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Optimal values range of interval polynomial programming problems

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Abstract

Uncertainty exists in many real-life engineering and mechanical problems. Here, we assume that uncertainties are caused by intervals of real numbers. In this paper, we consider the interval nonlinear programming (INLP) problems where the objective function and constraints include interval coefficients. So that the variables are deterministic and sign-restricted. Additionally, the constraints are considered in the form of inequalities. A basic task in INLP is calculating the optimal values range of objective function, which may be computationally very expensive. However, if the boundary functions are available, the problems become much easier to solve. By making these assumptions, an efficient method is proposed to compute the optimal values range using two classic nonlinear problems. Then, the optimal values range are obtained by direct inspection for a special kind of interval polynomial programming (IPP) problems. Two numerical examples are given to verify the effectiveness of the proposed method.

Keywords: interval uncertainty, interval nonlinear programming, optimal values range, interval polynomial, boundary functions
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1. Introduction

In practical structures, there always exists many kinds of uncertainties. Some of these uncertainties have a physical origin, resulting from loading conditions, material properties, or other predetermined characteristics and cannot be influenced by the designer. Other uncertainties are caused by people, with typical examples including manufacturing and measurement deviations. These types

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of parameters are generally estimated from historical data using probability theory or fuzzy set theory. But it is difficult to identify suitable probability distribution function or membership function. Some researchers have started to represent uncertain parameters as intervals to avoid the need for selection of probability distribution function and membership function. In the literature of interval optimization, may researchers have tried to address the interval uncertainties in different ways.

Solving interval linear and nonlinear programming problems has become an interesting area of research in recent years. Hladík et al. [1] discussed the optimal value range in interval linear programming (ILP). Mostafaee and Hladík [10] computed the exact range of the optimal value function in interval fractional linear programming.

Hladík [2] proposed a method to determine the optimal values range for convex quadratic and posynomial geometric programming problems. Jiang et al. [3, 4] have developed the efficient nonlinear interval number programming methods. Kumar and Panda [5] studied a general INLP model in which intervals are considered as random variables. Levin [6, 7] proposed a algorithm to determine the set of solutions of INLP problems. Liu et al. [8] developed a multi-objective optimization method for uncertain structures based on nonlinear interval number programming method. Li and Xu [9] investigated a different approach to the application of interval analysis which represents the range of a function on interval variables.

The aim of this paper is to determine the optimal values range for other classes of the INLP problems.

The remainder of this paper is organized as follows: section 2 includes the definitions and preliminaries. section 3 presents a general INLP problem and by some assumptions formulates a new method to obtain the optimal values range of the general INLP problem. section 4 introduces a special class of INLP problems, namely IPP problems and obtains the optimal values range by direct calculation. section 5 analyzes two numerical examples; and the conclusions are summarized in section 6.

Since the economical parameters of the real life problems (including the mentioned applications) are often imprecise, the results developed in this paper form a useful and efficient tool in decision making and analysis.

2. Definitions and preliminaries

In this section, we state some definitions of interval analysis.

Definition 2.1. *[11] A closed, real interval (or briefly an interval) is represented by $c^I = [a, b]$. That is, the set $\{c \mid a \leqslant c \leqslant b\}$ of all real numbers between and including the boundary points a and b.*

Definition 2.2. *[11] For two intervals $a^I = [\underline{a}, \overline{a}]$ and $b^I = [\underline{b}, \overline{b}]$, operations of addition, subtraction, multiplication and scalar multiplication are defined as follows:*

$$a^I + b^I = [\,\underline{a} + \underline{b},\, \overline{a} + \overline{b}\,],$$

$$a^I - b^I = [\,\underline{a} - \overline{b},\, \overline{a} - \underline{b}\,],$$

$$a^I \times b^I = [\,min\{\underline{a}\,\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\,\overline{b}\}, max\{\underline{a}\,\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\,\overline{b}\}\,],$$

and

$$k\,a^I = k\,[\underline{a}, \overline{a}] = \begin{cases} [k\underline{a}, k\overline{a}]\,, & k \geqslant 0, \\ [k\overline{a}, k\underline{a}]\,, & k \leqslant 0. \end{cases}$$

In this paper, we suppose that the variables are sign-restricted. Also, let X^{si} be the set of sign-restricted variables. Furthermore, we display the vector $(x_1, ..., x_n)^t$ as $\mathbf{x} = (x_1, ..., x_n)^t$.

Definition 2.3. *Consider the sign-restricted variable of x_j and an interval $a^I = [\underline{a}, \overline{a}]$. We define multiplication of interval a^I in variable x_j as*

$$a^I\, x_j = [\underline{a}, \overline{a}]\, x_j = \begin{cases} [\underline{a}\, x_j,\, \overline{a}\, x_j]\,, & x_j \geqslant 0, \\ [\overline{a}\, x_j,\, \underline{a}\, x_j]\,, & x_j \leqslant 0, \end{cases}$$

Now, we define the interval function.

Definition 2.4. *[6] An interval function is an interval whose upper and lower bounds are real function in the domain definition of $G \subseteq \mathbb{R}^n$ and is represented as $f^I(\boldsymbol{x}) = [h(\boldsymbol{x}), l(\boldsymbol{x})]$. That is, the set $\{f(\boldsymbol{x}) \mid h(\boldsymbol{x}) \leqslant f(\boldsymbol{x}) \leqslant l(\boldsymbol{x}),\ \boldsymbol{x} \in G\}$ of all real value functions between and including the boundary functions $h(\boldsymbol{x})$ and $l(\boldsymbol{x})$.*

Definition 2.5. *For two interval functions $f^I(\boldsymbol{x}) = [\underline{f}(\boldsymbol{x}), \overline{f}(\boldsymbol{x})]$ and $g^I(\boldsymbol{x}) = [\underline{g}(\boldsymbol{x}), \overline{g}(\boldsymbol{x})]$ in the definition domain of $G \subseteq \mathbb{R}^n$, operations of addition and subtraction are defined as follows:*

$$f^I(\boldsymbol{x}) + g^I(\boldsymbol{x}) = [\,\underline{f}(\boldsymbol{x}) + \underline{g}(\boldsymbol{x}),\ \overline{f}(\boldsymbol{x}) + \overline{g}(\boldsymbol{x})\,],$$
$$f^I(\boldsymbol{x}) - g^I(\boldsymbol{x}) = [\,\underline{f}(\boldsymbol{x}) - \overline{g}(\boldsymbol{x}),\ \overline{f}(\boldsymbol{x}) - \underline{g}(\boldsymbol{x})\,].$$

3. Interval nonlinear programming problems

A method for finding the optimal values range for some particular subclasses of the INLP problems such as convex quadratic programming and posynomial geometric programming is proposed by Hladík [2].

In this section, First, we mention the general form of an INLP problem that can be found in Levin [6, 7]. Then, we assume a situation where the associated intervals of the objective function and constraints are available for the general INLP problem and determine the optimal values range for the objective function.

Let an arbitrary nonlinear continuous function of n variables be as,

$$y = f(x_1, ..., x_n), \tag{3.1}$$

We will consider function (3.1) in the limited domain determined by the system of constraints:

$$g_i(x_1, ..., x_n) \geqslant b_i, \qquad i = 1, ..., m, \tag{3.2}$$

where all $g_i,\ i = 1, ..., m$ are arbitrary nonlinear continuous functions. Then, the determined optimization problem can be stated as follows:

$$\begin{array}{ll} \min & f(x_1, ..., x_n) \\ s.t. & g_i(x_1, ..., x_n) \geqslant b_i, \quad i = 1, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n, \end{array} \tag{3.3}$$

There are a lot of methods for solving nonlinear programming (NLP) problem (3.3) in the classic mathematical programming.

In NLP problem (3.3), assume that variable coefficients in the objective function, at the left and right side of constraints are shows with $c_k, k = 1, ..., l$, $a_{is}, s = 1, ..., t$ and $b_i, i = 1, ..., m$ parameters, respectively. Now, let c_k, a_{is}, and b_i coefficients are in the form of intervals:

$$c_k^I = [\underline{c}_k, \overline{c}_k],\ \ k = 1, ..., l, \qquad a_{is}^I = [\underline{a}_{is}, \overline{a}_{is}],\ \ s = 1, ..., t, \qquad b_i^I = [\underline{b}_i, \overline{b}_i],\ \ i = 1, ..., m.$$

Then, the real functions f and g_i, $i = 1, ..., m$ become interval functions (i.e., take the form f^I and g_i^I, $i = 1, ..., m$). Furthermore, the parameters b_i, $i = 1, ..., m$ become intervals (take the form b_i^I, $i = 1, ..., m$).

Thus, noninterval NLP problem (3.3) is converted to a general INLP problem:

$$\begin{array}{ll} \min & f^I(\mathbf{x}) \\ s.t. & g_i^I(\mathbf{x}) \geqslant b_i^I, \quad i = 1, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n. \end{array} \tag{3.4}$$

We state general INLP problem (3.4) as characteristic problem

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ s.t. & g_i(\mathbf{x}) \geqslant b_i, \quad i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n, \end{array} \tag{3.5}$$

where f, $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that the coefficients of the their variables are belonging to the corresponding interval coefficients in the interval functions of $f^I(\mathbf{x})$ and $g_i^I(\mathbf{x})$.

Let $f^\star$ and $\mathbf{x}^\star$ be the optimal value of the objective function and the optimal solution for characteristic problem with the form of (3.5), respectively. Note that, we represent $f(\mathbf{x}^\star)$ as $f^\star$.

In Hladík [2], lower and upper bound of $f^\star$ are defined as infimum and supremum on the optimal value of all characteristic problems, which we represent by $\underline{f}^\star$ and $\overline{f}^\star$.

Now, we assume a situation where the associated intervals of the objective function and constraints are available for general INLP problem (3.4):

$$\begin{aligned} f^I(\mathbf{x}) &= \ [\,\underline{f}(\mathbf{x}),\, \overline{f}(\mathbf{x})\,], \\ g_i^I(\mathbf{x}) &= \ [\,\underline{g}_i(\mathbf{x}),\, \overline{g}_i(\mathbf{x})\,], \quad i = 1, 2, ..., m, \\ b_i^I &= \ [\underline{b}_i, \overline{b}_i], \ \ i = 1, ..., m, \\ x_j &\in X^{si}, \quad j = 1, 2, ..., n, \end{aligned}$$

After the corresponding replacements, general INLP problem (3.4) can take the following interval form:

$$\begin{array}{ll} \min & [\,\underline{f}(\mathbf{x}),\, \overline{f}(\mathbf{x})\,] \\ s.t. & [\,\underline{g}_i(\mathbf{x}),\, \overline{g}_i(\mathbf{x})\,] \geqslant [\underline{b}_i, \overline{b}_i], \ \ i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n, \end{array} \tag{3.6}$$

where, $\underline{f}(\mathbf{x})$, $\overline{f}(\mathbf{x})$, $\underline{g}_i(\mathbf{x})$, and $\overline{g}_i(\mathbf{x})$ for all $i = 1, 2, ..., m$, are called the boundary functions for INLP problem (3.6).

Note that, each characteristic problem of (3.6) has the same form (3.5) such that $f(\mathbf{x}) \in [\,\underline{f}(\mathbf{x}),\, \overline{f}(\mathbf{x})\,]$, $g_i(\mathbf{x}) \in [\,\underline{g}_i(\mathbf{x}),\, \overline{g}_i(\mathbf{x})\,]$ and $b_i \in [\underline{b}_i, \overline{b}_i]$ for $i = 1, 2, ..., m$.

In the remainder of this section, we propose a method for finding the optimal values range of the objective function of INLP problem (3.6).

The idea to compute the optimal values range $[\underline{f}^\star, \overline{f}^\star]$ for INLP problem (3.6) is by solving two characteristic problems of INLP problem (3.6).

First, the following theorem is required.

Theorem 3.1. *In INLP problem* (3.6)*, consider the interval inequality constraint:*

$$[\,\underline{g}_i(\boldsymbol{x}),\, \overline{g}_i(\boldsymbol{x})\,] \geqslant [\underline{b}_i, \overline{b}_i], \tag{3.7}$$

the largest and the smallest feasible regions have the following forms, respectively:

$$\overline{g}_i(\boldsymbol{x}) \geqslant \underline{b}_i, \tag{3.8}$$

and

$$\underline{g}_i(\boldsymbol{x}) \geqslant \overline{b}_i. \tag{3.9}$$

Proof . *Consider characteristic inequality constraint from interval inequality constraint* (3.7) *as*

$$g_i(\boldsymbol{x}) \geqslant b_i, \tag{3.10}$$

where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that the coefficients of the its variables are belonging to the corresponding interval coefficients in the interval function of $g_i^I(\boldsymbol{x})$. Also, we have $g_i(\boldsymbol{x}) \in [\,\underline{g}_i(\boldsymbol{x}), \overline{g}_i(\boldsymbol{x})\,]$ and $b_i \in [\,\underline{b}_i, \overline{b}_i\,]$.

1. *(The largest feasible region) Every solution of* (3.10) *is also a solution of* (3.8)*. Since $\overline{g}_i(\boldsymbol{x}) \geqslant g_i(\boldsymbol{x}) \geqslant b_i \geqslant \underline{b}_i$. Hence, the feasible region* (3.8) *contains all the other feasible regions. Namely,* (3.8) *is the largest feasible region of the interval inequality constraint* (3.7)*.*
2. *(The smallest feasible region) Let the classic inequality constraint* (3.9) *holds. Every solution of* (3.9) *is also a solution of (3.10). Since $g_i(\boldsymbol{x}) \geqslant \underline{g}_i(\boldsymbol{x}) \geqslant \overline{b}_i \geqslant b_i$. So, the feasible region* (3.9) *is the intersection of all other feasible regions. Namely,* (3.9) *is the smallest feasible region of the interval inequality constraint (3.7).*

□

We now explain how to calculate the optimal values $\underline{f}^\star$ and $\overline{f}^\star$ of the objective function of INLP problem (3.6).

Theorem 3.2. *Consider INLP problem* (3.6)*. The optimal values range for the objective function are obtained by solving two classic nonlinear problems:*

$$\begin{aligned} \min \quad & \underline{f}(\boldsymbol{x}) \\ s.t. \quad & \overline{g}_i(\boldsymbol{x}) \geqslant \underline{b}_i, \quad i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n. \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \min \quad & \overline{f}(\boldsymbol{x}) \\ s.t. \quad & \underline{g}_i(\boldsymbol{x}) \geqslant \overline{b}_i, \quad i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n. \end{aligned} \tag{3.12}$$

Proof . *Let $f^\star$ be the optimal value of the objective function for characteristic problem* (3.5)*. Furthermore, suppose $\underline{f}^\star$ and $\overline{f}^\star$ are the optimal values of objective functions for classic nonlinear programming problems* (3.11) *and* (3.12) *respectively. We have to prove that $\underline{f}^\star \leqslant f^\star \leqslant \overline{f}^\star$. Suppose $\boldsymbol{x}'$, $\boldsymbol{x}''$, and $\boldsymbol{x}^\star$ are the optimal solutions of the problems* (3.11)*,* (3.12)*, and* (3.5)*, respectively.*

1. *(The lower bound) According to Theorem 3.1, the constraints of problem (3.11) form the largest feasible region, any feasible solution of problem (3.5) is a feasible solution of problem* (3.11)*. Now since* $\boldsymbol{x}^\star$ *is the optimal solution of problem* (3.5)*, therefore* $\boldsymbol{x}^\star$ *is a feasible solution of problem* (3.11)*. We conclude that* $\underline{f}^\star \leqslant \underline{f}(\boldsymbol{x}^\star)$*.* $\underline{f}(\boldsymbol{x}^\star) \leqslant f(\boldsymbol{x}^\star) = f^\star$ *results* $\underline{f}^\star \leqslant f^\star$*.*
2. *(The upper bound) According to Theorem 3.1, the constraints of problem* (3.12) *form the smallest feasible region, any feasible solution of problem* (3.12) *is a feasible solution of problem* (3.5)*. We conclude that* $f^\star \leqslant f(\boldsymbol{x}'')$*.* $f(\boldsymbol{x}'') \leqslant \overline{f}(\boldsymbol{x}'') = \overline{f}^\star$ *results* $f^\star \leqslant \overline{f}^\star$*.*

□

Similar to problem (3.6), for $\leqslant$ interval inequality constraints, we have the following problem:

$$\begin{aligned} \min \quad & [\,\underline{f}(\mathbf{x}),\, \overline{f}(\mathbf{x})\,] \\ s.t. \quad & [\,\underline{g}_i(\mathbf{x}),\, \overline{g}_i(\mathbf{x})\,] \leqslant [\underline{b}_i, \overline{b}_i], \;\; i = 1,2,...,m, \\ & x_j \in X^{si}, \quad j = 1,2,...,n, \end{aligned} \tag{3.13}$$

So, similar to the theorems 3.1 and 3.2, we have the theorems 3.3 and 3.4, respectively.

Theorem 3.3. *In INLP problem* (3.13)*, consider the interval inequality constraint:*

$$[\,\underline{g}_i(\boldsymbol{x}),\, \overline{g}_i(\boldsymbol{x})\,] \leqslant [\underline{b}_i, \overline{b}_i],$$

the largest and the smallest feasible regions have the following forms, respectively:

$$\underline{g}_i(\boldsymbol{x}) \leqslant \overline{b}_i,$$

and

$$\overline{g}_i(\boldsymbol{x}) \leqslant \underline{b}_i.$$

Theorem 3.4. *Consider INLP problem* (3.13)*. The optimal values range for the objective function are obtained by solving two nonlinear classical problems* (3.14) *and* (3.15)*:*

$$\begin{aligned} \min \quad & \underline{f}(\boldsymbol{x}) \\ s.t. \quad & \underline{g}_i(\boldsymbol{x}) \leqslant \overline{b}_i, \;\; i = 1,2,...,m, \\ & x_j \in X^{si}, \quad j = 1,2,...,n. \end{aligned} \tag{3.14}$$

$$\begin{aligned} \min \quad & \overline{f}(\boldsymbol{x}) \\ s.t. \quad & \overline{g}_i(\boldsymbol{x}) \leqslant \underline{b}_i, \;\; i = 1,2,...,m, \\ & x_j \in X^{si}, \quad j = 1,2,...,n. \end{aligned} \tag{3.15}$$

Without loss of generality, we will only deal with INLP problem (3.6) with minimize objective function and $\geqslant$ inequal constraints.

4. Interval polynomial programming problems

In this section, we introduce a special class of INLP problems, namely IPP problems. We then obtain the boundary functions and optimal values range of the objective function for these problems.

We defined an IPP problem as

$$\begin{aligned}\min \quad & f^I(\mathbf{x}) = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j^{n_j} \\ s.t. \quad & g_i^I(\mathbf{x}) = \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j^{n_{ij}} \geqslant [\underline{b}_i, \overline{b}_i], \quad i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n,\end{aligned} \tag{4.1}$$

where the powers of n_j and n_{ij} are the non-negative integer numbers.

We state IPP problem (4.1) as the characteristic version

$$\begin{aligned}\min \quad & f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j^{n_j} \\ s.t. \quad & g_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j^{n_{ij}} \geqslant b_i, \quad i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n,\end{aligned} \tag{4.2}$$

where $c_j \in [\underline{c}_j, \overline{c}_j]$, $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_i \in [\underline{b}_i, \overline{b}_i]$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Let $f^\star$ be the optimal value of the objective function for characteristic problem with the form of (4.2). Our aim is to find lower and upper bounds for $f^\star$. In other words, we want to compute the optimal values range for the objective function of IPP problem (4.1) by the method presented in previous section.

First, the following theorem is required.

Theorem 4.1. *For an interval inequality constraint*

$$\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}]\, x_j^{n_{ij}} \geqslant [\underline{b}_i, \overline{b}_i], \tag{4.3}$$

where for all j, $x_j \in X^{si}$, the largest and the smallest feasible regions are take the following forms, respectively:

$$\sum_{j=1}^{n} a'_{ij} x_j^{n_{ij}} \geqslant \underline{b}_i, \tag{4.4}$$

and

$$\sum_{j=1}^{n} a''_{ij} x_j^{n_{ij}} \geqslant \overline{b}_i, \tag{4.5}$$

where

$$a'_{ij} = \begin{cases} \overline{a}_{ij}, & x_j \geqslant 0 \\ \underline{a}_{ij}, & x_j \leqslant 0 \end{cases}, \qquad a''_{ij} = \begin{cases} \underline{a}_{ij}, & x_j \geqslant 0 \\ \overline{a}_{ij}, & x_j \leqslant 0 \end{cases}$$

Proof . *Let* $A_1 = \{j \mid x_j \geqslant 0\}$, $A_2 = \{j \mid x_j \leqslant 0\}$. *Also, let* $\sum_{j=1}^{n} a_{ij} x_j^{n_{ij}} \geqslant b_i$ *be the characteristic formula of the interval inequality constraint* (4.3).

1. *(The largest feasible region) For all* $j \in A_1$;

$$\sum_{j \in A_1} \overline{a}_{ij}\, x_j^{n_{ij}} \geqslant \sum_{j \in A_1} a_{ij} x_j^{n_{ij}},$$

and for all $j \in A_2$;

$$\sum_{j \in A_2} \underline{a}_{ij}\, x_j^{n_{ij}} \geqslant \sum_{j \in A_2} a_{ij} x_j^{n_{ij}},$$

by adding the two above inequalities, we have

$$\sum_{j=1}^{n} a'_{ij} x_j^{n_{ij}} \geqslant \sum_{j=1}^{n} a_{ij} x_j^{n_{ij}}, \tag{4.6}$$

On the other hand,

$$\sum_{j=1}^{n} a_{ij} x_j^{n_{ij}} \geqslant b_i \geqslant \underline{b}_i,$$

therefore

$$\sum_{j=1}^{n} a'_{ij} x_j^{n_{ij}} \geqslant \sum_{j=1}^{n} a_{ij} x_j^{n_{ij}} \geqslant b_i \geqslant \underline{b}_i,$$

Namely, any solution which satisfies in characteristic formula of the interval inequality constraint (4.3) *will also be satisfied in classic inequality constraint* (4.4). *So, the relation* (4.4) *is the largest feasible region.*

2. *(The smallest feasible region) Let the classic inequality constraint (4.5) holds. For all* $j \in A_1$;

$$\sum_{j \in A_1} \underline{a}_{ij}\, x_j^{n_{ij}} \leqslant \sum_{j \in A_1} a_{ij} x_j^{n_{ij}},$$

and for all $j \in A_2$;

$$\sum_{j \in A_2} \overline{a}_{ij}\, x_j^{n_{ij}} \leqslant \sum_{j \in A_2} a_{ij} x_j^{n_{ij}},$$

by adding two above inequalities, we have

$$\sum_{j=1}^{n} a''_{ij} x_j^{n_{ij}} \leqslant \sum_{j=1}^{n} a_{ij} x_j^{n_{ij}}, \tag{4.7}$$

On the other hand, we assumed that the classic inequality constraint (4.5) holds. We also have $b_i \leqslant \overline{b}_i$, *hence*

$$b_i \leqslant \overline{b}_i \leqslant \sum_{j=1}^{n} a''_{ij} x_j^{n_{ij}} \leqslant \sum_{j=1}^{n} a_{ij} x_j^{n_{ij}},$$

Therefore, any solution which satisfies in classic inequality constraint (4.5) *will also be satisfied in all possible characteristics of the interval inequality constraint* (4.3) *simultaneously. Consequently, the relation* (4.5) *is the smallest feasible region.*

□

Similarly, for $\leqslant$ interval inequality constraints exists a similar theorem.

Now, we compute $\underline{f}^\star$ and $\overline{f}^\star$ for IPP problem (4.1).

Theorem 4.2. *Consider IPP problem* (4.1)*. The optimal values range for the objective function are obtained by solving the following two classic nonlinear problems, respectively:*

$$\begin{aligned} \min \quad & \underline{f}(\boldsymbol{x}) = \sum_{j=1}^{n} c''_j x_j^{n_j} \\ s.t. \quad & \overline{g}_i(\boldsymbol{x}) = \sum_{j=1}^{n} a'_{ij} x_j^{n_{ij}} \geqslant \underline{b}_i,\ i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \min \quad & \overline{f}(\boldsymbol{x}) = \sum_{j=1}^{n} c'_j x_j^{n_j} \\ s.t. \quad & \underline{g}_i(\boldsymbol{x}) = \sum_{j=1}^{n} a''_{ij} x_j^{n_{ij}} \geqslant \overline{b}_i,\ i = 1, 2, ..., m, \\ & x_j \in X^{si}, \quad j = 1, 2, ..., n, \end{aligned} \tag{4.9}$$

where a'_{ij} and a''_{ij} have been defined in the Theorem 4.1, and

$$c''_j = \begin{cases} \underline{c}_j, & x_j \geqslant 0 \\ \overline{c}_j, & x_j \leqslant 0 \end{cases}, \qquad c'_j = \begin{cases} \overline{c}_j, & x_j \geqslant 0 \\ \underline{c}_j, & x_j \leqslant 0 \end{cases}$$

Proof . *Let $f^\star$ be the optimal value of the objective function for characteristic problem* (4.2)*. Furthermore, let $\underline{f}^\star$ and $\overline{f}^\star$ be the optimal values of objective function for classic nonlinear programming problems* (4.8) *and* (4.9) *respectively. We have to prove that $\underline{f}^\star \leqslant f^\star \leqslant \overline{f}^\star$. Suppose $\boldsymbol{x}'$, $\boldsymbol{x}''$, and $\boldsymbol{x}^\star$ are the optimal solutions of problems* (4.8), (4.9), *and* (4.2)*, respectively.*

1. *(The lower bound) According to Theorem 4.1, the constraints of problem* (4.8) *form the largest feasible region, any feasible solution of problem (4.2) is a feasible solution of problem* (4.8)*. Now since $\boldsymbol{x}^\star$ is the optimal solution of problem (4.2), therefore $\boldsymbol{x}^\star$ is a feasible solution of problem* (4.8)*. We conclude that*

$$\underline{f}^\star \leqslant \sum_{j=1}^{n} c''_j (x_j^\star)^{n_j}. \tag{4.10}$$

Now, since for all j, $\underline{c}_j \leqslant c_j$ therefore

$$\sum_{j \in A_1} \underline{c}_j\, (x_j^\star)^{n_j} \leqslant \sum_{j \in A_1} c_j (x_j^\star)^{n_j},$$

and since for all j, $\overline{c}_j \geqslant c_j$ therefore

$$\sum_{j \in A_2} \overline{c}_j\, (x_j^\star)^{n_j} \leqslant \sum_{j \in A_2} c_j (x_j^\star)^{n_j},$$

by adding two above inequalities, we obtain

$$\sum_{j=1}^{n} c''_j (x^\star_j)^{n_j} \leqslant \sum_{j=1}^{n} c_j (x^\star_j)^{n_j}, \tag{4.11}$$

But, we have

$$\sum_{j=1}^{n} c_j (x^\star_j)^{n_j} = f^\star,$$

therefore

$$\sum_{j=1}^{n} c''_j (x^\star_j)^{n_j} \leqslant f^\star, \tag{4.12}$$

On the other hand, according to relations (4.10) *and* (4.12) *we obtain* $\underline{f}^\star \leqslant f^\star$.

2. *(The upper bound) According to Theorem 4.1, the constraints of problem* (4.9) *form the smallest feasible region, any feasible solution of problem* (4.9) *is a feasible solution of problem (4.2). Hence*

$$f^\star \leqslant \sum_{j=1}^{n} c_j (x''_j)^{n_j}, \tag{4.13}$$

If we prove that

$$\sum_{j=1}^{n} c_j (x''_j)^{n_j} \leqslant \overline{f}^\star, \tag{4.14}$$

Therefore, the second part of theorem will be proven. We know that for all j, $c_j \leqslant \overline{c}_j$ *and* $c_j \geqslant \underline{c}_j$, *we can obtain the following relations respectively:*

$$\sum_{j\in A_1} c_j (x''_j)^{n_j} \leqslant \sum_{j\in A_1} \overline{c}_j\ (x''_j)^{n_j},$$

$$\sum_{j\in A_2} c_j (x''_j)^{n_j} \leqslant \sum_{j\in A_2} \underline{c}_j\ (x''_j)^{n_j},$$

by adding two above inequalities, we obtain

$$\sum_{j=1}^{n} c_j (x''_j)^{n_j} \leqslant \sum_{j=1}^{n} c'_j (x''_j)^{n_j}, \tag{4.15}$$

But, we have

$$\sum_{j=1}^{n} c'_j (x''_j)^{n_j} = \overline{f}^\star,$$

therefore

$$\sum_{j=1}^{n} c_j (x''_j)^{n_j} \leqslant \overline{f}^\star,$$

Namely, the relationship (4.14) holds. So, the proof of second part is now complete.

□

Similarly, for $\leqslant$ interval inequality constraints exists a similar theorem.

Without loss of generality, we will deal only with IPP problem (4.1) with minimize objective function and $\geqslant$ inequal constraints.

5. Numerical examples

In this section, we will illustrate the method presented in this paper by giving two examples. Remember that in these examples, we solve the classic nonlinear programming problems using MATLAB software.

Example 5.1. *Consider the following INLP problem:*

$$\begin{aligned} \min \quad & f^I(\boldsymbol{x}) = \left([1,2]e^{[1,2]x_1}\right) \times \left([2,3]e^{[4,5]x_2}\right) \\ s.t. \quad & g^I(\boldsymbol{x}) = [0.8,1]x_1^3 + [1,2]x_2 \geqslant [1,3], \\ & x_1, x_2 \geqslant 0. \end{aligned} \tag{5.1}$$

We compute the optimal values range for the objective function.

We can rewrite $f^I(\boldsymbol{x}) = \left([1,2]e^{[1,2]x_1}\right) \times \left([2,3]e^{[4,5]x_2}\right)$ *as follows*

$$\begin{aligned} f^I(\boldsymbol{x}) &= ([1,2] \times [2,3])\left(e^{[1,2]x_1} \times e^{[4,5]x_2}\right) \\ &= [2,6] \times e^{[1,2]x_1 + [4,5]x_2} \\ &= [2,6] \times e^{[x_1+4x_2, 2x_1+5x_2]} \\ &= [2e^{x_1+4x_2}, 6e^{2x_1+5x_2}] \end{aligned}$$

After the corresponding replacement, INLP problem (5.1) *can take the following interval form:*

$$\begin{aligned} \min \quad & [\underline{f}(\boldsymbol{x}), \overline{f}(\boldsymbol{x})] = [2e^{x_1+4x_2}, 6e^{2x_1+5x_2}] \\ s.t. \quad & [\underline{g}(\boldsymbol{x}), \overline{g}(\boldsymbol{x})] = [0.8x_1^3 + x_2, x_1^3 + 2x_2] \geqslant [1,3], \\ & x_1, x_2 \geqslant 0. \end{aligned} \tag{5.2}$$

Using Theorem 3.2, we obtain the optimal values range for INLP problem (5.2) *by solving following two classic nonlinear problems:*

$$\begin{aligned} \min \quad & \underline{f}(x) = 2e^{x_1+4x_2} \\ s.t. \quad & \overline{g}(x) = x_1^3 + 2x_2 \geqslant 1, \\ & x_1, x_2 \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} \min \quad & \overline{f}(x) = 6e^{2x_1+5x_2} \\ s.t. \quad & \underline{g}(x) = 0.8x_1^3 + x_2 \geqslant 3, \\ & x_1, x_2 \geqslant 0, \end{aligned}$$

The former has an optimal value $\underline{f}^\star = 5.44$ *and the latter* $\overline{f}^\star = 134.15$*, which yields* $[\underline{f}^\star, \overline{f}^\star] = [5.44, 134.15]$*. Namely, the optimal value* $f^\star$ *for each characteristic problem from INLP problem (5.1) is in the optimal interval* $[5.44, 134.15]$*. For instance*

$$\begin{aligned} \min \quad & f(x) = (2e^{x_1}) \times (2e^{4.5x_2}) \\ s.t. \quad & g(x) = 0.8x_1^3 + 2x_2 \geqslant 1.5, \\ & x_1, x_2 \geqslant 0, \end{aligned}$$

which has the optimal value $f^\star = 13.73 \in [5.44, 134.15]$.

Example 5.2. *Consider the following IPP problem:*

$$\begin{aligned} \min\quad & f^I(\boldsymbol{x}) = [2,3]x_1 + [5,6]x_2^3 - [2,3]x_3^3 \\ s.t.\quad & g_1^I(\boldsymbol{x}) = [1,2]x_1^2 + [2,3]x_2^2 + [-1,5]x_3 \geqslant [4,6], \\ & g_2^I(\boldsymbol{x}) = [2,3]x_1^5 + [2,4]x_2 + [-3,-2]x_3^2 \geqslant [-26,-20], \\ & x_1, x_3 \leqslant 0,\ x_2 \geqslant 0. \end{aligned} \tag{5.3}$$

We compute the optimal values range for the objective function.

We can rewrite IPP problem (5.3) *as follows*

$$\begin{aligned} \min\quad & [\underline{f}(\boldsymbol{x}), \overline{f}(\boldsymbol{x})] = [3x_1 + 5x_2^3 - 2x_3^2, x_1 + 6x_2^3 - 3x_3^3] \\ s.t.\quad & [\underline{g}_1(\boldsymbol{x}), \overline{g}_1(\boldsymbol{x})] = [x_1^2 + 2x_2^2 + 5x_3, 2x_1^2 + 3x_2^2 - x_3] \geqslant [4,6], \\ & [\underline{g}_2(\boldsymbol{x}), \overline{g}_2(\boldsymbol{x})] = [3x_1^5 + 2x_2 - 3x_3^2, 2x_1^5 + 4x_2 - 2x_3^2] \geqslant [-26,-20], \\ & x_1, x_3 \leqslant 0,\ x_2 \geqslant 0. \end{aligned} \tag{5.4}$$

Using Theorem 4.2, we obtain the optimal values range for IPP problem (5.4) *by solving following two classic nonlinear problems:*

$$\begin{aligned} \min\quad & \underline{f}(\boldsymbol{x}) = 3x_1 + 5x_2^3 - 2x_3^2 \\ s.t.\quad & \overline{g}_1(\boldsymbol{x}) = 2x_1^2 + 3x_2^2 - x_3 \geqslant 4, \\ & \overline{g}_2(\boldsymbol{x}) = 2x_1^5 + 4x_2 - 2x_3^2 \geqslant -26, \\ & x_1, x_3 \leqslant 0,\ x_2 \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} \min\quad & \overline{f}(\boldsymbol{x}) = 2x_1 + 6x_2^3 - 3x_3^3 \\ s.t.\quad & \underline{g}_1(\boldsymbol{x}) = x_1^2 + 2x_2^2 + 5x_3 \geqslant 6, \\ & \underline{g}_2(\boldsymbol{x}) = 3x_1^5 + 2x_2 - 3x_3^2 \geqslant -20, \\ & x_1, x_3 \leqslant 0,\ x_2 \geqslant 0, \end{aligned}$$

The former has an optimal value $\underline{f}^\star = -5.02$ *and the latter* $\overline{f}^\star = 12.42$, *which yields* $[\underline{f}^\star, \overline{f}^\star] = [-5.02, 12.42]$. *Namely, the optimal value* $f^\star$ *for each characteristic problem from IPP problem* (5.3) *is in the optimal interval* $[-5.02, 12.42]$. *For instance*

$$\begin{aligned} \min\quad & \underline{f}(\boldsymbol{x}) = 3x_1 + 5x_2^3 - 2x_3^2 \\ s.t.\quad & \underline{g}_1(\boldsymbol{x}) = x_1^2 + 2x_2^2 + 5x_3 \geqslant 6, \\ & \underline{g}_2(\boldsymbol{x}) = 3x_1^5 + 2x_2 - 3x_3^2 \geqslant -20, \\ & x_1, x_3 \leqslant 0,\ x_2 \geqslant 0, \end{aligned}$$

which has the optimal value $f^\star = 8.35 \in [-5.02, 12.42]$.

6. Conclusion

The proposed method measures the optimal values range for the objective function in INLP and IPP problems. Which is important and applied, because it solves the INLP (and also IPP) problems using two deterministic problems.

In [2] the optimal values range of the objective function for two special formats of INLP problems was obtained based on duality theory. In this paper, first we calculate the optimal values range for general INLP problems in a case where the associated intervals of the objective function and constraints are available. Then, we directly calculate the optimal values range of objective function for IPP problems.

Knowing the optimal values range for the objective function (where the objective function includes the total cost or final profit) is important in INLP problems because the decision maker can easily have the maximum and minimum values for the objective function and thus make more suitable decisions.

In fact,two approaches can be used when dealing with INLP problems. First is to determine the optimal values range while the second one is finding the set of optimal solutions for INLP problems. We used the first approach in the current study.

The application of the proposed method was also illustrated using two examples.

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