ_\lambda^{B_2}Ax_n\rangle \\
&\quad -2(1-\alpha_n)\gamma\langle TJ_\lambda^{B_2}Ax_n - Ap, JAx_n - TJ_\lambda^{B_2}Ax_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2 \\
&\quad -2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad \times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| \\
&\quad -2(1-\alpha_n)\gamma\langle JAx_n - TJ_\lambda^{B_2}Ax_n, Ax_n - Ap\rangle \\
&\quad -2(1-\alpha_n)\gamma\langle Ax_n - TJ_\lambda^{B_2}Ax_n, JAp - TJ_\lambda^{B_2}Ax_n\rangle \\
&\quad -2(1-\alpha_n)\gamma\langle TJ_\lambda^{B_2}Ax_n - Ap, JAx_n - TJ_\lambda^{B_2}Ax_n\rangle \qquad (3.19)
\end{aligned}
$$

Since T is $J-$quasi pseudocontractive, that is, $\langle JAx_n - TJ_\lambda^{B_2}Ax_n, Ax_n - Ap\rangle \geq 0$, we have

$$\begin{aligned}
&\Phi(p,x_n) = \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\
&-2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&\times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&-2(1-\alpha_n)\gamma\langle JAx_n - TJ_\lambda^{B_2}Ax_n, Ap - TJ_\lambda^{B_2}Ax_n\rangle\\
&-2(1-\alpha_n)\gamma\langle TJ_\lambda^{B_2}Ax_n - Ap, JAx_n - TJ_\lambda^{B_2}Ax_n\rangle\\
= \;&\Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\
&-2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&\times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&-2(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|\|Ap - TJ_\lambda^{B_2}Ax_n\|\\
&-2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|\|JAx_n - TJ_\lambda^{B_2}Ax_n\|\\
= \;&\Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\
&-2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&\times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&-(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2 - (1-\alpha_n)\gamma\|Ap - TJ_\lambda^{B_2}Ax_n\|^2\\
&-(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2 - (1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2\\
= \;&\Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\
&-2(1-\alpha_n)\|x_n - JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&\times\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&-2(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2 - 2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\|x_n - JJ_\lambda^{B_1}(Jx_n \;-\; &\gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| = \|x_n - p + p\\
&-JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
\leq\; &\|x_n - p\| + \|Jx_n - p - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
\leq\; &\|x_n - p\| + \|Jx_n - p\| + \gamma\|A^*(J-TJ_\lambda^{B_2})Ax_n)\|
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\|JJ_\lambda^{B_1}(Jx_n \;-\; &\gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\| = \|J_\lambda^{B_1}(Jx_n\\
&-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - p + p\\
&-JSJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
\leq\; &\|JJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - p\|\\
&+\|SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - p\|\\
\leq\; &\|Jx_n - p\| + \gamma\|A^*(J-TJ_\lambda^{B_2})Ax_n)\|\\
&+\|SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J-TJ_\lambda^{B_2})Ax_n) - p\|\\
\leq\; &\|Jx_n - p\| + \gamma\|A^*(J-TJ_\lambda^{B_2})Ax_n\|\\
&+L^2\|Jx_n - p\| + \gamma L^2\|A^*(J-TJ_\lambda^{B_2})Ax_n\|\\
=\; &(1+L^2)\|Jx_n - p\| + (\gamma + \gamma L^2)\|A^*(J-TJ_\lambda^{B_2})Ax_n\|
\end{aligned} \tag{3.22}$$

Substitute (3.21) and (3.22) into (3.20)

$$\begin{aligned}
\Phi(p,y_n) \le\ & \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
& -2(1-\alpha_n)\{\|x_n - p\| + \|Jx_n - p\| + \gamma\|A^*(J - TJ_\lambda^{B_2})Ax_n)\|\} \\
& \times\{(1+L^2)\|Jx_n - p\| + (\gamma + \gamma L^2)\|A^*(J - TJ_\lambda^{B_2})Ax_n\|\} \\
& -2(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2 \\
& -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2
\end{aligned}$$

$$\begin{aligned}
=\ & \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
& -(1-\alpha_n)(1+L^2)\|x_n - p\|^2 - (1-\alpha_n)(1+L^2)\|Jx_n - p\|^2 \\
& -2(1-\alpha_n)(1+L^2)\|Jx_n - p\|^2 - (1-\alpha_n)(1+L^2)\gamma\|A^*(J - TJ_\lambda^{B_2})Ax_n)\|^2 \\
& -(1-\alpha_n)(1+L^2)\gamma\|x_n - p\|^2 - (1-\alpha_n)(\gamma+\gamma L^2)\gamma\|A^*(J - TJ_\lambda^{B_2})Ax_n\|^2 \\
& -(1-\alpha_n)(\gamma+\gamma L^2)\|A^*(J - TJ_\lambda^{B_2})Ax_n)\|^2 \\
& -(1-\alpha_n)(\gamma+\gamma L^2)\|Jx_n - p\|^2 - 2(1-\alpha_n)(\gamma+\gamma L^2)\|A^*(J - TJ_\lambda^{B_2})Ax_n\|^2 \\
& -2(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2 - 2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2
\end{aligned}$$

$$\begin{aligned}
=\ & \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
& -(1-\alpha_n)(1+L^2)(3+\gamma)\|Jx_n - p\|^2 \\
& -(1-\alpha_n)(1+L^2)(1+\gamma)\|x_n - p\|^2 \\
& -(1-\alpha_n)(1+L^2)(3\gamma+2\gamma^2)\|A^*(J - TJ_\lambda^{B_2})Ax_n)\|^2 \\
& -2(1-\alpha_n)\gamma\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2 \\
& -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\|A^*(J - TJ_\lambda^{B_2})Ax_n\|^2 &= \langle A^*(J - TJ_\lambda^{B_2})Ax_n, A^*(J - TJ_\lambda^{B_2})Ax_n\rangle \\
&= \langle AA^*(J - TJ_\lambda^{B_2})Ax_n, (J - TJ_\lambda^{B_2})Ax_n\rangle \\
&= \|A\|^2\|JAx_n - TJ_\lambda^{B_2}Ax_n\|^2
\end{aligned} \tag{3.24}$$

Substitute (3.24) into (3.23)

$$\begin{aligned}
\Phi(p,y_n) =\ & \Phi(p,x_n) - 2(1-\alpha_n)^2\|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n))\|^2 \\
& -(1-\alpha_n)(1+L^2)(3+\gamma)\|Jx_n - p\|^2 \\
& -(1-\alpha_n)(1+L^2)(1+\gamma)\|x_n - p\|^2 \\
& -(1-\alpha_n)[(1+L^2)(3\gamma+2\gamma^2)\|A\|^2 + 2]\|TJ_\lambda^{B_2}Ax_n - JAx_n\|^2 \\
& -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n - Ap\|^2
\end{aligned} \tag{3.25}$$

Substitute (3.25) into (3.16)

$$\begin{aligned}\Phi(p,x_{n+1}) &\leq \beta_n\Phi(p,x_0)+(1-\beta_n)\{\Phi(p,x_n)\\ &\quad -2(1-\alpha_n)^2\|Jx_n-SJ_\lambda^{B_1}(Jx_n-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\ &\quad -(1-\alpha_n)(1+L^2)(3+\gamma)\|Jx_n-p\|^2\\ &\quad -(1-\alpha_n)(1+L^2)(1+\gamma)\|x_n-p\|^2\\ &\quad -(1-\alpha_n)[(1+L^2)(3\gamma+2\gamma^2)\|A\|^2+2]\|TJ_\lambda^{B_2}Ax_n-JAx_n\|^2\\ &\quad -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n-Ap\|^2\}\\ &\leq \beta_n\Phi(p,x_0)+(1-\beta_n)\Phi(p,x_n)\\ &\quad -2(1-\beta_n)(1-\alpha_n)^2\|Jx_n-SJ_\lambda^{B_1}(Jx_n-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)(1+L^2)(3+\gamma)\|Jx_n-p\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)(1+L^2)(1+\gamma)\|x_n-p\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)[(1+L^2)(3\gamma+2\gamma^2)\|A\|^2+2]\|TJ_\lambda^{B_2}Ax_n-JAx_n\|^2\\ &\quad -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n-Ap\|^2 \end{aligned} \tag{3.26}$$

$$\begin{aligned}\Phi(p,x_{n+1}) &\leq (1-\beta_n)\Phi(p,x_n)+\beta_n\Phi(p,x_0)\\ &\quad -2(1-\beta_n)(1-\alpha_n)^2\|Jx_n-SJ_\lambda^{B_1}(Jx_n-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)(1+L^2)(3+\gamma)\|Jx_n-p\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)(1+L^2)(1+\gamma)\|x_n-p\|^2\\ &\quad -(1-\beta_n)(1-\alpha_n)[(1+L^2)(3\gamma+2\gamma^2)\|A\|^2+2]\|TJ_\lambda^{B_2}Ax_n-JAx_n\|^2\\ &\quad -2(1-\alpha_n)\gamma\|TJ_\lambda^{B_2}Ax_n-Ap\|^2 \end{aligned} \tag{3.27}$$

By condition (ii) $\sum_{n=1}^{\infty}\beta_n=\infty$ and from Lemma 2.7 that following limit exists

$$\lim_{n\to\infty}\Phi(p,x_n)=0. \tag{3.28}$$

From equation (3.27)

$$\begin{aligned}2(1-\beta_n)(1 &- \alpha_n)^2\|Jx_n-SJ_\lambda^{B_1}(Jx_n-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n))\|^2\\ &\quad +(1-\beta_n)(1-\alpha_n)[(1+L^2)(3\gamma+2\gamma^2)\|A\|^2+2]\|TJ_\lambda^{B_2}Ax_n-JAx_n\|^2\\ &\leq \Phi(p,x_n)-\beta_n\Phi(p,x_n)+\beta_n\Phi(p,x_0)\\ &\quad -\Phi(p,x_{n+1})\to 0 \text{ (as } n\to\infty). \end{aligned} \tag{3.29}$$

This implies that

$$\lim_{n\to\infty}\|(J-TJ_\lambda^{B_2})Ax_n\|=0 \tag{3.30}$$

$$\lim_{n\to\infty}\|SJ_\lambda^{B_1}(Jx_n-\gamma A^*(J-TJ_\lambda^{B_2})Ax_n-Jx_n\|=0 \tag{3.31}$$

Also,

$$\lim_{n\to\infty}\Phi(x_n,x_{n+1})=0, \tag{3.32}$$

It follows from equations (3.15)

$$\begin{aligned}
\Phi(x_n, x_{n+1}) &= \Phi(x_n, J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n)) \\
&= \|x_n\|^2 - 2\langle x_n, \beta_n Jx_0 + (1-\beta_n)Jy_n\rangle \\
&\quad + \|\beta_n Jx_0 + (1-\beta_n)Jy_n\|^2 \\
&= \|x_n\|^2 - 2\langle p, \beta_n Jx_0\rangle - 2(1-\beta_n\langle x_n, Jy_n\rangle \\
&\quad + \|\beta_n Jx_0 + (1-\beta_n)Jy_n\|^2 \\
&\leq \|x_n\|^2 - 2\langle x_n, \beta_n Jx_0\rangle - 2(1-\beta_n\langle x_n, Jy_n\rangle + \beta_n\|x_0\|^2 \\
&\quad + (1-\beta_n)\|y_n\|^2 - \beta_n(1-\beta_n)\|y_n - x_0\|^2 + \|x_n\|^2 - \|x_n\|^2 \\
&\leq \beta_n\Phi(x_n, x_0) + (1-\beta_n)\Phi(x_n, y_n) \\
&\leq \beta_n[\Phi(x_n, p) + \Phi(p, x_0)] + (1-\beta_n)[\Phi(x_n, p) + \Phi(p, y_n)] \qquad (3.33)
\end{aligned}$$

From equations (3.28), (3.30) and (3.31), $\lim_{n\to\infty} \Phi(x_n, x_{n+1}) = 0$.

By Lemma 2.5, we have $F(S)$, $F(J_\lambda^{B_1})$, $F(T)$ and $F(J_\lambda^{B_2})$ are closed and convex, and hence Γ is also closed and convex. Let $p = P_\Gamma u$. By characterization of the generalised projection, we get

$$\langle u - p, Jz - p\rangle \leq 0, \ \ \forall z \in \Gamma. \qquad (3.34)$$

Since $p \in \Gamma$, we obtain $Tp = Jp$, $J_\lambda^{B_1}p = Jp$ and $J_\lambda^{B_2}Ap = JAp$.

$$\lim_{n\to\infty} \sup\langle u - p, Jx_n - p\rangle \leq 0.$$

To show this, let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle u - p, Jx_{n_i} - p\rangle = \lim_{n\to\infty} \sup\langle u - p, Jx_n - p\rangle$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ and $z \in H_1$ such that $x_{n_{i_j}} \rightharpoonup Jz$. Without loss of generality, we can assume that $\{x_{n_i}\} \rightharpoonup Jz$. Since A is a bounded linear operator, we have $\langle q, Ax_{n_i} - JAz\rangle = \langle A^*q, x_{n_i} - Jz\rangle$ as $i \to \infty$, for all $q \in H_2$, this implies that $Ax_{n_i} \rightharpoonup JAz$. By the demiclosedness of $J - T$ and $J - J_\lambda^{B_2}$ at zero, then $J - TJ_\lambda^{B_2}$ is also demiclosed at zero, and from equation (3.30), we get $JAz \in F(TJ_\lambda^{B_2}) = F(T) \cap B_2^{-1}0$. Since $x_{n_i} \rightharpoonup Jz$ and $\|SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) - Jx_n\| \to 0$ as $n \to \infty$, we have $x_{n_i} \rightharpoonup Jz$. Also, by the demiclosedness of $J - S$ and $J - J_\lambda^{B_1}$ at zero, then $J - SJ_\lambda^{B_2}$ is also demiclosed at zero, and from equation (3.31), we get $Jz \in F(SJ_\lambda^{B_1}) = F(S) \cap B_1^{-1}0$.

Now let us show that $Jz \in B_1^{-1}0$. Let $\omega_n = J_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n)$, then we can easily prove that

$$\frac{1}{\lambda}\left(Jx_n - \omega_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n\right) \in B_1\omega_n$$

By the monotonicity of B_1, we have

$$\left\langle \omega_n - v, \frac{1}{\lambda}\left(Jx_n - \omega_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n\right) - w \right\rangle$$

for all $(v, w) \in G(B_1)$. Thus, we also have

$$\left\langle \omega_{n_i} - v, \frac{1}{\lambda}\left(Jx_{n_i} - \omega_{n_i} - \gamma A^*(J - TJ_\lambda^{B_2})Ax_{n_i}\right) - w \right\rangle \qquad (3.35)$$

for all $(v, w) \in G(B_1)$. Since $\omega_{n_i} \rightharpoonup Jz$, $\|\omega_{n_i} - J_\lambda^{B_1}(Jx_{n_i} - \gamma A^*(J - TJ_\lambda^{B_2})Ax_{n_i})\| \to 0$. $(J - TJ_\lambda^{B_2})Ax_{n_i} \to 0$ as $i \to \infty$, then by taking the limit as $i \to \infty$ in equation (3.35) yields

$$\langle Jz - v, -w \rangle \leq 0$$

for all $(v, w) \in G(B_1)$. By the maximal monotonicity of B_1, we get $0 \in B_1(Jz)$, that is, $Jz \in B_1^{-1}0$.

Also, let us show that $Jz \in B_2^{-1}0$. Let $\varepsilon_n = J_\lambda^{B_2}Ax_n$, then we can easily prove that

$$\frac{1}{\lambda}(JAx_n - \varepsilon_n) \in B_2\varepsilon_n$$

By the monotonicity of B_2, we have

$$\left\langle \varepsilon_n - \varrho, \frac{1}{\lambda}(JAx_n - \varepsilon_n) - \vartheta \right\rangle$$

for all $(\varrho, \vartheta) \in G(B_2)$. Thus, we also have

$$\left\langle \varepsilon_{n_i} - \varrho, \frac{1}{\lambda}(JAx_{n_i} - \varepsilon_{n_i}) - \vartheta \right\rangle \tag{3.36}$$

for all $(\varrho, \vartheta) \in G(B_2)$. Since $\varepsilon_{n_i} \rightharpoonup JAz$, $\|\varepsilon_{n_i} - J_\lambda^{B_2}Ax_{n_i}\| \to 0$, then by taking the limit as $i \to \infty$ in equation (3.36) yields

$$\langle JAz - \varrho, -\vartheta \rangle \leq 0$$

for all $(\varrho, \vartheta) \in G(B_2)$. By the maximal monotonicity of B_2, we get $0 \in B_2(JAz)$, that is, $JAz \in B_2^{-1}0$.

Now we prove that $z \in F(S)$. Otherwise, assume that $z \notin Fix(S)$, that is, $z \notin Sz$. Opial's condition that

$$\begin{aligned} \liminf \|x_{n_i} - z\| &< \liminf \|x_{n_i} - Sz\|, \\ &= \liminf \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - Sz\|, \\ &= \liminf \|Sx_{n_i} - Sz\|, \\ &\leq \liminf \|x_{n_i} - z\|, \end{aligned}$$

This is a contradiction.Thus, $z \in F(S)$.

Again, we prove that $Az \in F(T)$. Otherwise, assume that $Az \notin Fix(T)$, that is, $Az \notin TAp$. Opial's condition that

$$\begin{aligned} \liminf \|Ax_{n_i} - JAz\| &< \liminf \|x_{n_i} - Tz\|, \\ &= \liminf \|Ax_{n_i} - TAx_{n_i} + TAx_{n_i} - TJAz\|, \\ &= \liminf \|TAx_{n_i} - TJAz\|, \\ &\leq \liminf \|Ax_{n_i} - JAz\|, \end{aligned}$$

This is a contradiction. Thus, $Az \in F(T)$.

Therefore, $z \in \Gamma$. Since z satisfies the inequality (3.34) as

$$\lim_{n\to\infty} \sup\langle u - p, Jx_n - p \rangle = \lim_{i\to\infty} \langle u - p, Jx_{n_i} - p \rangle = \langle u - p, Jz - p \rangle \leq 0$$

and also satisfies the Opial's condition in Definition 2.1, it follows from Lemma 2.7 that $\{x_n\}$ converges strongly to $p \in \Gamma$.

Corollary 3.1. *Let E_1 and E_2 be real Banach spaces. Let B_1 and B_2 be maximal monotone operators of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be generalized resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J - T)$ is demiclosed at zero. If the solution set of SCFPNPP is nonempty (that is, $\Gamma = \{Jx \in B_1^{-1}0, JAx \in F(T) \cap B_2^{-1}0\} \neq \emptyset$). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ be real sequences satisfying the conditions: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (ii) $\lim_{n\to\infty} \beta_n = 0$ and (iii) $\sum_{n=1}^{\infty} \beta_n = \infty$. Let x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by* (3.15), *then*

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|JAx_n - TJ_\lambda^{B_2}Ax_n\| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$.

Corollary 3.2. *Let E_1 and E_2 be real Banach spaces. Let B_1 and B_2 be maximal monotone operators of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be generalized resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator. If the solution set of SNPP is nonempty (that is, $\Gamma = \{Jx \in B_1^{-1}0, JAx \in B_2^{-1}0\} \neq \emptyset$). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ be real sequences satisfying the conditions: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (ii) $\lim_{n\to\infty} \beta_n = 0$ and (iii) $\sum_{n=1}^{\infty} \beta_n = \infty$. Let x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by* (3.15)*,then*

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a point* $p \in \Gamma$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$.

Corollary 3.3. *Let E_1 and E_2 be real Banach spaces. Let B_1 and B_2 be maximal monotone operators of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be generalized resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator and $T : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-seudocontractive self maps of E_1 such that $(J - T)$ is demiclosed at zero. If the solution set of SCFPNPP is nonempty (that is, $\Gamma = \{Jx \in F(S) \cap B_1^{-1}0, JAx \in B_2^{-1}0\} \neq \emptyset$). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ be real sequences satisfying the conditions: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (ii) $\lim_{n\to\infty} \beta_n = 0$ and (iii) $\sum_{n=1}^{\infty} \beta_n = \infty$. Let x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by*(3.15)*,then*

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - J_\lambda^{B_2})Ax_n)\| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$.

Corollary 3.4. *Let E_1 and E_2 be real Banach spaces. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J-S)$ and $(J-T)$ are demiclosed at zero. If the solution set of SCFPP is nonempty (that is, $\Gamma = \{Jx \in F(S), JAx \in F(T)\} \neq \emptyset$). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ be real sequences satisfying the conditions: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (ii) $\lim_{n\to\infty} \beta_n = 0$ and (iii) $\sum_{n=1}^{\infty} \beta_n = \infty$. Let x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by* (3.15)*,then*

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - S(Jx_n - \gamma A^*(J - T)Ax_n)\| = \lim_{n\to\infty} \|JAx_n - TAx_n\| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$.

4. Applications

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\bar{x} \in C$ such that $f(\bar{x}, y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics reduce to finding a solution to the equilibrium problem (see [7]).

Lemma 4.1. *For solving the equilibrium problem, they assumed that the bifunction f satisfies the following conditions:*

(A1) $f(x,x) = 0$ *for all* $x \in C$,
(A2) f *is monotone, that is,* $f(x,y) + f(y,x) \leq 0$ *for all* $x, y \in C$,
(A3) for every $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$,
(A4) $f(x, \cdot)$ *is convex and lower semicontinuous for each* $x \in C$.

Equilibrium problems have been studied extensively; see [7, 23, 24, 60].

Lemma 4.2. *(see [7]). Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). If $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C. \tag{4.1}$$

Lemma 4.3. *(see [56]). Let C be a nonempty closed convex subset of E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). For $r > 0$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \ \forall y \in C\}. \tag{4.2}$$

Then the following hold:

(i) T_r *is single valued,*
(ii) T_r *is firmly nonexpansive, that is, for any* $x, y \in E$

$$\langle Jx - Jy, T_r x - T_r y\rangle \geq \|JT_r x - JT_r y\|^2, \tag{4.3}$$

(iii) $Fix(T_r) = EP(f)$,
(iv) $EP(f)$ *is closed and convex.*

Lemma 4.4. *Let C be a nonempty closed convex subset of E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). Define A_f as follows:*

$$A_f(x) = \begin{cases} \{z \in E_1 : f(z,y) \geq \langle y - x, Jz\rangle, \ \ \forall y \in C\}, & \text{if } x \in C \\ \emptyset & x \text{ if } \notin C \end{cases} \tag{4.4}$$

Then the following hold:

(i) A_f *is msximal monotone,*
(ii) $EP(f) = A_f^{-1}0$,
(ii) $T_r^f = (J + rA_f)^{-1}0, \ \ r > 0.$

Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : E_1 \to E_2$ a bounded linear operator, then the SEP is to find a point $x^* \in C$ such that

$$\left.\begin{array}{l} f_1(x^*, x) \geq 0 \ \forall x \in C \ \text{ and} \\ y^* := Ax^* \in Q \ \text{ solves } \ f_2(y^*, y) \geq 0 \ \forall y \in Q \end{array}\right\} \tag{4.5}$$

Then above problem is to find a point $x^* \in C$ such that

$$Jx^* \in EP(f_1) \ \text{ and } \ JAx^* \in EP(f_2) \tag{4.6}$$

Let E be a real Banach space, and let f be a proper lower semicontinuous convex function of E into $(-\infty, +\infty]$. Then the subdifferential ∂f of f is defined as

$$\partial f(x) = \{z \in E : f(y) - f(x) \geq \langle Jz, y - x\rangle, \ \ \forall y \in E\} \tag{4.7}$$

for all $x \in E$. [44] claimed that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of E, and let δ_C be the indicator function of C. That is,

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} \tag{4.8}$$

Since δ_C is a proper lower semicontinuous convex function on E, the subdifferential ∂_{δ_C} of δ_C is a maximal monotone operator. The resolvent J_λ of ∂_{δ_C} for $\lambda > 0$ is defined by

$$J_\lambda x = (J + \lambda \partial_{\delta_C})^{-1} Jx, \ \ \ \forall x \in E. \tag{4.9}$$

they have

$$\begin{aligned} u = (J + \lambda\partial_{\delta_C})^{-1} Jx \ &\Leftrightarrow \ Jx \in Ju + \lambda \partial_{\delta_C} u \\ &\Leftrightarrow \ Jx \in Ju + \lambda N_C u \Leftrightarrow Jx - Ju \in \lambda N_C u \\ &\Leftrightarrow \ \frac{1}{\lambda}\langle Jx - Ju, y - u\rangle \leq 0, \ \ \forall y \in C \\ &\Leftrightarrow \ u = \prod_C x \end{aligned} \tag{4.10}$$

where $N_C u = \{z \in E_1 : \langle Jz, x - u\rangle \leq 0 \ \ \forall x \in C\}$.

Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. Let $S : E_1 \to E$ and $T : E_2 \to E_2$ be two Lipschitzian quasi pseudocontractiv mappings and $A : E_1 \to E_2$ a bounded linear operator from E_1 to E_2.

The Split Variational Inequality Problem denoted by SVIP is to find a point $u^* \in C$ such that

$$\left.\begin{array}{l} \langle Ju - Ju^*, SJ_\lambda^{B_1}(Ju^* - \gamma A^*(J - TJ_\lambda^{B_2})Au^*)\rangle \geq 0 \ \ \forall u \in C \ \text{ and} \\ v^* := JAx^* \in Q \ \text{ such that} \\ \langle v - v^*, A^*(J - TJ_\lambda^{B_2})Av^*\rangle \geq 0 \ \ \forall v \in Q. \end{array}\right\} \tag{4.11}$$

Let D be the solution setof the SVIP given by

$$D = \{Ju^* \in VI(C, S) : JAu^* \in VI(Q, T)\} \tag{4.12}$$

We observe that $Ju^* \in SVIP$ if and only if $Ju^* = SJ_\lambda^{B_1}(Ju^* - \gamma A^*(J - TJ_\lambda^{B_2})Au^*)$ and $JAu^* = A^*(J - TJ_\lambda^{B_2})Au^*$.

Let E_1 and E_2 be real Banach spaces. Let $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ be proper, lower semicontinuous and convex functions. Let $A : E_1 \to E_2$ be a bounded linear operator, the Split Optimizatin problem (SOP) is the problem of finding $x^* \in E_1$ such that

$$Jx^* \in \text{Argmin } f \text{ and } JAx^* \in \text{Argmin } g. \tag{4.13}$$

Denote by $\partial f = B_1$ and $\partial g = B_2$. Since Jx^* and JAx^* are the minimum of f on E_1 and g on E_2, respectively for any $\lambda > 0$, we have

$$\begin{aligned} x^* = F(S) \cap (\partial f)^{-1}0 = Fix(SJ_\lambda^{\partial f}) \text{ and} \\ Ax^* = F(T) \cap (\partial g)^{-1}0 = Fix(TJ_\lambda^{\partial g}). \end{aligned} \tag{4.14}$$

Also, this implies that the split optimization problem (4.13) is equivalent to the split common fixed point and null point problem SCFPNPP.

4.1. Split feasibility Problem (SFP)

Theorem 4.1. *Let E_1 and E_2 be real Banach spaces and C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J - S)$ and $(J - T)$ are demiclosed at zero. If the solution set of SFP is nonempty (that is, $\Gamma = \{Jx \in F(S) \cap C \ : JAx \in F(T) \cap Q\} \neq \emptyset$). Suppose that x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by*

$$\begin{aligned} x_{n+1} &= J^{-1}(\beta_n x_0 + (1 - \beta_n) y_n) \\ y_n &= J^{-1}(\alpha_n x_n + (1 - \alpha_n) S \textstyle\prod_C (Jx_n - \gamma A^*(J - T \prod_C) Ax_n)) \end{aligned} \quad n \geq 1. \tag{4.15}$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the conditions: (i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ exists for each $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - S \prod_C (Jx_n - \gamma A^*(J - T \prod_Q) Ax_n)\| = \lim_{n\to\infty} \|JAx_n - T \prod_Q Ax_n\| = 0$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in \Gamma$.

Proof: Set $B_1 := \partial\delta_C$ and $B_2 := \partial\delta_Q$. Then B_1 and B_2 are maximal monotone such that $J_\lambda^{B_1} = \prod_C$ and $J_\lambda^{B_2} = \prod_Q$ for $\lambda > 0$. We also have $B_1^{-1}0 = C$ and $B_2^{-1}0 = Q$. Hence the result is obtained directly by Theorem 3.1.

4.2. Split Equilibrium Problem (SEP)

Theorem 4.2. *Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (A1) – (A4) and let $T_{r_1}^{f_1}$ and $T_{r_2}^{f_2}$ be resolvents of A_{f_1} and A_{f_2} in Lemma 4.4, respectively for r_1, $r_2 > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of H_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi-pseudocontractive self maps of E_2 such that $(J - S)$ and $(J - T)$ are demiclosed at zero. If the solution set of SEP* (4.6) *is nonempty (that is,*

$\Gamma = \{Jx \in F(S) \cap EP(f_1), JAx \in F(T) \cap EP(f_2)\} \neq \emptyset$). *Suppose that* $x_0, x_1 \in E_1$ *be arbitrary, the iterative sequence* $\{x_n\}$ *generated by*

$$\begin{array}{rcl} x_{n+1} & = & J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ y_n & = & J^{-1}(\alpha_n Jx_n + (1-\alpha_n)ST_{r_1}^{f_1}(Jx_n - \gamma A^*(J - TT_{r_2}^{f_2})Ax_n)) \end{array} \quad n \geq 1. \tag{4.16}$$

where the parameter γ *and the sequences* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfying the conditions: (i)* $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, *(ii)* $\sum_{n=1}^{\infty} \alpha_n < \infty$, *(iii)* $\lim_{n\to\infty} \beta_n = 0$ *and (iv)* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Then,*

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ *exists for each* $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - ST_{r_1}^{f_1}(Jx_n - \gamma A^*(J - TT_{r_2}^{f_2})Ax_n)\| = \lim_{n\to\infty} \|JAx_n - TT_{r_2}^{f_2}Ax_n\| = 0$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$.

Proof : We set $B_1 := A_{f_1}$ and $B_2 := A_{f_2}$. By Lemma 4.4, we know that B_1 and B_2 are maximal monotone, $EP(f_1) = B_1^{-1}0$, $EP(f_2) = B_2^{-1}0$, $T_{r_1}^{f_1} = J_\lambda^{B_1}$ and $T_{r_2}^{f_2} = J_\lambda^{B_2}$, so the result is obtained directly by Theorem 3.1.

4.3. Split Variational inequality Problem (SVIP)

Theorem 4.3. *Let* E_1 *and* E_2 *be Banach spaces,* $A : E_1 \to E_2$ *be a bounded linear operator, and* $S : E_1 \to 2^{E_1^*}$ *be Lipschitzian* $J-$*quasi-pseudocontractive self maps of* E_1 *and* $T : E_2 \to 2^{E_2^*}$ *be Lipschitzian* $J-$*quasi-pseudocontractive self maps of* E_2 *such that* $(J-S)$ *and* $(J-T)$ *are demiclosed at zero. Let* A^* *denotes the adjoint ofA. Let* $B_1 : E_1 \to 2^{E_1^*}$ *and* $B_2 : E_2 \to 2^{E_2^*}$ *be two set valued maximal monotone mappings and* γ, $\lambda > 0$. *Given any* $x^* \in E_1$,

(i) if Jx^* *is a solution of SVIP, then* $SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) = Jx^*$,

(ii) Suppose that $SJ_\lambda^{B_1}(Jx_n - \gamma A^*(J - TJ_\lambda^{B_2})Ax_n) = Jx^*$ *and the solution set of SVIP is not empty, then* Jx^* *is a solution of SVIP.*

Proof : (i) Suppose that $x^* \in E_1$ is a solution of SVIP, then $x^* \in F(S) \cap B_1^{-1}0$ and $Ax^* \in F(T) \cap B_2^{-1}0$. By Lemma 3.2, it can be seen that $SJ_\lambda^{B_1}(Jx^* - \gamma A^*(J - TJ_\lambda^{B_2})Ax^*) = Jx^*$.
(ii) Suppose that Jw^* is the solution of SVIP and $SJ_\lambda^{B_1}(Jx^* - \gamma A^*(J - TJ_\lambda^{B_2})Ax^*) = Jx^*$ by Lemma 3.2, we have

$$\langle Jx^* - \gamma A^*(J - TJ_\lambda^{B_2})Ax^* - Jx^*, Jx^* - Jw^* \rangle \geq 0$$

for each $Jw^* \in F(S) \cap B_1^{-1}0$, that is,

$$\langle A^*(J - TJ_\lambda^{B_2})Ax^*, Jx^* - Jw^* \rangle \leq 0$$

for each $Jw^* \in F(S) \cap B_1^{-1}0$,

$$\langle JAx^* - TJ_\lambda^{B_2}Ax^*, JAx^* - JAw^* \rangle \leq 0$$

Jw^* is the solution of SVIP.
We set $B_1 := A_{f_1}$ and $B_2 := A_{f_2}$. By Lemma 4.4, we know that B_1 and B_2 are maximal monotone, $S = I$, $T = I$, so the result is obtained directly by Theorem 3.1.

4.4. Split Optimization Problem (SOP)

Theorem 4.4. *Let E_1 and E_2 be real Banach spaces. Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be proper lower semicontinuous convex function of E into $(-\infty, +\infty]$. Let $A : E_1 \to E_2$ be a bounded linear operator. Let $A : E_1 \to E_2$ be a bounded linear operator, and $S : E_1 \to 2^{E_1^*}$ be Lipschitzian $J-$quasi pseudocontractive self maps of E_1 and $T : E_2 \to 2^{E_2^*}$ be Lipschitzian $J-$quasi pseudocontractive self maps of H_2 such that $(J - S)$ and $(J - T)$ are demiclosed at zero. If the solution set of SOP (4.13) is nonempty (that is, $\Gamma = \{Jx \in F(S) \cap B_1^{-1}0, JAx \in F(T) \cap B_2^{-1}0\} \neq \emptyset$). Suppose that x_0, $x_1 \in E_1$ be arbitrary, the iterative sequence $\{x_n\}$ generated by*

$$\begin{aligned} x_{n+1} &= J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)SJ_\lambda^{\partial f}(Jx_n - \gamma A^*(J - TJ_\lambda^{\partial g})Ax_n)) \end{aligned} \quad n \geq 1. \tag{4.17}$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the conditions: (i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi(p, x_n)$ exists for each $p \in \Gamma$,

(b) $\lim_{n\to\infty} \|Jx_n - SJ_\lambda^{\partial f}(Jx_n - \gamma A^*(J - TJ_\lambda^{\partial g})Ax_n)\| = \lim_{n\to\infty} \|JAx_n - TJ_\lambda^{\partial g} Ax_n\| = 0$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in \Gamma$.

Proof: Set $B_1 := \partial f$ and $B_2 := \partial g$. Hence the result is obtained directly by Theorem 3.1.

Conclusion

We study the split common fixed point and null point problem between Banach spaces outside Hilbert spaces. We propose a Halpern-type algorithm with self-adaptive stepsize and prove a strong convergence theorem without imposing demi-compactness condition the nonlinear mappings. We apply our main results to split feasibility problem (SFP), split equilibrium problem (SEP), split variational inequality problem (SVIP) and split optimization problem (SOP). Some existing results are derived from our main results and their proofs are given. Lastly, our results resolved some of the open problems in literature.

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